STRONG CONVERGENCE OF FUNCTIONS ON KÖTHE SPACES

BY
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Abstract. Let $\Lambda$ be a rearrangement invariant Köthe space over a nondiscrete group $G$ with Haar measure $\mu$. For a function $f \in \Lambda$ and relatively compact 0-neighborhood $U$ in $G$ the function

$$T_{Uf}(x) = \frac{1}{\mu(U)} \int_{U+x} f \, d\mu$$

is continuous and also belongs to $\Lambda$. The convergence $T_{Uf} \to f$ (as $U \to 0$) for the strong Köthe topology on $\Lambda$ is involved in establishing compactness criteria for subsets of a Köthe space. The main result of this paper is a necessary and sufficient condition for convergence $T_{Uf} \to f$ in the strong topology on $\Lambda$.

1. In [3], [4] and [5] Köthe studied pairs of subspaces of real sequences that were in weak duality. Dieudonné later generalized the theory to subspaces of locally integrable functions over a locally compact measure space $E$ with Radon measure $\mu$. If $E$ is a $\sigma$-compact, locally compact Hausdorff space with regular Radon measure $\mu$, we let $\Omega$ be the space of all functions which are integrable on each compact set in $E$. For a subset $\Gamma$ of $\Omega$, the Köthe space associated with $\Gamma$ is $\Lambda = \Lambda(\Gamma) = \{f \in \Omega : \int_{E} |fg| \, d\mu < \infty \text{ for all } g \in \Gamma\}$ and the Köthe dual is $\Lambda^* = \Lambda^*(\Gamma) = \Lambda(\Lambda(\Gamma))$. The pair $(\Lambda, \Lambda^*)$ is in weak duality; an example of such a pair is $(L^p, L^q)$. The set $\Lambda$ can be made into a complete locally convex topological vector space under the strong topology $S(\Lambda, \Lambda^*) = S$ defined by the seminorms

$$S_H(f) = \sup_{g \in H} \int |fg| \, d\mu$$

as $H$ runs through the weakly bounded subsets of the Köthe dual $\Lambda^*$.

For $E = G$ an additive topological group, $f \in \Omega$, and $U$ a relatively compact 0-neighborhood in $G$, we define the continuous function

$$T_{Uf}(x) = \frac{1}{\mu(U)} \int_{U+x} f \, d\mu.$$

In [2] these functions were used in giving compactness criteria for subsets of Köthe spaces over $G$. The importance of the convergence of $T_{Uf}$ to $f$ (for the strong

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topology) as $U$ runs through the relatively compact neighborhoods of 0 in $G$ for each function $f$ in a Köthe space will appear in papers by Welland and Goes which are as yet unpublished. In this paper a necessary and sufficient condition is established for the convergence of $T_u f$ to $f$ for the strong topology on a rearrangement invariant Köthe space.

2. Two functions $f$ and $g$ are said to be equimeasurable or rearrangement invariant if

$$
\mu(\{x : |f(x)| > r\}) = \mu(\{x : |g(x)| > r\})
$$

for all nonnegative $r$. A Köthe space $\Lambda$ is rearrangement invariant if $f' \in \Lambda$ whenever $f'$ is equimeasurable with some $f \in \Lambda$. It is known that such a $\Lambda$ is contained in the direct sum of $L^1(E, \mu)$ and $L^\infty(E, \mu)$, and that the Köthe dual $\Lambda^*$ is also rearrangement invariant (proved in [7]).

A set of functions $H$ is normal if $g \in H$ and $|h| \leq |g|$ implies $h \in H$. In [6] it was proven that if $E$ has no atoms (i.e. a set $S \subseteq E$ of positive measure such that $S_i \subseteq S$ implies $\mu(S_i) = 0$ or $\mu(S_i) = \mu(S)$) there is a fundamental system of normal, rearrangement invariant, weakly bounded subsets $\{H\}$ of $\Lambda^*$ for which the seminorms $S_H$, which generate the strong topology of $\Lambda$, have the property that $S_H(f) = S_H(f')$ for $f'$ equimeasurable with $f$.

In this paper $G$ will be a $\sigma$-compact, locally compact, Hausdorff, nondiscrete topological group; $\mu$ will denote invariant Haar measure on $G$. The family of relatively compact neighborhoods of 0 in $G$ will be denoted by $\mathcal{U}$. In addition, we will often write $\int_G f(x) \, dx$ to mean $\int f \, d\mu$, and $f_y$ to be the function $f_y(x) = f(y+x)$ for $y \in G$.

3. The following lemma will enable us to use the information we know about rearrangement invariant Köthe spaces over a nonatomic space $G$.

**Lemma 1.** If $G$ does not have the discrete topology, then Haar measure $\mu$ is nonatomic; that is $G$ has no atoms.

**Proof.** We first show that if $G$ contains an atom $S$, then $\mu(S)$ must be finite. If not, and $\{K_n\}_{n=1}^\infty$ is the increasing sequence of compact sets whose union is $G$, then $\mu(K_n \cap S)$ is finite ($n=1, 2, \ldots$) and strictly less than $\mu(S) = \infty$. It follows that $\mu(K_n \cap S) = 0$ for each $n$, and $\mu(S) = 0$; this is a contradiction. Therefore we must assume that $\mu(S) < \infty$. Now since $\mu(S) = \sup \{\mu(K) : K \subseteq S, K$ is compact$\}$ and $S$ is an atom, there is a compact set $K$ which is an atom satisfying $\mu(K) = \mu(S)$. By the nondiscreteness of $G$, there is a nonempty open set $U$ containing 0 such that $\mu(U) < \mu(K)$. Since $\{U + x : x \in K\}$ is an open cover of the compact $K$, there must be a finite number of elements $\{x_1, x_2, \ldots, x_n\}$ in $K$ such that $\bigcup_{i=1}^n U + x_i \supseteq K$. 

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But $\mu(U + x_i \cap K) \leq \mu(U) < \mu(K)$ ($i = 1, 2, \ldots, n$) implies $\mu(U + x_i \cap K) = 0$. It then follows that $\mu(S) = \mu(K) = 0$.

**Lemma 2.** If $\Lambda$ is a rearrangement invariant Köthe space over $G$, $U$ is a compact neighborhood of 0 in $G$ and $f$ is a function in $\Lambda$, then

(i) $T_0 f \in \Lambda$;
(ii) $T_0 f$ is a uniformly continuous function on $G$;
(iii) $\rho(T_0 f) \leq \rho(f)$ as $\rho$ runs through a certain family of seminorms that generate the strong topology of $\Lambda$.

**Proof.** If $f \in \Lambda$ and $U$ is compact in $G$, we show that $T_0 f \cdot g$ is integrable for every $g \in \Lambda^*$; that is, $T_0 f \in \Lambda$. As $\Lambda^*$ is rearrangement invariant, there is a normal, rearrangement invariant, weakly bounded subset $H$ of $\Lambda^*$ which $g$ belongs to, and satisfying $S_H(f') = S_H(f)$ whenever $f'$ is equimeasurable with $f$. Since $f_x$ is equimeasurable with $f$ for each $y \in G$, we have

$$
\int_G |T_0 f(x) \cdot g(x)| \,dx = \int_G \frac{1}{\mu(U)} \left| \int_U f(x+y) \cdot g(x) \,dy \right| \,dx
\leq \frac{1}{\mu(U)} \int_G \int_U |f(x+y) \cdot g(x)| \,dy \,dx
= \frac{1}{\mu(U)} \int_G \int_U |f(x+y) \cdot g(x)| \,dx \,dy
\leq \frac{1}{\mu(U)} \int_G \sup_{h \in H} \int_U |f(x+y) \cdot h(x)| \,dx \,dy
= \frac{1}{\mu(U)} \int_G S_H(f_y) \,dy
= \frac{1}{\mu(U)} \int_G S_H(f) \,d\mu = S_H(f) < \infty.
$$

Thus $T_0 f \in \Lambda$. Furthermore, it is clear that $\int_G T_0 f \cdot g \cdot d\mu \leq S_H(f)$ for all functions $g \in H$. Taking the supremum on all $g \in H$, we obtain $S_H(T_0 f) \leq S_H(f)$. Since the seminorms $S_H$ generate the strong topology of $\Lambda$, (iii) is proved. In addition, this also shows that $T_0 : \Lambda \to \Lambda$ is a strongly continuous linear function.

In order to show that $T_0 f$ is uniformly continuous for $U$ compact and $f \in \Lambda$, we observe first that $f = h + g$ where $h \in L^1(G, \mu)$ and $g \in L^\infty(G, \mu)$ (since $\Lambda$ is rearrangement invariant); we then have $T_0 f = T_0 h + T_0 g$. We must show that for any $\epsilon > 0$ there is a 0-neighborhood $V$ in $G$ such that $x - y \in V$ implies $|T_0 f(x) - T_0 f(y)| < \epsilon$.

Let $\epsilon > 0$ be given. Since $h$ is integrable, there is a $\delta > 0$ such that $A$ measurable and $\mu(A) < \delta$ implies $\int_A |h| \,d\mu < \epsilon/4$. By the compactness of $U$ in $G$ and the regularity of $\mu$, there is a symmetrical 0-neighborhood $V$ such that $\mu(V + U \backslash U)$
<min \{ 4 \cdot \epsilon / \| g \|_\infty ; \delta \}. Then for \( x - y \in V \), we will have

\[
\mu(U) \cdot |T_{U}f(x) - T_{U}f(y)|
\]

\[
= \mu(U) \cdot |T_{U}h(x) - T_{U}h(y) + T_{U}g(x) - T_{U}g(y)|
\]

\[
\leq \left[ \int_{U + x} h(t) \, dt - \int_{U + y} h(t) \, dt \right] + \left[ \int_{U + x} g(t) \, dt - \int_{U + y} g(t) \, dt \right]
\]

\[
\leq \int_{x - y + U} |h(t)| \, dt + \int_{x - y + U} |g(t)| \, dt + \int_{x + U} |h(t)| \, dt + \int_{x + U} |g(t)| \, dt
\]

\[
= \int_{x - y + U} |h(t+x)| \, dt + \int_{x - y + U} |g(t+y)| \, dt
\]

\[
+ \int_{x + U} |h(t)| \, dt + \int_{x + U} |g(t)| \, dt
\]

\[
< \frac{\epsilon}{4} + \frac{\epsilon}{4} + 2 \cdot \| g \|_\infty \cdot \frac{\epsilon}{\| g \|_\infty} \cdot \frac{1}{4} = \epsilon.
\]

Thus \( T_{U}f \) is uniformly continuous.

**Remark.** On the real line such functions \( T_{U}f \) are of the form

\[
T_{h}f(x) = \frac{1}{h} \int_{x-h/2}^{x+h/2} f(t) \, dt \quad \text{for } h > 0.
\]

The \( T_{h} \) operation takes a function \( f \) and smoothes it out to a function that approximates the original function; in fact, \( \lim_{h \to 0} T_{h}f(x) = f(x) \) a.e. whenever \( f \) is locally integrable. As an example let us consider \( f = \chi_{[a,b]} \) where \( a \) and \( b \) are real numbers with \( a < b \). Then

\[
T_{h}f(x) = \frac{x - (a - h/2))}{h} \quad \text{for } a - h/2 \leq x \leq a + h/2;
\]

\[
= 1 \quad \text{for } a + h/2 \leq x \leq b - h/2;
\]

\[
= \frac{(b + h/2 - x)}{h} \quad \text{for } b - h/2 \leq x \leq b + h/2.
\]

Obviously, \( \lim_{h \to 0} T_{h}f(x) = 1 \) for \( x \in (a, b) \).

We now give an example of two locally integrable functions \( f \) and \( g \) on \( R^1 \) such that \( \int_{R^1} f(t) \cdot g(t) \, dt = 0 \), but \( \int_{R^1} T_{h}f(t) \cdot g(t) \, dt = \infty \) for all \( h > 0 \). This example will show that there is a Köthe space over \( R^1 \), \( \Lambda = L_{\infty}^1 = \{ f \in \Omega : \int_{R^1} |f| \, d\mu < \infty \} \), such that \( f \in \Lambda \), but \( T_{h}f \notin \Lambda \) for any \( h \).

For each integer \( n \geq 5 \) we choose numbers \( a_n, b_n, c_n \) and \( d_n \) such that \( n < a_n < b_n \)

\[
< c_n < d_n, \quad b_n - a_n = c_n - b_n = d_n - c_n = 1/n, \quad \text{and} \quad n + 1 = a_n = n - 1.
\]

Set \( A_n = (a_n, b_n) \)

\[
\cup (c_n, d_n), \quad B_n = (b_n, c_n) \quad (n \geq 5), \quad f = \sum_{n=1}^{\infty} n \cdot \chi_{A_n} \quad \text{and} \quad g = \sum_{n=1}^{\infty} n \cdot \chi_{B_n}.
\]

Clearly,
\[ \int_{-\infty}^{\infty} T^h f(x) \cdot g(x) \, dx = \int_{-\infty}^{\infty} \frac{1}{h} \int_{-h/2}^{h/2} f(x+t) \cdot g(x) \, dt \, dx \]
\[ = \frac{1}{h} \int_{-h/2}^{h/2} \int_{-\infty}^{\infty} f(x+t) \cdot g(x) \, dx \, dt \]
\[ = \frac{1}{h} \int_{-h/2}^{h/2} \sum_{n=0}^{\infty} \int_{-h/2}^{h/2} n^2 \cdot \chi_{A_n}(x+t) \cdot \chi_{B_n}(x) \, dx \, dt \]
\[ = \sum_{n=0}^{\infty} \frac{1}{h} \int_{-h/2}^{h/2} n^2 \cdot \mu(A_n-t \cap B_n) \, dt \]
\[ = \sum_{n=0}^{\infty} \frac{1}{h} \int_{-h/2}^{h/2} n^2 \cdot \mu(A_n+t \cap B_n) \, dt \]
\[ = \frac{1}{h} \sum_{n=0}^{\infty} n^2 \cdot \int_{0}^{1/n} t \cdot dt = \frac{1}{h} \sum_{n=0}^{\infty} n^2 \cdot \frac{1}{n^2} \cdot \frac{1}{2} = \infty. \]

In the following theorem \( C_0 = C_0(G) \) will denote the continuous functions of compact support on \( G \) and \( S(\Lambda, \Lambda^*) = S \) will denote the strong Köthe topology on \( \Lambda \) generated by the rearrangement invariant seminorms

\[ S_B(f) = \sup_{g \in B} \int_G fg \, d\mu \quad \text{for} \quad f \in \Lambda \]

where \( B \) runs through the normal, rearrangement invariant, weakly bounded subsets of \( \Lambda^* \). Recall that \( \lim_{u \in \mathcal{U}} T u f = f \) for the topology \( S \) means that for each weakly bounded \( B \subset \Lambda^* \) there is a relatively compact \( 0 \)-neighborhood \( U_0 \) in \( G \) such that \( S_B(T u f - f) \leq 1 \) for all \( 0 \)-neighborhoods \( U \subset U_0 \).

**Theorem.** If \( \Lambda \) is a rearrangement invariant Köthe space over \( G \) with the strong topology \( S \), then the following are equivalent:

1. \( C_0 \) is dense in \( \Lambda \);
2. \( \lim_{u \in \mathcal{U}} T u f = f \) for each \( f \in \Lambda \); \( \lim_{n \to \infty} \varphi \cdot \chi_{K_n} = \varphi \) for each nonnegative uniformly continuous \( \varphi \) in \( \Lambda \).

**Proof.** The implication (1) implies (2) will be proven first. We begin by showing that \( \lim_{u \in \mathcal{U}} T u \varphi = \varphi \) for each \( \varphi \in C_0 \). Let \( \varphi \in C_0 \) have its support on \( K \) compact, and let \( B \) be a normal, weakly bounded subset of \( \Lambda^* \). If \( U \) is a compact and symmetric \( 0 \)-neighborhood in \( G \), then \( U + K = \{u + k : u \in U, k \in K\} \) is compact, \( d = S_B(\chi_U + x) \) is a finite number, and the support of \( T u \varphi \) is contained in \( U + K \). In order to see this last statement we observe that for \( x \in E \setminus [U + K] \) we will have \( U + x \cap K = \varnothing \); for if there was a \( u \in U \) and \( k \in K \) satisfying \( u + x = k \), we would have \( x = k - u \in K + U \) (as \( U \) is symmetrical), a contradiction. Thus for \( x \in E \setminus [U + K] \), we obtain

\[ T u \varphi(x) = \frac{1}{\mu(U)} \int_{U + x} \varphi(y) \, dy = \frac{1}{\mu(U)} \int_{U + x \cap K} \varphi(y) \, dy = 0. \]
If $d = 0$ the above argument shows that $S_B(T_v \varphi - \varphi) = 0$ for all 0-neighborhoods $V \subset U$. We can then assume that $d > 0$. Now since $\varphi$ is uniformly continuous, there is a symmetric 0-neighborhood $V$ with $V + V \subset U$ such that $x - y \in V$ implies $|\varphi(x) - \varphi(y)| < 1/d$. The 0-neighborhood $V$ is relatively compact, and for any 0-neighborhood $V' \subset V$ we have

$$S_B(T_v \varphi - \varphi) = S_B(\chi_{K+u} \cdot (T_v \varphi - \varphi))$$

$$= \sup_{g \in B} \int_G |\chi_{K+u}(x) \cdot (T_v \varphi(x) - \varphi(x))| \cdot g(x) \, dx$$

$$\leq \sup_{g \in B} \int_G |\chi_{K+u}(x) \cdot \left\{ \frac{1}{\mu(V')} \int_{V'} |\varphi(x) - \varphi(x+y)| \, dy \right\} \cdot g(x) \, dx$$

$$< \sup_{g \in B} \int_G |\chi_{K+u} \cdot \frac{1}{d} g \cdot d\mu = \frac{1}{d} S_B(\chi_{K+u}) = 1.$$  

Thus $\lim_{V \to \emptyset} T_v \varphi = \varphi$ for each $\varphi \in C_0$. Fix $f \in \Lambda$. We now show $T_v f = f$ in $\Lambda$. Let $B$ be a normal, rearrangement invariant, weakly bounded subset of $\Lambda^*$ whose associated seminorm $S_B$ is rearrangement invariant. Since $C_0$ is strongly dense in $\Lambda$, there is a $\varphi \in C_0$ such that $S_B(\varphi - f) < \frac{1}{d}$.

The convergence of $T_v \varphi$ to $\varphi$ implies there is a relatively compact 0-neighborhood $U_0$ such that $S_B(T_v \varphi - \varphi) < \frac{1}{d}$ for all 0-neighborhoods $U \subset U_0$. By Lemma 2 we have

$$S_B(T_v f - f) \leq S_B(T_v f - T_v \varphi) + S_B(T_v \varphi - \varphi) + S_B(\varphi - f) \leq 2 \cdot S_B(\varphi - f) + S_B(T_v \varphi - \varphi) < 1$$

for $U \subset U_0$. Thus $\lim_{V \to \emptyset} T_v f = f$ for each $f \in \Lambda$.

For the second part of (1) implies (2) we suppose that $\varphi \in \Lambda$ is a nonnegative and uniformly continuous function. If $\varphi \in \Lambda^*$ is weakly bounded and normal, there is a function $\varphi \in C_0$ such that $S_B(\varphi - \varphi) < 1$. Suppose the support of $\varphi$ is contained in $K_m$ for some integer $m$. If $n \geq m$ we have

$$0 = |\varphi(x) - \chi_{K_n} \cdot \varphi(x)| \leq |\varphi(x) - \varphi(x)| \quad \text{for any } x \in K_n;$$

$$|\varphi(x) - \chi_{K_n} \cdot \varphi(x)| = |\varphi(x)| = |\varphi(x) - \varphi(x)| \quad \text{for } x \in E \setminus K_n.$$  

Therefore, for $n \geq m$ we have

$$S_B(\varphi - \chi_{K_n} \cdot \varphi) \leq S_B(\varphi - \varphi) < 1,$$

which was to be shown.

In order to prove (2) implies (1) we must first prove that if $B$ is a normal, rearrangement invariant, weakly bounded subset of $\Lambda^*$ whose associated seminorm $S_B$ is rearrangement invariant, and $\epsilon > 0$ is arbitrary, then there is a $\delta > 0$ (dependent upon $\epsilon$) such that $S_B(\chi_A) < \epsilon$ for any measurable set $A$ satisfying $\mu(A) < \delta$. Since $G$ is nondiscrete, there is a compact set $K$ with nonempty interior $K^o$ for which the boundary of $K$, $\partial K = K \setminus K^o$, contains a point $x_0$ with the property that every open set about $x_0$ has a nonempty intersection with the open sets $K^o$ and $E \setminus K$. Since we have assumed $\lim_{V \to \emptyset} T_v \chi_K = \chi_K$ for the strong topology of $\Lambda$, there is a compact, symmetric 0-neighborhood $U_0$ in $G$ such that $S_B(T_v \chi_K - \chi_K) < \epsilon/2$ for any 0-neighborhood $U \subset U_0$. Now we show that there is a 0-neighborhood $V_0 \subset U_0$ such
that $T_{\nu \chi_K}(x) > \frac{1}{2}$ for all $x$ in some nonempty open set contained in $E \setminus K$. As $U_0 + x_0$ is an open neighborhood of $x_0$ meeting $K^\circ$ in a nonempty open set and $\mu$ is a regular measure, there is a symmetric $0$-neighborhood $U' \subset U_0$ such that $0 < \mu(U') < \mu(U_0 + x_0 \cap K)$. The $0$-neighborhood $V_0 = (U' + x_0 \cap E \setminus K) \cup (U_0 + x_0 \cap K) - x_0 \supset (U' + x_0 \cap E \setminus K) \cup (U' + x_0 \cap K) - x_0 = U'$ is contained in $U_0$.

Since $\mu(U' + x_0 \cap E \setminus K) \leq \mu(U') = \mu(U_0 + x_0 \cap K)$, we have

\[
T_{\nu \chi_K}(x_0) = \frac{1}{\mu(V_0)} \int_{V_0 + x_0} \chi_K \, d\mu = \frac{\mu(V_0 + x_0 \cap K)}{\mu(U_0 + x_0 \cap K)}.
\]

The fact that $T_{\nu \chi_K}$ is continuous and $T_{\nu \chi_K}(x_0) > \frac{1}{2}$ implies that there is an open set $V(x_0)$ about $x_0$ such that $T_{\nu \chi_K}(x) > \frac{1}{2}$ for all $x \in V(x_0)$. By the choice of $x_0$, $V = V(x_0) \cap E \setminus K$ is a nonempty open set contained in $E \setminus K$ for which $x \in V$ implies $T_{\nu \chi_K}(x) > \frac{1}{2}$. It follows that $\frac{1}{2} \cdot \chi_V \leq T_{\nu \chi_K} \cdot \chi_{E \setminus K}$. Since $V_0 \subset U_0$, for any measurable set $A \subset V$ we have

\[
\frac{1}{2} \cdot S_B(\chi_A) \leq S_B(\frac{1}{2} \chi_V) \leq S_B(T_{\nu \chi_K} \cdot \chi_{E \setminus K}) = S_B(\chi_{E \setminus K} \cdot (T_{\nu \chi_K} - \chi_K)) \leq S_B(T_{\nu \chi_K} - \chi_K) < \frac{1}{2}.
\]

Set $\delta = \frac{1}{2} \mu(V) > 0$, and let $A$ be a measurable set with $\mu(A) < \delta$. Since $G$ has no atoms, there is a measurable set $A' \subset A$ satisfying $\mu(A') = \mu(A)$. The equimeasurability of $\chi_K$ with $\chi_A$ implies

\[
S_B(\chi_A) = S_B(\chi_A') \leq S_B(\chi_V) < \epsilon.
\]

We now show that $C_0$ is dense in $\Lambda$. Let $f \in \Lambda$; we assume without loss of generality that $f \geq 0$. Given $B \subset \Lambda$ normal, rearrangement invariant and weakly bounded, there is a compact $0$-neighborhood $U_0$ such that $S_B(T_{U_0} f - f) < \frac{1}{4}$. The function $\varphi = T_{U_0} f$ is nonnegative, uniformly continuous and contained in $\Lambda$. By the hypothesis of (2), there is an integer $n$ for which $S_B(\varphi - \varphi \chi_{K_n}) < \frac{1}{4}$. Setting $d = \sup_{x \in E \setminus K_1} \varphi(x)$, we can find a $\delta > 0$ such that $\mu(A) < \delta$ implies $S_B(\chi_A) < 1/3d$. Let $U$ be an open set such that $K_n \subset U \subset K_{n+1}$ and $\mu(U \setminus K_n) < \delta$, and let $g$ be a continuous Urysohn function with its support contained in $U$ and having the properties that $g \equiv 1$ on $K_n$ and $g(x) \leq 1$ for all $x \in G$. The function $g \cdot \varphi = \psi$ is a continuous function whose compact support is contained in $U$ satisfying

\[
S_B(\varphi \cdot \chi_{K_n} - \psi) = S_B(\varphi \cdot \chi_{K_n} - \varphi \cdot g) = S_B(\varphi \cdot (g - \chi_{K_n})) = S_B(\varphi \cdot \chi_{U \setminus K_n} (g - \chi_{K_n})) = S_B(\chi_{U \setminus K_n} (g - \chi_{K_n})) \leq d \cdot 1 \cdot S_B(\chi_{U \setminus K_n}) < \frac{d}{3d} = \frac{1}{3}.
\]
We finally have

\[ S_B(f - \psi) \leq S_B(f - \varphi) + S_B(\varphi \chi_{K_n}) + S_B(\varphi \chi_{K_n} - \psi) < 1. \]

Thus \( C_0 \) is dense in \( \Lambda \) for the strong topology. The Theorem is proved.

**Corollary.** If \( \Lambda \) is a rearrangement invariant Köthe space over \( G \) which contains all the constant functions, then \( T_{Uf} \to f \) \( (U \in \mathcal{U}) \) strongly for each \( f \in \Lambda \) if and only if the uniformly continuous functions are strongly dense in \( \Lambda \).

**Proof.** To show sufficiency we observe that \( S_B(\chi_G) < \infty \) for \( B \subseteq \Lambda^* \) weakly bounded. From this we can show, as in the Theorem, that \( T_{U\varphi} \to \varphi \) for any uniformly continuous \( \varphi \in \Lambda \). We again use the denseness of the uniformly continuous functions and the fact that \( S_B(T_{Uf}) \leq S_B(f) \) for seminorms \( S_B \) generating the topology on \( \Lambda \) to show \( T_{Uf} \to f \) \( (U \in \mathcal{U}) \) strongly. The necessity part of the Corollary is obvious since \( T_{Uf} \) is uniformly continuous for \( U \) compact.

**Remark.** The \( L^p \) spaces for \( 1 \leq p < +\infty \) are spaces in which \( T_{Uf} \to f \) \( (U \in \mathcal{U}) \). \( L^\infty \) is a space which does not have this property; the continuous functions are not dense for the strong (norm) topology. We now give an example of a Köthe space that is not rearrangement invariant, in which the continuous functions of compact support are dense, and for which \( f \in \Lambda \) implies \( T_{Uf} \in \Lambda \), but \( T_{Uf} \) does not converge to \( f \) for the strong topology.

Let \( G = \mathbb{R}^1 \) and let \( \mu \) be Lebesgue measure on \( \mathbb{R}^1 \). Construct sequences of positive numbers \( \{a_n\}_{n=1}^{\infty} \), \( \{b_n\}_{n=1}^{\infty} \), and \( \{c_n\}_{n=1}^{\infty} \) such that the following is true:

- \( a_n < b_n < c_n < a_{n+1} \) \( \ldots \);
- \( c_n - b_n = b_n - a_n = 1/n^2 \) for each \( n \);
- if \( A_n = (a_n, b_n) \) and \( B_n = (b_n, c_n) \) \( (n = 1, 2, \ldots) \), then \( \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} B_n \) is contained in a compact interval.

Set \( f = \sum_{n=1}^{\infty} h^{3/4} \chi_{A_n} \) and \( g = \sum_{n=1}^{\infty} n^{3/4} \chi_{B_n} \). Both \( f \) and \( g \) are integrable, of compact support, and \( f \cdot g = 0 \). Thus \( f \in L^1 \) which is Köthe space in which the continuous functions of compact support are dense, and in which \( T_{Uf} \in L^1g \) whenever \( f' \in L^1g \) (as \( L^1g \) contains all the continuous functions). We show \( \lim_{h \to 0} \int_{-\infty}^{\infty} T_{Uf}(x) \cdot g(x) \ dx = \infty \). Choosing \( h \) sufficiently small, we will have

\[
\int_{-\infty}^{\infty} T_{Uf}(x) \cdot g(x) \ dx = \sum_{n=1}^{\infty} \int_{B_n}^{B_{n+1}} \frac{1}{h^{3/2}} \ f(x+t) \cdot g(x) \ dt \ dx \\
\geq \sum_{n=1}^{\infty} \frac{1}{h^{3/2}} \int_{B_n}^{B_{n+1}} n^{3/4} \chi_{A_n}(x+t) \ dx \ dt \\
\geq \sum_{n \leq n(h)} \frac{1}{h^{3/2}} \int_{B_n}^{B_{n+1}} n^{3/4} \chi_{A_n}(x+t) \ dx \ dt \quad (\text{where } n(h) = \lfloor \sqrt{2/h} \rfloor + 1) \\
= \sum_{n \leq n(h)} n^{3/4} \int_{-h/2}^{h/2} t \cdot dt = \frac{h}{8} \sum_{n \leq n(h)} n^{3/2} \geq \frac{1}{10} \int_{h^{1/4}}^{\infty} \to \infty \quad \text{as } h \to 0.
\]
REFERENCES


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