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SLICES OF MAPS AND LEBESGUE AREA(1)

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Abstract. For a large class of k dimensional surfaces, S, it is shown that the Lebesgue area of S can be essentially expressed in terms of an integral of the $k-1$ area of a family, F, of $k-1$ dimensional surfaces that cover S. The family F is regarded as being composed of the slices of F. The definition of the $k-1$ area of a surface restricted to one of its slices is formulated in terms of the theory developed by H. Federer, [F3].

1. **Introduction.** There are many results in geometric measure theory that yield an inequality between the k dimensional measure of a set $R \subset E^n$ and the integral of the $k-1$ dimensional measure of the intersection of R with the level surfaces of certain real-valued functions defined on E^n. There are similar results in the theory of Lebesgue area.

In this paper we will show that the Lebesgue area of a large class of k dimensional surfaces, S, can be essentially expressed in terms of an integral of the $k-1$ area of a family, F, of $k-1$ dimensional surfaces that cover S. An analogous result is obtained for the k dimensional Hausdorff measure of a Hausdorff k-rectifiable set. We regard the family F as being composed of the slices of the surface and one of the essential parts of our problem is to determine the appropriate definition of the $k-1$ area of a surface restricted to one of its slices. The definition that we employ is expressed in terms of the theory developed by H. Federer, [F3]. In case $k=2$, there are two other definitions considered and we show that they lead to the same results. One definition was created by L. Cesari [C] and it employs the concepts of the Carathéodory theory of prime ends. The other definition depends upon the notion of the length of a light mapping that was introduced in [F2].

For 2 dimensional surfaces, R. Rishel [RL] and R. Fullerton [FN] obtained results that are in the same spirit as ours by employing Cesari's definition of generalized length. However, Rishel's definition of the slice of a surface is different from ours and Fullerton's premature death precluded him from developing his results beyond polyhedral parametric surfaces and continuously differentiable non-parametric surfaces.

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2. **Preliminaries and definitions.** Euclidean n-space will be denoted by E^n. Lebesgue measure on E^n will be denoted by L_n and H^k will stand for k dimensional Hausdorff measure. If $A \subset E^n$, then the measure $H^k \llcorner A$ is defined by $H^k \llcorner A(E) = H^k(A \cap E)$. A set $R \subset E^n$ is called *Hausdorff k-rectifiable* if there is a Lipschitzian function f on E^k to E^n such that

$$H^k(R - \text{range} f) = 0.$$

We refer the reader to [F4] for a thorough investigation of the properties possessed by Hausdorff k-rectifiable sets. Many of these properties will be employed below and the following will be especially useful. If $R \subset E^n$ is a Hausdorff k-rectifiable set, $g\colon R \to E^m$ is Lipschitzian, and $j = \min [k, m]$, then g possesses an $(H^k \llcorner R, k)$ approximate j dimensional Jacobian $H^k \llcorner R$ almost everywhere, [F4, 3.2.19]. Denote this Jacobian by $\text{ap}\, J_j g$ and in case $k > m = 1$, we will use the notation $\text{ap}\, |\nabla g|$. The following theorem is of particular importance, [F4, 3.2.20, 3.2.22].

2.1. THEOREM. *If $R \subset E^n$ is Hausdorff k-rectifiable, $g\colon R \to E^m$ is Lipschitzian, and $j = \min [k, m]$, then*

$$\int_R \text{ap}\, J_j g \, dH^k = \int_{E^m} H^\alpha[g^{-1}(y) \cap R] \, dH^j(y)$$

where $\alpha = \max [0, k - m]$.

Finally, we recall the definition of Hausdorff k dimensional density

$$\Theta^k(H^k \llcorner R, y) = \lim_{r \to 0} \alpha(k)^{-1} r^{-k} H^k[R \cap B(y, r)]$$

where $B(y, r)$ denotes the open n-ball of radius r with center at y and $\alpha(k)$ is the volume of the unit k-ball in E^k. If R is Hausdorff k-rectifiable, then

$$\Theta^k(H^k \llcorner R, y) = 1 \tag{1}$$

for H^k almost all $y \in R$.

We will now show that the Hausdorff k measure of a rectifiable set can be expressed in terms of the integral of the $k-1$ measure of its slices. The proof of this result will establish the method that is basic in the demonstration of the main theorem, 3.3, that appears below.

2.2. THEOREM. *Suppose $R \subset E^n$ is a Hausdorff k-rectifiable set. Then*

$$H^k(R) = \sup \left\{ \int_{E^1} H^{k-1}[u^{-1}(r) \cap R] \, dL_1(r) \right\}$$

where the supremum is taken over all Lipschitz functions $u\colon E^n \to E^1$ that have Lipschitz constant 1.

Proof. If u is such a function, then $\text{ap}\, |\nabla u|$ exists and is dominated by 1 at $H^k \llcorner R$ almost all points. Therefore, by appealing to Theorem 2.1 it is clear that the above supremum is no more than $H^k(R)$.

In order to establish the opposite inequality, it will suffice to show that for every $\varepsilon>0$, there is a function $u: E^n \to E^1$ with Lipschitz constant 1 which satisfies

$$\int_R \operatorname{ap} |\nabla u| \, dH^k > (1-\varepsilon)H^k(R). \tag{2}$$

To this end, first recall that R has an H^k approximate tangent k plane, $P(y)$, at H^k almost all $y \in R$, [F4, 3.3.18]. Moreover, there exists a $H^k \llcorner R$ measurable function α defined on R with values in the space of Grassmann simple k vectors of unit norm. In addition, for H^k almost all $y \in R$, $P(y)$ is the k dimensional vectorsubspace of E^n associated with $\alpha(y)$, [F4, 3.2.25]. Consequently, from Lusin's Theorem, the density theorem [F4, 2.9.11], and (1), it follows that at H^k almost all $y \in R$ there is a set $A \subset R$ such that

$$\begin{aligned} &\text{(i)} \quad \Theta^k(H^k \llcorner A, y) = 1 \text{ and} \\ &\text{(ii)} \quad \alpha \text{ is continuous at } y \text{ relative to } A. \end{aligned} \tag{3}$$

Consider a point $y \in R$ where $P(y)$ exists and let $N(y)$ be the $n-k$ plane passing through y that is perpendicular to $P(y)$. For every real number $\beta>0$, let

$$C(\beta) = \{x : \operatorname{dist}[x, N(y)] > \beta \operatorname{dist}[x, P(y)]\}.$$

It follows from the definition of the H^k approximate tangent k plane, $P(y)$, that

$$\lim_{r\to 0} r^{-k}H^k[R \cap B(y, r) - C(\beta)] = 0 \tag{4}$$

for every $\beta>0$.

Choose $y \in R$ such that $P(y)$ exists, and where (1) and (3) hold. This will be true at H^k almost all $y \in R$. Let $\varepsilon>0$. Choose $\zeta(\varepsilon)=\zeta$ (which will be determined later) and select β large enough to ensure that

$$|\pi_y[(z-y)/|z-y|]| > 1-\zeta \tag{5}$$

whenever $z \in C(\beta)$. Here $\pi_y: E^n \to P(y)$ denotes the orthogonal projection. From (4), (3), and the equality

$$\Theta^k(H^k \llcorner R, y) = \Theta^k(H^k \llcorner A, y) = 1$$

follows the existence of $r^*(y, \varepsilon)=r^*$ such that, for $0<r<r^*$,

$$\begin{aligned} &H^k[A \cap B(y, r) \cap C(\beta)] > (1-\varepsilon)^{1/2}H^k[R \cap B(y, r)] \quad \text{and} \\ &|\alpha(x)-\alpha(y)| < \zeta \quad \text{whenever } x \in A \cap B(y, r). \end{aligned} \tag{6}$$

Select $0<r<r^*$ and define $u(x)=\operatorname{dist}[x, E^n-B(y, r)]$. Since $|\nabla u(x)|=1$ in $B(y, r)$, (5) implies that

$$|\pi_y[\nabla u(x)]| > 1-\zeta \quad \text{for } x \in B(y, r) \cap C(\beta). \tag{7}$$

If ζ is chosen sufficiently small, then from (6) and (7) clearly follows $|\pi_x[\nabla u(x)]| > (1-\varepsilon)^{1/2}$ for $x \in A \cap B(y, r) \cap C(\beta)$ where $P(x)$ exists. Therefore,

$$\int_{R\cap B(y,r)} \text{ap } |\nabla u| \, dH^k \geqq (1-\varepsilon)^{1/2} H^k[A \cap B(y, r) \cap C(\beta)]$$

and thus (6) leads to

$$\int_{R\cap B(y,r)} \text{ap } |\nabla u| \, dH^k \geqq (1-\varepsilon) H^k[R \cap B(y, r)] \tag{8}$$

whenever $0 < r < r^*$.

For H^k almost every $y \in R$, consider the family of all n-balls $B(y, r)$ where $0 < r < r^*(y, \varepsilon)$. Then by a covering theorem due to Besicovitch and A. P. Morse that has been generalized in [F4, 2.8.15], there exist a countable number of n-balls, $B_1, B_2, \ldots$, whose closures are disjoint such that $H^k \llcorner R(E^n - \bigcup_{i=1}^{\infty} B_i) = 0$. With each n-ball B_i is associated the function $u_i(x) = \text{dist } [x, E^n - B_i]$ so that (8) is satisfied. Letting $u(x) = \sum_{i=1}^{\infty} u_i(x)$, $x \in E^n$, it is clear that u has Lipschitz constant 1 and that

$$\begin{aligned} (1-\varepsilon)H^k(R) &= (1-\varepsilon)H^k \llcorner R(E^n) = (1-\varepsilon) \sum_{i=1}^{\infty} H^k \llcorner R(B_i) \\ &\leqq \sum_{i=1}^{\infty} \int_{R\cap B_i} \text{ap } |\nabla u_i| \, dH^k = \int_R \text{ap } |\nabla u| \, dH^k. \end{aligned}$$

According to (2), this concludes the proof.

3. **Slices and Lebesgue area.** In this section we will establish a result concerning Lebesgue area that is analogous to Theorem 2.2. *Throughout this section we will consider a continuous map $f\colon X \to E^n$ of finite Lebesgue area, where X is a k dimensional smooth manifold, $k \leqq n$. We will also assume that $k = 2$ or $H^{k+1}[f(X)] = 0$.*

Our treatment relies heavily on the work of Federer [F3] and the notation and results of that paper will be employed here without change. Thus, the *monotone-light factorization* $f = l_f \circ m_f$ will be considered where l_f is defined on the *middle space*, M_f. Moreover, there is a unique current-valued measure μ over M_f whose total variation, $\|\mu\|$, is equal to the Lebesgue area of f. If T is a current, then $M(T)$ is the *mass* of T and $F(T)$ denotes the *flat norm* of T. The *boundary* of T is denoted by ∂T. Define a measure ν over E^n by $\nu(A) = \|\mu\|(l_f^{-1}(A))$.

One of the main results of [F3] states that there is an integer valued function, K, such that

$$\mathscr{L}(f) = \int_{E^n} K(y) \, dH^k(y) \tag{9}$$

where $\mathscr{L}(f)$ is the Lebesgue area of f. For $y \in E^n$ and $r > 0$, let $Z(r)$ be the family of components of $l_f^{-1}[B(y, r)]$. For ν almost all $y \in E^n$, there are a finite number of "essential" points, z, of $l_f^{-1}(y)$ with the property that if r is sufficiently small, then with each $V \in Z(r)$ that contains an essential z is associated an integer $m(V)$ such that

$$K(y) = \sum_{V \in Z(r)} |m(V)|. \tag{10}$$

Moreover, there is an oriented k dimensional plane, $P(y)$, passing through y such that, for each such $V \in Z(r)$,

$$\lim_{r \to 0} r^{-k} F[\mu(V) - m(V) \cdot P(y) \cap B(y, r)] = 0. \tag{11}$$

3.1. Definition. Let $u: E^n \to E^1$ be Lipschitz and let $Z(r)$ be the set of components of $M_f \cap \{z : u \circ l_f(z) < r\}$. Define

$$\lambda(f; u, r) = \sum_{V \in Z(r)} M[\partial\mu(V)].$$

As a function of r, $\lambda(f; u, r)$ is L_1 measurable; indeed,

$$\liminf_{t \to r^-} \lambda(f; u, t) \geqq \lambda(f; u, r).$$

To see this, choose $r > 0$ and let $V_1, V_2, \ldots$ be the components that constitute $Z(r)$. For $t > r$ let $W_i(t)$ be the union of those components of $Z(t)$ that are contained in V_i, $i = 1, 2, \ldots$. The sets $W_i(t)$ are nested and increase with t. Therefore, since $\bigcup_{t<r} W_i(t) = V_i$, it follows that, as currents, $\mu[W_i(t)] \to \mu(V_i)$ weakly as $t \to r^-$. From the facts that ∂ is continuous and mass is lower semicontinuous with respect to weak convergence follows

$$\liminf_{t \to r^-} M[\partial\mu(W_i(t))] \geqq M[\partial\mu(V_i)], \qquad i = 1, 2, \ldots.$$

Hence,

$$\begin{aligned} \liminf_{t \to r^-} \lambda(f; u, t) &\geqq \liminf_{t \to r^-} \sum_{i=1}^{\infty} M[\partial\mu(W_i(t))] \\ &\geqq \sum_{i=1}^{\infty} M[\partial\mu(V_i)] = \lambda(f; u, r). \end{aligned}$$

3.2. Lemma. *If $f: X \to E^n$ and $u: E^n \to E^1$ has Lipschitz constant N, then*

$$\int_{-\infty}^{\infty} \lambda(f; u, r)\, dL_1(r) \leqq N\mathscr{L}(f).$$

Proof. Let

$$\gamma(r) = \|\mu\|(\{z : u \circ l_f(z) < r\}).$$

Observe, for L_1 almost all r, that $\gamma'(r) < \infty$ and for each $V \in Z(r)$ that $\mu(V)$ is an integral current, [F3, 3.4]. Now by applying [FF, 3.9] to $T = \mu(V)$, the proof proceeds as in [F3, 3.2] and we obtain

$$M[\partial\mu(V)] \leqq N \liminf_{h \to 0^+} h^{-1} \|\mu(V)\|(\{y : r - h \leqq u(y) < r\}).$$

From this follows

$$\begin{aligned}\lambda(f; u, r) &\leqq N \liminf_{h\to 0+} h^{-1} \sum_{V\in Z(r)} \|\mu(V)\|(\{y : r-h \leqq u(y) < r\})\\ &\leqq N \liminf_{h\to 0+} h^{-1}\|\mu(V)\|(\{z : r-h \leqq u\circ l_f(z) < r\}) \leqq N\gamma'(r).\end{aligned}$$

Our purpose now is to show that the supremum of the left side of 3.2 over all functions u with $N=1$ equals the Lebesgue area of f.

To this end let R be the set of those $y \in E^n$ for which (10) and (11) hold and for which $0<K(y)<\infty$. Then R is a Hausdorff k-rectifiable set and, without loss of generality, we may assume that

$$\Theta^k(H^k \llcorner R, y) = 1, \qquad y \in R. \tag{12}$$

Choose $y \in R$ and to simplify notation, take $y=0$. Assume r to be taken small enough so that (10) holds. Select an essential $z \in l_f^{-1}(0)$ and consider those $V \in Z(r)$ that contain z. Let $T_r=\mu(V)$ and $P=P(0)$. Let $\pi: E^n \to P$ be the orthogonal projection and define $h_r: P \to P$ by $h_r(x)=r^{-1}\cdot x$. The flat norm of a current is not increased under a projection and, for k dimensional currents in E^k, the flat norm and the mass norm agree. Therefore, it follows from (11) that

$$\lim_{r\to 0} r^{-k}M[\pi_\#(T_r)-m(V)\cdot B(0, r)\cap P] = 0.$$

This implies

$$\lim_{r\to 0} M[(h_r\circ\pi)_\#(T_r)-m(V)\cdot B(0, 1)\cap P] = 0.$$

Again, from the continuity of ∂ and the lower semicontinuity of mass, for $\eta>0$ and all sufficiently small r,

$$M[(h_r\circ\pi)_\#(\partial T_r)] \geqq |m(V)|M[\partial(B(0, 1)\cap P)]-\eta.$$

Thus, for all small r,

$$\begin{aligned}M(\partial T_r) &\geqq M[\partial\pi_\#(T_r)] = M[(h_r\circ\pi)_\#(\partial T_r)]r^{k-1}\\ &\geqq [|m(V)|-\eta\alpha(k-1)^{-1}]M[\partial(B(0, r)\cap P)].\end{aligned} \tag{13}$$

Choose $\delta>0$. It follows from (10) and (13) that, for $y \in R$, there is $r^*=r^*(y, \delta)$ such that if $B=B(y, r)$, $0<r<r^*$, and $u(x)=-\operatorname{dist}(x, E^n-B)$, then

$$\lambda(f; u, -t) = \sum_{V\in Z(-t)} M[\partial\mu(V)] \geqq [K(y)-\delta]M[\partial(B(y, t)\cap P)]$$

where $Z(-t)$ is the set of components of $\{z : u\circ l_f(z)<-t\}$ and $0<t<r$. Therefore,

$$\int_{-\infty}^{\infty} \lambda(f; u, t)\, dL_1(t) \geqq [K(y)-\delta]\alpha(k)r^k. \tag{14}$$

Define a measure ζ over E^n by

$$\zeta(E) = \int_{E\cap R} K(y)\, dH^k(y)$$

for every Borel set E. Then

$$\zeta(E^n) = \mathscr{L}(f); \tag{15}$$

indeed, $\zeta=\nu$, but we will not use this fact. Appealing to [F4, 2.9.8] we have for H^k almost all $y \in R$, that

$$\lim_{r\to 0} \zeta[B(y, r)]/H^k \llcorner R[B(y, r)] = K(y). \tag{16}$$

For $\varepsilon>0$ and $y \in R$, it follows from (12), (14), and (16) that there exists $r^*(y, \varepsilon)=r^*$ such that

$$\int_{-\infty}^{\infty} \lambda(f; u, t)\, dL_1(t) \geqq (1-\varepsilon)\zeta[B(y, r)] \tag{17}$$

whenever $0<r<r^*$. Consider the family of n-balls, $B(y, r)$, $y \in R$ and $0<r<r^*(y, \varepsilon)$. Appealing to the covering theorem [F4, 2.8.15], there exist balls $B_1, B_2, \ldots$ whose closures are disjoint and that have the property

$$\zeta\left(E^n - \bigcup_{i=1}^{\infty} B_i\right) = 0. \tag{18}$$

As in the proof of 2.2 we define

$$u(x) = \sum_{i=1}^{\infty} u_i(x), \qquad x \in E^n,$$

where $u_i(x)=-\operatorname{dist}(x, E^n-B_i)$, $i=1, 2, \ldots$. Then u has Lipschitz constant 1 and $\lambda(f; u, r)=\sum_{i=1}^{\infty} \lambda(f; u_i, r)$, $r \in E^1$. Therefore, (17) and (18) yield

$$\begin{aligned}\int_{-\infty}^{\infty} \lambda(f; u, r)\, dL_1(r) &= \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} \lambda(f; u_i, r)\, dL_1(r) \\ &= \sum_{i=1}^{\infty} (1-\varepsilon)\zeta(B_i) = (1-\varepsilon)\zeta(E^n) = (1-\varepsilon)\mathscr{L}(f).\end{aligned}$$

Thus, the following theorem has been established.

3.3. THEOREM. *Suppose $f: X \to E^n$ has finite Lebesgue area and suppose $k=2$ or $H^{k+1}[f(X)]=0$. Then $\sup\{\int_{-\infty}^{\infty} \lambda(f; u, r)\, dL_1(r)\}=\mathscr{L}(f)$ where the supremum is taken over all functions $u: E^n \to E^1$ that have Lipschitz constant* 1.

4. **Mappings from a 2-cell.** The results of [F3] are also valid in case X is a manifold with boundary, [G], [M], and therefore Theorem 3.3 also holds in this case. In this section we consider the special situation when X is the unit square, Q, in E^2 for then there are two other reasonable definitions for the 1-area or "length" of f restricted to a slice determined by u. In order for 3.3 to remain valid, Definition 3.1 of $\lambda(f; u, r)$ is modified so that only those components $V \in Z(r)$ are considered for which closure $V \cap m_f(\text{bdry } Q)=0$. Let $C(r)$ be the set of such components.

4.1. DEFINITION. If U is a subset of a separable metric space, let

$$\delta(U) = \text{bdry } U \cap \{z : \text{dimension}(\text{bdry } U, z) > 0\}.$$

4.2. DEFINITION. For $V \in C(r)$, let

$$|V| = \int_{E^n} N[l_f, \delta(V), y]\, dH^1(y)$$

where $N[l_f, \delta(V), y]$ denotes the number of points (possibly ∞) in $l_f^{-1}(y) \cap \delta(V)$. Define

$$\rho(f; u, r) = \sum_{V \in C(r)} |V|.$$

Similarly, for $V \in C(r)$, let $\|V\|$ denote the length of f restricted to the boundary of $m_f^{-1}(V)$ as defined by Cesari in [C, 20.2]. Define

$$\sigma(f; u, r) = \sum_{V \in C(r)} \|V\|.$$

Finally, define

$$\mathscr{C}(f) = \sup\left\{\int_{-\infty}^{\infty} \sigma(f; u, r)\, dL_1(r)\right\}$$

where the supremum is taken over all functions $u: E^n \to E^1$ that have Lipschitz constant 1. By replacing $\sigma(f; u, r)$ by $\rho(f; u, r)$, we define $\mathscr{D}(f)$ in a similar manner.

The following result is an immediate consequence of [C, 20.5] and [RL, 1.6].

4.2. THEOREM. *If $f: Q \to E^n$ is continuous, then $\mathscr{L}(f) \geqq \mathscr{C}(f) \geqq \mathscr{D}(f)$.*

In order to prove that $\mathscr{D}(f) \geqq \mathscr{L}(f)$, we first observe that if f and g are Fréchet equivalent maps of Q into E^n, then $\mathscr{D}(f) = \mathscr{D}(g)$. The same is true for the functionals $\mathscr{L}$ and $\mathscr{C}$, [C, 31.7]. These facts will be very useful since we later employ Morrey's representation theorem. Undoubtedly, it is possible to prove $\mathscr{D}(f) \geqq \mathscr{L}(f)$ without resorting to Morrey's theorem, but we prefer this method since we anticipate an application of (24) below in future work.

In this regard recall that a continuous map $f: Q \to E^n$ is *almost conformal* provided that

(i) the coordinate functions of f are absolutely continuous in the sense of Tonelli (ACT) on Q, and their partial derivatives are square integrable on Q, and

(ii) the partial derivatives $D_1 f$, $D_2 f$ satisfy $D_1 f(x) \cdot D_2 f(x) = 0$ for L_2 almost all $x \in Q$.

The formal differential, df, which is defined as the linear transformation associated with the matrix of partial derivatives of f, exists for L_2 almost all points in Q. Thus, for all such x, $df(x)$ is a linear transformation from E^2 into E^n and it induces a linear transformation on the space of Grassmann 2-vectors. Thus, if α is a 2-vector in E^2, then $df(x)(\alpha) = df(x, \alpha)$ is a 2-vector in E^n. Later, we will also use the fact that if $\{f_i\}$ is a sequence of mollifiers of f, then

(19)
(i) $f_i \to f$ uniformly on compact subsets of Q and
(ii) $\int_Q |df_i - df|^2\, dL_2 \to 0$ where $|df(x)|$ denotes the norm of $df(x)$.

If $u: E^n \to E^1$ is Lipschitz and $f: Q \to E^n$ is almost conformal, then it is not difficult to show that $F = u \circ f$ is ACT on Q and $|\nabla F|$ is square integrable on Q. Moreover, the following facts were established in [Z]:

(20)

(i) If g is square integrable, then

$$\int_Q |\nabla F| g \, dL_2 = \int_{E^1} \int_{F^{-1}(r)} g(y) \, dH^1(y) \, dL_1(r).$$

(ii) If W is a component of $\{x : F(x) < r\}$, then for L_1 almost all r, W has finite perimeter. That is, if W is considered as a current, then in fact, it is an integral current.

(iii) Let $\beta(W)$ be the reduced boundary of W, that is, $\beta(W)$ consists of those points x where $n(x)$, the exterior normal to W, exists at x. Observe that $\beta(W) \subset \operatorname{bdry} W \subset F^{-1}(r)$. Following the proof of [Z, 3.3], we have $H^1[\beta(W) - \delta(W)] = 0$.

4.3. Lemma. *If $f: Q \to E^n$ is almost conformal and $u: E^n \to E^1$ is Lipschitz, then for L_1 almost all r, $f^{-1}(y)$ is totally disconnected for H^1 almost all $y \in u^{-1}(r)$.*

Proof. In view of the fact that the coordinate functions of f are ACT on Q, it follows that there is a countable set of vertical line segments in Q, Λ_1, such that Λ_1 is dense in Q and if $\lambda \in \Lambda_1$ then $H^1[f(\lambda)] < \infty$. Likewise, there is a set Λ_2 corresponding to the horizontal direction. Setting $\Lambda = \Lambda_1 \cup \Lambda_2$, it follows that if $N = f(\Lambda)$ then $H^2[N] = 0$. However, [F1, 3.2] or 2.1 implies that $H^1[N \cap u^{-1}(r)] = 0$ for L_1 almost all r. Hence, if $y \notin N \cap u^{-1}(r)$, then $f^{-1}(y)$ has only point components.

4.4. Lemma. *With the same hypotheses as* 4.3, *$\lambda(f; u, r) \leq \rho(f; u, r)$ for L_1 almost all $r \in E^1$.*

Proof. Let α be a continuous function on Q whose values are unit Grassmann 2-vectors in E^2. Select $r \in E^1$ so that the results of (20) and 4.3 hold, and let W be a component of $\{x : F(x) < r\}$ such that (closure W) $\cap$ bdry $Q = 0$. Since the mollifiers, f_i, of f are C^∞, (19) implies for every C^∞ differential 2-form φ that

$$\begin{aligned} \lim_{i \to \infty} f_{i\#}(W)(\varphi) &= \lim_{i \to \infty} \int_W \varphi[f_i(x)] \cdot df_i(x, \alpha(x)) \, dL_2(x) \\ &= \int_W \varphi[f(x)] \cdot df(x, \alpha(x)) \, dL_2(x). \end{aligned} \tag{21}$$

On the other hand, [F3, 3.4] implies that, for L_1 almost all r, $\lim_{i \to \infty} f_{i\#}(W)(\varphi) = \mu(V)(\varphi)$ where $V = m_f(W)$. Thus,

$$\mu(V)(\varphi) = \int_W \varphi[f(x)] \cdot df(x, \alpha(x)) \, dL_2(x). \tag{22}$$

Since $\int_Q |df_i - df|^2 \, dL_2 \to 0$ it follows from [(20)(i)] that there exists a subsequence (which will still be denoted as $\{df_i\}$) such that

$$\int_{F^{-1}(r)} |df_i - df| \, dH^1 \to 0. \tag{23}$$

From Hölder's inequality and (i) of (20), it is clear that (23) holds for L_1 almost all r. Since $\beta(W) \subset F^{-1}(r)$ for all r under consideration, (19), (iii) of (20), (21), (22), (23), and the Gauss-Green Theorem [F4, 4.5.6] imply that, for L_1 almost all r,

$$\begin{aligned} \partial\mu(V)(\varphi) &= \lim_{i\to\infty} \partial f_{i\#}(W)(\varphi) = \lim_{i\to\infty} f_{i\#}(\partial W)(\varphi) \\ &= \lim_{i\to\infty} \int_{\beta(W)} \varphi[f_i(x)] \cdot df_i(x, v(x)) \, dH^1 \\ &= \int_{\beta(W)} \varphi[f(x)] \cdot df(x, v(x)) \, dH^1(x) \\ &= \int_{\beta(W)\cap\delta(W)} \varphi[f(x)] \cdot df(x, v(x)) \, dH^1(x) \end{aligned} \tag{24}$$

where $v(x)$ is the unit vector perpendicular to $n(x)$ chosen so that $v(x) \wedge n(x) = \alpha(x)$. Since the partial derivatives of the coordinate functions of f exist L_2 almost everywhere in Q, we apply [F4, 3.1.8] to find disjoint sets $Q = \bigcup_{i=0}^{\infty} A_i$ such that $L_2(A_0) = 0$ and f restricted to A_i, $i > 0$, is Lipschitzian. From [FU, Theorem 3(d)] and (i) of (20), we may assume that $H^1[\delta(W) \cap A_0] = 0$. Thus, [F4, 3.2.20] and (24) yield

$$M[\partial\mu(V)] \leqq \int_{E^n} N[f, \delta(W), y] \, dH^1(y).$$

However, 4.3 allows us to assume that

$$\int_{E^n} N[f, \delta(W), y] \, dH^1(y) = \int_{E^n} N[l_f, \delta(V), y] \, dH'(y)$$

and the lemma follows directly from this.

4.5. Theorem. *Suppose $f: Q \to E^n$ has finite Lebesgue area and, in addition, suppose f has the property that for each $y \in E^n$, no component of $f^{-1}(y)$ disconnects Q. Then $\mathscr{C}(f) = \mathscr{D}(f) = \mathscr{L}(f)$.*

Proof. With the conditions imposed on f it follows that the middle space, M_f, is either a 2-cell or a 2-sphere, [R, II.2.91].

In the event that M_f is a 2-cell, Morrey's representation theorem [MO1], [MO2] asserts the existence of an almost conformal map $g: Q \to E^n$ that is Fréchet equivalent to f. Now $\mathscr{D}(f) = \mathscr{D}(g)$ and $\mathscr{L}(f) = \mathscr{L}(g)$ while 4.2, 4.4, and 3.3 imply that $\mathscr{D}(g) = \mathscr{L}(g)$.

If M_f is a 2-sphere, then f has a Fréchet equivalent $g: Q \to E^n$ with the property that g is constant on the boundary of Q while f is not constant on any nondegenerate continuum in the interior of Q, [R, II.3.28]. However, this case is treated essentially the same way as the preceding one and, thus, the proof is complete.

In order to establish Theorem 4.5 for any map $f: Q \to E^n$ of finite Lebesgue area, it will suffice to prove that $\mathscr{D}$ is cyclically additive because it is known that Lebesgue area possesses this property, [R, V.2.55].

Let $C_1, C_2, \ldots$ be the cyclic elements of M_f and let $r_i: M_f \to C_i$ be the monotone retraction, [R, II.2.40]. The mappings $f_i = l_f \circ r_i \circ m_f$ are the cyclic components of f and they satisfy the hypotheses of Theorem 4.5. Let $C_1, C_2, \ldots, C_k$ be any finite number of cyclic elements of M_f. It will be sufficient to show that

$$\mathscr{D}(f) \geqq \sum_{i=1}^{k} \mathscr{D}(f_i) \tag{25}$$

in view of the fact that $\mathscr{D}(f) \leqq \sum_{i=1}^{\infty} \mathscr{D}(f_i)$, which follows from the cyclic additivity of $\mathscr{L}$, 4.2, and 4.5.

Choose $\varepsilon > 0$. From the construction that appears in the proof of Theorem 3.3, it follows that there is a function $u: E^n \to E^1$ with Lipschitz constant 1 such that, for $i = 1, 2, \ldots, k$,

$$\int_{-\infty}^{\infty} \lambda(f_i; u, r)\, dL_1(r) > \mathscr{L}(f_i) - \varepsilon/k. \tag{26}$$

In order to see this we will only consider the case of $k=2$, since the general case is handled in the same way and is no more difficult except for complications in notation. Let ζ_i be the measure associated with f_i as in (15), $i=1, 2$, and let R_i be the Hausdorff 2-rectifiable set that is determined by f_i as in the proof of Theorem 3.3. Choose compact sets $K_1 \subset R_1 - R_2$, $K_2 \subset R_2 - R_1$, and $K_3 \subset R_1 \cap R_2$ such that

$$\zeta_1(K_1) > \zeta_1(R_1 - R_2) - \varepsilon/3, \qquad \zeta_2(K_2) > \zeta_2(R_2 - R_1) - \varepsilon/3,$$
$$\zeta_i(K_3) > \zeta_i(R_1 \cap R_2) - \varepsilon/3, \qquad i = 1, 2.$$

Let U_i be open sets that are mutually disjoint that contain K_i, $i=1, 2, 3$. For each $y \in K_i$ and all sufficiently small balls centered at y, (17) holds with f and ζ replaced by f_i and ζ_i, $i=1, 2$. Similarly, at H^2 almost all $y \in K_3$, (17) is satisfied simultaneously for $i=1, 2$ for all sufficiently small balls centered at y. We may assume that all balls considered are contained in some U_i, $i=1, 2, 3$. Thus, we have a Vitali cover of $K = \bigcup_{i=1}^{3} K_i$ and, therefore, there exist balls $B_1, B_2, \ldots$ such that

$$H^2 \llcorner K\left[E^n - \bigcup_{i=1}^{\infty} B_i\right] = 0.$$

As in the proof of Theorem 3.3, we define a Lipschitz function $u: E^n \to E^1$ in terms of the B_i. Since ζ_i is absolutely continuous with respect to H^2, it follows for $i=1, 2$ that

$$\begin{aligned}\int_{-\infty}^{\infty} \lambda(f_i; u, r)\, dL_1(r) &\geqq \sum_{j=1}^{\infty} (1-\varepsilon)\zeta_i(B_j) = (1-\varepsilon)\zeta_i(K_i \cup K_3) \\ &\geqq (1-\varepsilon)[\zeta_i(R_i) - 2\varepsilon/3] = (1-\varepsilon)[\mathscr{L}(f_i) - 2\varepsilon/3].\end{aligned}$$

By redefining ε appropriately, (26) is now established. Therefore, (26), 4.4, and 4.5 imply that

$$\int_{-\infty}^{\infty} \rho(f_i; u, r)\, dL_1(r) > \mathscr{D}(f_i) - \varepsilon/k, \qquad i = 1, 2, \ldots, k. \tag{27}$$

Let W be a component of $\{x : u \circ f(x) < r\}$ and let $V = m_f(W)$. Now $r_i(V) = V \cap C_i$ and the boundary of $V \cap C_i$ relative to C_i is contained in (bdry V) $\cap\, C_i$. Consequently, if $V_i = V \cap C_i$, $\delta(V_i) \subset \delta(V)$, $i = 1, 2, \ldots, k$. Any two cyclic elements intersect at most in one point, [R, II.2.24], and therefore

$$N[l_f, \delta(V), y] \geqq \sum_{i=1}^{k} N[l_f, \delta(V_i), y]$$

for all but finitely many $y \in E^n$. Hence,

$$\rho(f; u, r) \geqq \sum_{i=1}^{k} \rho(f_i; u, r)$$

and (27) now leads to

$$\begin{aligned} \mathscr{D}(f) &\geqq \int_{-\infty}^{\infty} \rho(f; u, r)\, dL_1(r) \\ &\geqq \sum_{i=1}^{k} \int_{-\infty}^{\infty} \rho(f_i; u, r)\, dL_1(r) > \sum_{i=1}^{k} \mathscr{D}(f_i) - \varepsilon. \end{aligned}$$

Since ε was arbitrarily chosen, (25) is now established and we have established that if $f\colon Q \to E^n$ has finite Lebesgue area, then

$$\mathscr{L}(f) = \mathscr{C}(f) = \mathscr{D}(f). \tag{28}$$

Moreover, by considering $u\colon E^n \to E^1$ as the distance function from a fixed hyperplane in E^n, it follows from Theorems 4.6 and 7.16 of [F2] that there is a constant k such that $\mathscr{L}(f) \leqq k\mathscr{D}(f)$. Thus, if $\mathscr{D}(f) < \infty$ then so is $\mathscr{L}(f) < \infty$(2). Therefore 4.2 and (28) yield the following result.

4.6. THEOREM. *If $f\colon Q \to E^n$ is continuous, then $\mathscr{L}(f) = \mathscr{C}(f) = \mathscr{D}(f)$.*

The results and techniques employed by Slepian [S1], [S2] will allow the extension of this theorem to maps defined on any 2-dimensional manifold.

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