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DECOMPOSITIONS OF 3-MANIFOLDS AND PSEUDO-ISOTOPIES

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Abstract. In this paper we construct pseudo-isotopies which realize certain cellular decompositions of 3-manifolds. In general we show that the pseudo-isotopy may be defined so as to leave points fixed outside of a given open set containing the nondegenerate elements of the decomposition. For nondegenerately continuous decompositions it is shown that the pseudo-isotopy does not move the nondegenerate elements far from their original positions.

Introduction. In this paper we shall investigate the realization of suitable cellular decompositions of 3-manifolds by pseudo-isotopies. If G is a decomposition of a metric space M, then we say that G may be realized by a pseudo-isotopy in case there exists a continuous function $H\colon M\times[0,1]\to M$ such that

(1) $H_0(x)=x$, for each $x\in M$,

(2) if $g\in G$, then $H_1(g)$ is a point in M,

(3) if $g, g'\in G$, $g\neq g'$, then $H_1(g)\neq H_1(g')$,

(4) for $0\leqq t<1$, H_t is a homeomorphism, and H_1 is onto.

The first pseudo-isotopy theorem of this nature was established by Price in [3] where he proved that if G is a cellular decomposition of S^3 such that $S^3/G=S^3$, then G may be realized by a pseudo-isotopy. This result was extended to arbitrary 3-manifolds in [5]. Siebenmann [4] has since given a proof of this theorem for all $n\neq 4$. In the following we shall give a simplified proof of [5] and, in addition, show that considerable control may be exercised over the pseudo-isotopies. In Theorem 1 the pseudo-isotopy which realizes the decomposition is constructed in such a way that it is the identity outside of any given open set containing the nondegenerate elements of the decomposition. For nondegenerately continuous cellular decompositions of 3-manifolds, we show in Theorem 2 that a pseudo-isotopy may be found which does not move the nondegenerate elements far from their original positions. We would like to thank Professor S. Armentrout for many helpful suggestions in connection with this paper.

Notation and definitions. Suppose G is an upper semicontinuous decomposition (henceforth, referred to as a decomposition) of a topological space X. Then the decomposition space associated with G will be denoted by X/G and the natural

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projection mapping from X onto X/G by P. H_G will denote the collection of nondegenerate elements of G, and H_G^* the union of these sets.

A 3-manifold is a separable metric space with the property that each point has a neighborhood which is a 3-cell. If M is a 3-manifold, a point p of M is an interior point of M if and only if p has an open neighborhood in M which is an open 3-cell. The interior of M, Int M, is the set of all interior points and the boundary of M is $M-\text{Int } M$. A subset A of M is said to be cellular in case there exists a sequence $C_1, C_2, \ldots$ of 3-cells in M such that $C_i \subset \text{Int } C_{i-1}$ and $\bigcap_{i=1}^{\infty} C_i = A$. Cellular subsets then must lie in the interior of the manifold. G is said to be a cellular decomposition of M if each $g \in G$ is cellular.

If M is a manifold, by a triangulation of M is meant a simplicial complex T such that (1) $M = \bigcup \{t : t \in T\}$ and (2) T is locally finite in the sense that each point of M has a neighborhood which intersects only finitely many sets of T. For a given triangulation T of M and a subset A of M, then

$N[A] = \bigcup \{t \in T : t \cap A \neq \varnothing\}$,

$O[A] = N[A] - \bigcup \{t \in T : t \cap A = \varnothing\}$,

$N^2[A] = N[N[A]]$ and $N^3[A] = N[N^2[A]]$.

The natural metric, d_N, for T is that metric which assigns a length $2^{1/2}$ to each 1-simplex and makes each 2-simplex an equilateral triangle, and each 3-simplex a regular tetrahedron.

For a given positive number, ε, an isotopy $H\colon X \times [a, b] \to Y$ is called an ε-isotopy if diam $H[\{x\} \times [a, b]] < \varepsilon$, for each $x \in X$. If x is a point in a metric space X, then $S_\varepsilon(x)$ denotes the open ε-neighborhood about x, and if f and g are functions from X into a metric space (Y, d), then $d(f, g) < \varepsilon$ means that $d(f(x), g(x)) < \varepsilon$ for each $x \in X$. The identity function is denoted by id, and the unit interval $[0, 1]$ is denoted by I.

Preliminary lemmas.

LEMMA 1 (ARMENTROUT [1]). *Suppose G is a cellular decomposition of a 3-manifold M such that M/G is a 3-manifold. Then for each positive number, ε, there exists a homeomorphism h from M onto M/G such that $d(P, h) < \varepsilon$.*

LEMMA 2 (KISTER [2]). *Suppose M is a 3-manifold having triangulation T, and d_N is the natural metric for T. Then for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if f and g are homeomorphisms from M onto M and $d_N(f, g) < \delta$, there is an ε-isotopy of M taking g onto f.*

LEMMA 3. *Suppose G is a decomposition of M and $\{h_i\}$ is a sequence of homeomorphisms from M onto M/G which converges (uniformly) to P. Suppose $\delta > 0$. Then there exists a positive integer N such that if $k > j > N$, $d(h_k h_j^{-1}, \text{id}) < \delta$.*

Proof. Corresponding to $\delta/2$, there is a positive integer N such that if $i > N$, then $d(h_i, P) < \delta/2$. Suppose $k > j > N$ and $y \in M/G$. Then

$$d(h_k h_j^{-1}(y), y) \leqq d(h_k h_j^{-1}(y), P h_j^{-1}(y)) + d(P h_j^{-1}(y), h_j h_j^{-1}(y)) < \delta/2 + \delta/2,$$

and the proof is completed.

Suppose G is a cellular decomposition of a 3-manifold M such that $M/G=M$. Let U be an open set containing H_G^*. Then by [1], U/G is homeomorphic to $P(U)=V$. Let T be a triangulation of the 3-manifold V, and let $\phi: V\to V$ be defined by $\phi(x)=x$, where the domain has the relative metric d_R induced by the metric of M/G and the range has the natural metric d_N for T. Let $Q=P|_U$.

LEMMA 4. *Assume the notation above, and suppose h is a homeomorphism from U onto V such that $d_N(h, Q)<\varepsilon$, where ε is chosen so that $S_\varepsilon(x)\subset N^2[x]$ for each x in V. Suppose $x^*\in \operatorname{Bd} V$ and $\{x_i\}\subset V$ converges to x^* (in the d_R metric). Then*

(1) *There exists $y^*\in \operatorname{Bd} U$ such that $\{P^{-1}(x_i)\}$ converges to y^*.*

(2) *$\{h^{-1}(N[\phi(x_i)])\}$ converges to y^*.*

Proof. (1) This follows from the fact that $\{x_i\}\cup x^*$ is compact and P is a compact map.

(2) For each positive integer i, let $y_i\in N^2[\phi(x_i)]$. Note $d_N(Qh^{-1}(y_i), hh^{-1}(y_i))<\varepsilon$ and, hence, $Q(h^{-1}(y_i))\in N^2[y_i]\subset N^3[x_i]$. Since $\{x_i\}$ converges to x^*, $\{\phi^{-1}(N^3[\phi(x_i)])\}$ will also converge to x^*. Let $z_i=Q(h^{-1}(y_i))$. Thus we have that $\{P^{-1}\phi^{-1}(z_i)\}$ converges to y^*. But $h^{-1}(y_i)\in P^{-1}\phi^{-1}(z_i)$ and the desired result follows.

The main results.

THEOREM 1. *Suppose G is a cellular decomposition of a 3-manifold M such that M/G is a 3-manifold. Let U be an open set containing H_G^*. Then there exists a pseudo-isotopy H which realizes the decomposition and such that $H|_{M-U}$ is the identity.*

Proof. For a given positive number ε, the δ obtained in Lemma 2 will be referred to as the Kister number, and for a given positive number δ the integer N of Lemma 3 will be referred to as the Lemma 3 integer. Let $V=P(U)$. We may assume that $U\subset \operatorname{Int} M$. Let T be a triangulation of V, d_N the natural metric for T and d_R the relative metric for V considered as a subspace of M/G. Let $Q=P|_U$ and $\{h_i\}$ be a sequence of homeomorphisms from U onto V converging uniformly to Q in the d_N metric [1]. Choose ε small enough so that $S_\varepsilon(x)\subset N^2[x]$ for each $x\in M/G$.

Let δ_1 be the Kister number which corresponds to $\varepsilon/2$, and then let N_1 be the Lemma 3 integer which corresponds to δ_1. Choose $i_1>N_1$ such that $d_N(Q, h_{i_1})<\varepsilon$. Let δ_2 be the Kister number for $\varepsilon/4$, and $N_2>N_1$ the Lemma 3 integer which corresponds to δ_2. Choose $i_2>\max\{i_1, N_2\}$. Then $d_N(h_{i_2}h_{i_1}^{-1}, \mathrm{id})<\delta_1$ and these two homeomorphisms may be connected by an $\varepsilon/2$-isotopy.

Let δ_3 be the Kister number for $\varepsilon/8$ and $N_3>N_2$ the Lemma 3 integer corresponding to δ_3. Choose $i_3>\max\{i_2, N_2\}$. Thus $d_N(h_{i_3}h_{i_2}^{-1}, \mathrm{id})<\delta_2$, and these two homeomorphisms may be connected by an $\varepsilon/4$-isotopy. Continuing in this manner and combining the resulting isotopies in the obvious way we obtain a pseudo-isotopy $F: U\times I\to V$ such that $F_0=h_{i_1}$ and $F_1=Q$. Now define $H: U\times I\to U$ by $H(x, t)=h_{i_1}^{-1}F(x, t)$. Extend H to all of M by setting $H(x, t)=x$ for $x\in M-U$, $t\in I$. To complete the proof we need only check that if $\{x_i\}$ is a sequence of points in U converging to a point $x^* \in \operatorname{Bd} U$, and $\{t_i\}$ converges to $t \in I$, then $\{H(x_i, t_i)\}$ converges to x^*. Since $\{P(x_i)\}$ converges to $P(x^*)$ (in the d_R metric), it follows from Lemma 4 that $\{h_{i_1}^{-1}(N^2[\phi(P(x_i))])\}$ converges to x^*. Since by our construction of F, points in M/G were never moved more than a distance of ε, we have that $F(x_i, t_i) \in N^2[\phi P(x_i)]$ for each i and the theorem follows.

Suppose M is a metric space and K is a collection of mutually disjoint subsets of M. If $g \in K$, then K is said to be continuous at g in case for each positive number ε, there exists an open subset V of M containing g such that if $g' \in K$ and $g' \cap V$, then $g \subset S_\varepsilon(g')$ and $g' \subset S_\varepsilon(g)$. If G is a decomposition of a metric space M, then G is said to be nondegenerately continuous if, for each $g \in H_G$, H_G is continuous at g. Results concerning nondegenerately continuous decompositions may be found in [7] and [8]. The next lemma follows easily from the definition of a nondegenerately continuous decomposition.

Lemma 5. *Suppose G is a nondegenerately continuous decomposition of a metric space M, and let ε be a positive number. Then for each $g \in H_G$, there exists an open set U_g such that if $g' \in H_G$ and $g' \cap U_g \neq \varnothing$, then $S_{\varepsilon/4}(g) \subset S_{\varepsilon/2}(g')$.*

A proof of Lemma 6 is given in [6].

Lemma 6. *Suppose G is a cellular decomposition of a 3-manifold M such that $M/G = M$. Suppose $\mathcal{U}$ is a saturated ($P^{-1}P(U) = U$, for each $U \in \mathcal{U}$) open cover of H_G^*, and ε is a positive number. Then there exists a homeomorphism h from M onto M such that*

(1) *for each $g \in H_G$,* $\operatorname{diam} h(g) < \varepsilon$,

(2) $h(x) = x$, *for $x \in M - \bigcup \{U : U \in \mathcal{U}\}$,*

(3) *for each $g \in H_G$, there exists a $U \in \mathcal{U}$ such that $g \cup h(g) \subset U$.*

Standard techniques may be employed to establish the following lemma.

Lemma 7. *Suppose G is a cellular decomposition of a 3-manifold M such that $M/G = M$. Let ε be a positive number, and suppose that each $g \in H_G$ has diameter less than $\varepsilon/2$. Then there exists a triangulation T of M/G such that $P^{-1}(N^3[x]) \subset S_\varepsilon(P^{-1}(x))$ for each $x \in M/G$.*

Theorem 2. *Suppose G is a cellular nondegenerately continuous decomposition of a 3-manifold M such that M/G is a 3-manifold, and let ε be a positive number. Then there exists a pseudo-isotopy H which realizes the decomposition and has the property that, for $g \in H_G$ and $t \in I$, $H(g, t) \subset S_\varepsilon(g)$.*

Proof. For each $g \in H_G$, there exists by Lemma 5 a saturated open set U_g such that $g \subset U_g \subset S_{\varepsilon/4}(g)$ and if $g' \in H_G$, $g' \cap U_g \neq \varnothing$, then $S_{\varepsilon/4}(g) \subset S_{\varepsilon/2}(g')$. Thus, $\{U_g : g \in H_G\}$ is a saturated open covering of H_G^*, and, hence, by Lemma 6, there exists a homeomorphism h from M onto M such that

(1) for each $g \in H_G$, $\operatorname{diam} h(g) < \varepsilon/4$,

(2) $h(x) = x$, for $x \in M - \{U_g : g \in H_G\}$,

(3) for each $g \in H_G$, there exists $U_{g'}$ such that $g \cup h(g) \subset U_{g'}$.

Note that since $g \cap U_{g'} \neq \varnothing$, $S_{\varepsilon/4}(g') \subset S_{\varepsilon/2}(g)$, and, hence, $U_{g'} \subset S_{\varepsilon/2}(g)$. Let $h[G]$ be the decomposition of M whose elements are of the form $h(g)$ for $g \in G$. Clearly $M/h[G]$ is homeomorphic to M. Let P be the natural projection from M onto $M/h[G]$. Choose a triangulation T of $M/h[G]$ which satisfied Lemma 7 with respect to $\varepsilon/2$ and give T the natural metric, d_N. As indicated in [3], it is possible to find a sequence of homeomorphisms $\{h_i\}$ from M onto $M/h[G]$ which converges uniformly to P and with the property that $h_i^{-1}(t) \subset P^{-1}(O[\sigma])$ for each $\sigma \in T$. Choose γ small enough that if $x \in M/h[G]$ then $S_\gamma(x) \subset O[\sigma]$ for any 3-simplex σ containing x.

We now proceed as in Theorem 1. Let δ_1 be the Kister number for $\gamma/4$, and N_1 the Lemma 3 integer corresponding to δ_1. Choose $i_1 > N_1$. Let δ_2 be the Kister number corresponding to $\gamma/8$ and $N_2 > N_1$ the Lemma 3 integer for δ_2. Choose $i_2 > \max\{i_1, N_2\}$. Thus $d_N(h_{i_2}h_{i_1}^{-1}, \mathrm{id}) < \delta_1$ and these two homeomorphisms may be connected by a $\gamma/4$-isotopy. Continuing we obtain a sequence $\{h_{i_n}\}$ of homeomorphisms which are connected by small isotopies. Combining the isotopies in the appropriate manner, we construct a pseudo-isotopy F: $M \times I \to M/h[G]$ such that $F_0 = h_{i_1}$, $F_1 = P$, and no point in $M/h[G]$ moves more than a distance of $\gamma/2$ as t runs from 0 to 1. As before let H: $M \times I \to M$ be defined by $H(x, t) = h_{i_1}^{-1}F(x, t)$.

To complete the proof we need to show that if $g \in H_G$ and $t \in I$, then $H(g, t) \subset S_\varepsilon(g)$. Since g and $h(g)$ both lie in some $U_{g'}$ which in turn is contained in $S_{\varepsilon/2}(g)$, we have that $S_{\varepsilon/2}(h(g)) \subset S_\varepsilon(g)$. Therefore, it suffices to show that $H(g, t) \subset S_{\varepsilon/2}(h(g))$. By our construction, $F(g, t) \subset S_\gamma(P(g))$. Let $z \in F(g, t)$ and choose a 3-simplex σ such that $z \in \sigma$. Note that $O[\sigma] \subset N^3[P(g)]$. Thus $h_{i_1}^{-1}(z) \in h_{i_1}^{-1}(\sigma) \subset P^{-1}(O[\sigma]) \subset P^{-1}(N^3[P(g)]) \subset S_{\varepsilon/2}(h(g))$, the last inclusion due to our selection of T. This completes the proof.

A decomposition G of a metric space M is said to be continuous in case, for each $g \in G$, G is continuous at g.

COROLLARY. *Suppose G is a continuous cellular decomposition of a 3-manifold M such that $M/G = M$, and let ε be a positive number. Then there exists a pseudo-isotopy H which realizes the decomposition with the property that, for $g \in G$ and $t \in I$, $H(g, t) \subset S_\varepsilon(g)$.*

We conclude with an example of a cellular decomposition of R^3 which shows that Theorem 2 does not hold for arbitrary cellular decompositions of 3-manifolds. Let A be the closed interval from $(0, 0, 0)$ to $(1, 0, 0)$, and for each positive integer n, let A_n be the closed interval from $(0, 1/n, 0)$ to $(1/4, 1/n, 0)$ and B_n the closed interval from $(3/4, 1/n, 0)$ to $(1, 1/n, 0)$. If G is the decomposition of R^3 with nondegenerate elements, $A, A_1, B_1, A_2, B_2, \ldots$ then $M/G = M$, but obviously G cannot be realized by a pseudo-isotopy which satisfies the conditions of Theorem 2.

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