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SCHAUDER BASES IN THE BANACH SPACES $C^k(T^q)$

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Abstract. A Schauder basis is constructed for the space $C^k(T^q)$ of k-times continuously differentiable functions on T^q, the product of q copies of the one-dimensional torus. This basis has the property that is also a basis for the spaces $C^1(T^q)$, $C^2(T^q)$, $\ldots$, $C^{k-1}(T^q)$, and an interpolating basis for $C(T^q)$.

1. **Notation and introduction.** In this note we will simply say "basis" instead of "Schauder basis". Let I denote the closed unit interval [0, 1]. Let T denote the interval I with the endpoints identified (i.e. T is the one-dimensional torus). For a natural number q, let f be a real-valued function on I^q or T^q. For a multi-index $\nu=(\nu_1, \nu_2, \ldots, \nu_q)$, $\nu_i \in \{0, 1, 2, \ldots\}$, let $D^\nu f$ denote the partial derivative $f_{\nu_1\nu_2\cdots\nu_q}$. We let $|\nu|=\sum_{i=1}^q \nu_i$ denote the order of the partial derivative $D^\nu f$. In case f is a function of a single variable, then $D^m f$ will denote the mth derivative of f, $m=1, 2, \ldots$, with the conventions $Df=D^1 f$ and $D^0 f=f$. Let $C^k(I^q)$ denote the space of all real-valued functions on I^q for which all the partial derivatives of order no greater than k exist and are continuous. Writing this in symbols, we have

$$C^k(I^q) = \{f\colon I^q \to R;\ D^\nu f \text{ is continuous for } |\nu| \leqq k\} \quad \text{for } k = 0, 1, 2, \ldots \text{ and } q = 1, 2, \ldots.$$

Similarly, we have

$$C^k(T^q) = \{f\colon T^q \to R;\ D^\nu f \text{ is continuous for } |\nu| \leqq k\} \quad \text{for } k = 0, 1, 2, \ldots \text{ and } q = 1, 2, \ldots.$$

The linear spaces $C^k(I^q)$ and $C^k(T^q)$ are Banach spaces when endowed with the norm

$$\|f\|_{(q,k)} = \sum_{|\nu|\leqq k} \|D^\nu f\|_\infty$$

where $\|\cdot\|_\infty$ denotes the usual supremum norm and the sum extends over all multi-indices ν with $|\nu|\leqq k$. We use the natural conventions $C(T^q)=C^0(T^q)$, $C^k(T)=C^k(T^1)$, etc.

Ciesielski [3] and this author [7] independently resolved a problem of Banach by proving that the space $C^1(I^2)$ has a basis. These proofs did not easily generalize

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to show the existence of a basis for $C^k(I^2)$ when $k \geqq 2$. The proofs involved using "special" bases $\{f_n\}$ and $\{g_n\}$ for $C^1(I)$ to construct a basis for $C^1(I^2)$. Each of these "special" bases has the property that it is also a basis for the space $C^0(I)$. In what follows, we restrict our attention to the spaces $C^k(T^q)$.

DEFINITION. A basis $\{f_n\}$ for $C^k(T^q)$ is said to be a *simultaneous basis for* $C^k(T^q)$ if $\{f_n\}$ is a basis for each of the spaces $C^0(T^q), C^1(T^q), \ldots, C^k(T^q)$.

We will prove the existence of simultaneous bases for $C^k(T)$ and use this to prove the existence of a simultaneous basis for $C^k(T^q)$. It was the fact that the "special" bases $\{f_n\}$ and $\{g_n\}$ were simultaneous bases for $C^1(I)$ that permitted the construction in [7] to give a basis for $C^1(I^2)$.

Recently, Ciesielski and Domsta [4] have established the existence of a simultaneous basis for $C^k(I^q)$. In the present paper, the bases constructed have the extra property of being interpolating bases for $C(T^q)$ (in the sense of Semadeni [8]). The bases of Ciesielski and Domsta are not interpolating but have the extra property of being orthonormal in $L^2(I^q)$.

2. Simultaneous bases for $C^k(T)$.

THEOREM 2.1. *For each natural number k, there exists a simultaneous basis for the Banach space $C^k(T)$. Moreover, this simultaneous basis is an interpolating*(1) *basis for $C(T)$.*

The proof of Theorem 2.1 will follow from a number of lemmas. The proofs of these lemmas use techniques which are standard for those people who work with splines. However, for completeness, most of these proofs are given. The reader is referred to [2] for a relatively complete treatment of the theory of splines. Throughout the rest of this section we will use the following notation.

Theorem 2.1 will be proven for $k=2\kappa$, $\kappa=1, 2, \ldots$, and hence the result will follow for all k.

We permit the natural number N to take on *only* the values 2^m, $m=0, 1, 2, \ldots$.

By the partition Δ_N, we mean the set of points $\{0, 1/N, 2/N, \ldots, (N-1)/N\} \subset T$. Since we will be considering functions in $C(T)$, for ease of notation we will understand "the partition point $1+m/N$" to mean the partition point m/N for $m=0, 1, 2, \ldots, N-1$.

DEFINITION 2.2. A $(2\kappa+1)$-periodic spline on Δ_N is defined as an element of $C^{2\kappa}(T)$ whose restriction to each interval $(i/N, (i+1)/N)$, $i=0, 1, \ldots, N-1$, is a $(2\kappa+1)$-degree polynomial(2).

Of course, all functions mentioned in this section will be elements of $C(T)$.

It is well known [1, Theorem 2] that a $(2\kappa+1)$-periodic spline f on Δ_N is uniquely determined by specifying its values $f(i/N)$ for $i=0, 1, \ldots, N-1$.

(1) In the sense of Semadeni [8].

(2) *Note.* By "m-degree polynomial" we mean a polynomial of degree no greater than m.

Construction of the basis.

(2.1) Let $f_1 \equiv 1$; then, for $N = 1, 2, 4, 8, \ldots$ and $p = 1, 2, \ldots, N$, we define f_{N+p} to be *the* $(2\kappa+1)$-periodic spline on the partition Δ_{2N} which is zero at every point of the partition Δ_{2N} except $(2p-1)/2N$ and $f_{N+p}((2p-1)/2N) = 1$.

We now construct operators S_n, which turn out to be the partial sum operators for the basis $\{f_n\}$.

Enumerate the diadic rationals in the interval [0, 1) according to the following scheme:

$$\{r_n; n = 1, 2, \ldots\} = \left\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \ldots, \frac{2^{m-1}-1}{2^{m-1}}, \frac{1}{2^m}, \frac{3}{2^m}, \frac{5}{2^m}, \ldots, \frac{2^m-1}{2^m}, \ldots\right\}. \tag{2.2}$$

Given any $f \in C(\boldsymbol{T})$, $S_n f$ is defined inductively by the following:

$$\begin{aligned} a_1 &= f(r_1); \qquad S_n f = \sum_{i=1}^{n} a_i f_i; \\ a_{n+1} &= f(r_{n+1}) - S_n f(r_{n+1}), \qquad n = 1, 2, \ldots. \end{aligned} \tag{2.3}$$

We note that $S_N f$ is the (unique) $(2\kappa+1)$-periodic spline on Δ_N which "interpolates f on Δ_N". That is, $S_N f(i/N) = f(i/N)$, $i = 0, 1, \ldots, N-1$. In fact, we will show that $\{f_n\}$ is an interpolating basis for $C(\boldsymbol{T})$ with nodes $\{r_n\}$.

We recall the definition of the divided difference $f[x_0, x_1, \ldots, x_m]$ of a function f with respect to the points $x_0, x_1, \ldots, x_m$:

$$\begin{aligned} f[x_0] &= f(x_0) \\ f[x_0, x_1] &= \left(\frac{1}{x_m - x_0}\right)\{f[x_1] - f[x_0]\} = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} \\ &\ \ \vdots \\ f[x_0, x_1, \ldots, x_m] &= \left(\frac{1}{x_m - x_0}\right)\{f[x_1, x_2, \ldots, x_m] - f[x_0, x_1, \ldots, x_{m-1}]\} \\ &= \frac{f(x_0)}{DW_m(x_0)} + \frac{f(x_1)}{DW_m(x_1)} + \cdots + \frac{f(x_m)}{DW_m(x_m)} \end{aligned}$$

where $W_m(x) = (x - x_0)(x - x_1) \cdots (x - x_m)$. For the cases in which we are interested, $x_i - x_{i-1} = d$, $i = 1, 2, \ldots, m$, so that

$$\begin{aligned} |f[x_0, x_1, \ldots, x_m]| &\leqq (m+1) \frac{\max |f(x_i)|}{\min |DW_m(x_j)|} \\ &\leqq ((m+1)/d^m) \max \{|f(x_i)|; i = 0, \ldots, m\}. \end{aligned} \tag{2.4}$$

We will also use the following property of the divided difference: If

$$f \in C^m([x_0, x_m]),$$

then there exists a point $\xi \in [x_0, x_m]$ such that

$$(m!)f[x_0, x_1, \ldots, x_m] = D^m f(\xi). \tag{2.5}$$

The restriction of $S_n f$ to each interval $(i/N, (i+1)/N)$ is a $(2\kappa+1)$-degree polynomial, and so the function $D^{2\kappa}S_N f$ is linear on each of these intervals. Therefore, the values $M_i = D^{2\kappa}S_N f(i/N)$, $i=0, 1, \ldots, N-1$, uniquely determine the function $S_N f$. Finally, it has been shown that for $N > 2\kappa$, $S_N f$ may be uniquely determined by solving (for M_i) a matrix equation of the form:

$$A_N \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ M_{N-1} \end{bmatrix} = \begin{bmatrix} f[0, 1/N, \ldots, 2K/N] \\ f[1/N, 2/N, \ldots, (2k+1)/N] \\ \vdots \\ f[(N-1)/N, 1, \ldots, (N+2K-1)/N] \end{bmatrix} \tag{2.6}$$

where A_N is an $N \times N$ matrix depending only on κ and N. Ahlberg, Nilson and Walsh [1, pp. 239–243] have shown that the inverses A_N^{-1} exist and are uniformly bounded with respect to the norm on l_N^∞. That is, there exists a constant K'_κ (depending only on κ) such that

$$|M_j| \leqq K'_\kappa \left\{ \max_i \left| f\left[\frac{i}{N}, \frac{i+1}{N}, \ldots, \frac{1+2\kappa}{N}\right]\right| \right\} \tag{2.7}$$

for $j=0, 1, \ldots, N-1$ and $f \in C(\boldsymbol{T})$.

LEMMA 2.3. *There exists a constant K (depending only on κ) such that*

$$\|D^m S_N f\|_\infty \leqq K\left\{ \max_i \left| f\left[\frac{i}{N}, \frac{i+1}{N}, \ldots, \frac{i+m}{N}\right]\right| \right\}, \qquad m = 0, 1, \ldots, 2\kappa, \tag{2.8}$$

where the max *is taken over* $i=0, 1, \ldots, N-1$.

Proof. We show the existence of constants K_m, $m=0, 1, \ldots, 2\kappa$, such that

$$\|D^m S_N f\|_\infty \leqq K_m \left\{ \max_i \left| f\left[\frac{i}{N}, \frac{i+1}{N}, \ldots, \frac{i+m}{N}\right]\right| \right\}. \tag{2.9}$$

Inequality (2.7) gives (2.9) for $m=2\kappa$ since the function $D^{2\kappa}S_N f$ is piecewise linear. We work down to $m=0$. Assume (2.9) is true for some m such that $1 \leqq m \leqq 2\kappa$ then in each interval $[i/N, (i+m-1)/N]$ there exists a point $\xi = \xi(i, m)$ such that

$$\begin{aligned} D^m S_N f(\xi) &= [(m-1)!](S_N f)[i/N, (i+1)/N, \ldots, (i+m-1)/N] \\ &= [(m-1)!] f[i/N, (i+1)/N, \ldots, (i+m-1)/N]. \end{aligned}$$

Hence, for $x \in [i/N, (i+m)/N]$, we have

$$\begin{aligned} |D^{m-1}S_N f(x)| &= \left| D^{m-1}S_N f(\xi) + \int_\xi^x D^m S_N f(t)\, dt \right| \\ &\leqq [(m-1)!] \left| f\left[\frac{i}{N}, \frac{i+1}{N}, \ldots, \frac{i+m-1}{N}\right] \right| + \int_\xi^x |D^m S_N f(t)|\, dt. \end{aligned}$$

Working with the last term of this expression we see that

$$\int_{\xi}^{x} |D^m S_N f(t)|\, dt \leqq \int_{i/N}^{(i+m)/N} |D^m S_N f(t)|\, dt \leqq \left(\frac{m}{N}\right) \|D^m S_N f\|_\infty$$

$$\leqq \left(\frac{m}{N}\right) K_m \left\{ \max_i \left| f\left[\frac{i}{N}, \frac{i+1}{N}, \ldots, \frac{i+m}{N}\right] \right| \right\}$$

$$= \left(\frac{m}{N}\right) K_m \left\{ \max_i \left| \left(\frac{N}{m}\right) \left(f\left[\frac{i+1}{N}, \frac{i+2}{N}, \ldots, \frac{i+m}{N}\right] - f\left[\frac{i}{N}, \frac{i+1}{N}, \ldots, \frac{i+m-1}{N}\right] \right) \right| \right\}$$

$$\leqq 2K_m \left\{ \max_i \left| f\left[\frac{i}{N}, \frac{i+1}{N}, \ldots, \frac{i+m-1}{N}\right] \right| \right\}.$$

Hence the equalities $K_{m-1} = [(m-1)! + 2K_m]$, $m = 1, 2, \ldots, 2\kappa$, define K_m satisfying (2.9). Q.E.D.

LEMMA 2.4. *If $f \in C^m(T)$, $m \leqq 2\kappa$, then $\|D^m(S_N f - f)\|_\infty \to 0$ as N approaches infinity through powers of two.*

The proof of this lemma follows immediately from Theorem 4.41 of Ahlberg, Nilson, and Walsh [2, p. 136].

LEMMA 2.5. *$\|S_{2N} f - S_N f\|_\infty \leqq (m!/N^m) \|D^m(S_{2N} f - S_N f)\|_\infty$ for $m = 0, 1, \ldots, 2\kappa$ and $N = 1, 2, 4, \ldots$.*

Proof. The case $m=0$ is trivial. We define the auxiliary function h_N to be $S_{2N} f - S_N f$ and prove that

$$\|h_N\|_\infty \leqq (m!/N^m) \|D^m h_N\|_\infty, \tag{2.10}$$

by induction on m.

Assume (2.10) is true for $m < 2\kappa$, and notice that $h_N(i/N) = 0$, $i = 0, 1, \ldots, N-1$, so that $h_N[i/N, (i+1)/N, \ldots, (i+m)/N] = 0$. In each interval $[i/N, (i+m)/N]$ there exists a point $\eta = \eta(i, m)$ such that $D^m h_N(\eta) = (m!) h_N[i/N, (i+1)/N, \ldots, (i+m)/N] = 0$. Thus, for x in $[i/N, (i+m+1)/N]$, we have

$$|D^m h_N(x)| = \left| \int_{\eta}^{x} D^{m+1} h_N(t)\, dt \right| \leqq \int_{i/N}^{(i+m+1)/N} |D^{m+1} h_N(t)|\, dt.$$

Therefore, $\|D^m h_N\|_\infty \leqq ((m+1)/N) \|D^{m+1} h_N\|_\infty$. By substituting this last inequality into (2.10), we get $\|h_N\|_\infty \leqq ((m+1)!/N^{m+1}) \|D^{m+1} h_N\|_\infty$; hence, (2.10) is true for $m = 0, \ldots, 2\kappa$ by induction. Q.E.D.

We now prove Theorem 2.1 by showing that the functions $\{f_n\}$ form the desired basis.

We show that $\|D^m(S_n f - f)\|_\infty \to 0$ as $n \to \infty$, for each $f \in C^m(T)$. If $n = N + p$ $(1 \leqq p \leqq N)$, then

$$\|D^m(S_{N+p} f - f)\|_\infty \leqq \|D^m(S_{N+p} f - S_N f)\|_\infty + \|D^m(S_N f - f)\|_\infty. \tag{2.11}$$

We define the auxiliary function g to be $S_{N+p}f-S_Nf$. From the definitions of $\{f_n\}$ and $\{S_n\}$, we see that $S_{N+p}f$ agrees with the value of the function f at the points $i/2N$, $0 \leqq i < 2p$. For $2p \leqq i \leqq 2N$, we have $S_{N+p}f(i/2N) = S_Nf(i/2N)$, so that

$$\begin{aligned} |g(i/2N)| &= |S_{N+p}f(i/2N)-S_Nf(i/2N)| \\ &\leqq |S_{2N}f(i/2N)-S_Nf(i/2N)| \leqq \|S_{2N}f-S_Nf\|_\infty. \end{aligned} \tag{2.12}$$

It also follows from the definition of S_N that $S_{2N}g = g$. From Lemma 2.3 we get

$$\|D^m g\|_\infty = \|D^m S_{2N} g\|_\infty \leqq K\left\{\max_i \left| g\left[\frac{i}{2N},\ldots,\frac{i+m}{2N}\right]\right|\right\}.$$

Now by inequalities (2.4) and (2.12) we have

$$\begin{aligned} \left| g\left[\frac{i}{2N},\frac{i+1}{2N},\ldots,\frac{i+m}{2N}\right]\right| &\leqq (m+1)(2N)^m\left\{\max_j \left| g\left(\frac{j}{2N}\right)\right|\right\} \\ &\leqq (m+1)(2N)^m \|S_{2N}f-S_Nf\|_\infty. \end{aligned}$$

By Lemma 2.5, this last term is dominated by $[(m+1)!](2^m)\|D^m(S_{2N}f-S_Nf)\|_\infty$. This gives us

$$\|D^m g\|_\infty \leqq K[(m+1)!](2^m)\|D^m(S_{2N}f-S_Nf)\|_\infty.$$

Substituting in inequality (2.11), we get

$$\|D^m(S_{N+p}f-f)\|_\infty \leqq K[(2\kappa+1)!]2^{2\kappa}\|D^m(S_{2N}f-S_Nf)\|_\infty + \|D^m(S_Nf-f)\|_\infty,$$

which converges to zero as $N \to \infty$. So each function $f \in C^m(\boldsymbol{T})$, $m \leqq 2\kappa$, has an expansion $f = \sum_{i=1}^\infty a_i f_i$ which converges in the norm $\|\cdot\|_{(m,1)}$.

The uniqueness of the expansion follows from the usual applications of the continuous linear functionals ε_n, $n=1,2,\ldots$, where $\varepsilon_n(f) = f(r_n) =$ point evaluation at the nth diadic rational in the enumeration (2.2).

That the basis $\{f_n\}$ is interpolating with nodes $\{r_n\}$ is clear from the definitions of $\{f_n\}$ and $\{S_n\}$. Q.E.D.

3. Simultaneous bases for $C^k(\boldsymbol{T}^q)$.

THEOREM 3.1. *If $\{f_m\}$ and $\{g_n\}$ are simultaneous bases for $C^k(\boldsymbol{T})$, then the functions $\{f_m g_n\}$ (when arranged properly in a sequence) form a simultaneous basis for $C^k(\boldsymbol{T}^2)$.*

Before proving Theorem 3.1, we need to establish some more facts about simultaneous bases. Let $\{f_n\}$ be a simultaneous basis for $C^k(\boldsymbol{T}^q)$, then since $\{f_n\}$ is a basis for $C(\boldsymbol{T}^q)$ there exist functionals α_n in the conjugate space $[C(\boldsymbol{T}^q)]^*$ such that $\alpha_m(f_n) = \delta_{mn}$. Thus each $\alpha_n(\cdot)$ may be considered as a Radon measure on $\boldsymbol{T}^q$ and we write

$$\alpha_n(f) = \int_{\boldsymbol{T}^q} f(t)\, d\alpha_n(t), \qquad n = 1,2,\ldots. \tag{3.1}$$

Since $\{f_n\}$ is a basis for $C^m(\boldsymbol{T}^q)$, $m=1,2,\ldots,k$, given $f \in C^m(\boldsymbol{T}^q)$, there exists a unique sequence of scalars $\{a_i^m; i=1,2,\ldots\}$ such that $[f-\sum_{i=1}^n a_i^m f_i] \to 0$ as

$n \to \infty$, where the convergence is in the norm of the space $C^m(\boldsymbol{T}^q)$. It follows that $\alpha_i(f) = a_i^m$ for $m = 1, 2, \ldots, k$ and $i = 1, 2, \ldots$. That is, the ith coefficient functional for the basis $\{f_n\}$, when considered as a basis for $C^m(\boldsymbol{T}^q)$, is the restriction of $\alpha_i(\cdot)$ to the subspace $C^m(\boldsymbol{T}^q)$.

Let $\{f_m; \alpha_m\}$ and $\{g_n; \beta_n\}$ be simultaneous bases for $C^k(\boldsymbol{T})$, the partial sum operators S_n and T_n are defined by

$$S_n f = \sum_{i=1}^{n} \alpha_i(f) f_i; \quad T_n g = \sum_{i=1}^{n} \beta_i(g) g_i; \qquad f, g \in C(\boldsymbol{T}). \tag{3.2}$$

Since $\{f_m; \alpha_m\}$ and $\{g_n; \beta_n\}$ are simultaneous bases for $C^k(\boldsymbol{T})$, there exist constants L_i and M_i such that

$$\begin{aligned} \|S_n f\|_{(i,1)} &\leqq L_i \|f\|_{(i,1)}, \qquad f \in C^i(\boldsymbol{T}), \\ \|T_n g\|_{(i,1)} &\leqq M_i \|g\|_{(i,1)}, \qquad g \in C^i(\boldsymbol{T}), \end{aligned} \tag{3.3}$$

for $i = 0, 1, \ldots, k$ and $n = 1, 2, \ldots$.

Now for a function $h = h(x, y) \in C(\boldsymbol{T}^2)$ we define the operators

$$\begin{aligned} S_n h(x, y) &= \sum_{i=1}^{n} \alpha_i[h(\cdot, y)] f_i(x) = \sum_{i=1}^{n} \left[\int_T h(s, y)\, d\alpha_i(s) \right] f_i(x), \\ T_n h(x, y) &= \sum_{i=1}^{n} \beta_i[h(x, \cdot)] g_i(y) = \sum_{i=1}^{n} \left[\int_T h(x, t)\, d\beta_i(t) \right] g_i(y). \end{aligned}$$

That is, for each fixed y we consider $h(x, y)$ as a function of x and apply the operator S_n to this function defining $S_n h(x, y)$. (Similarly for $T_n h(x, y)$.)

There should be no confusion as to what $S_n h$ or $T_n h$ means, if one remembers the number of variables involved. For example, if f is a function of one variable x, then $S_n f$ will be defined by (3.2). If $h = h(x, y)$, then $S_n h$ will be given by (3.4).

At this stage we notice that if $D^{(0,j)} h \in C(\boldsymbol{T}^2)$, then

$$\begin{aligned} S_n D^{(0,j)} h(x, y) &= \sum_{i=1}^{n} \left[\int_T D^{(0,j)} h(s, y)\, d\alpha_i(s) \right] f_i(x) \\ &= D^{(0,j)} \sum_{i=1}^{n} \int_T h(s, y)\, d\alpha_i(s) f_i(x) = D^{(0,j)} S_n h(x, y). \end{aligned}$$

That is $D^{(0,j)} S_n h$ exists and equals $S_n D^{(0,j)} h$. Similarly, $D^{(i,0)} T_n h = T_n D^{(i,0)} h$.

Lemma 3.2. *Each of the operators S_n, T_n, defined in* (3.4), *is a continuous linear projection operator taking $C^k(\boldsymbol{T}^2)$ into itself.*

Proof. Let $h \in C^k(\boldsymbol{T}^2)$ and $i + j \leqq k$, then

$$\begin{aligned} \left| D^{(i,j)} \int_T h(s, y)\, d\alpha_n(s) f_n(x) \right| &= \left| \int_T D^{(0,j)} h(x, y)\, d\alpha_n(s) D^{(i,0)} f_n(x) \right| \\ &\leqq \|D^{(0,j)} h\|_\infty \|\alpha_n\| \, \|D^i f_n\|_\infty \leqq \|h\|_{(k,2)} \|\alpha_n\| \, \|f_n\|_{(k,1)}. \end{aligned}$$

Adding the last inequality over the $\frac{1}{2}(k+1)(k+2)$ distinct pairs (i,j) such that $0 \leqq i+j \leqq k$, we get

$$\left\| \int_T h(s, \cdot)\, d\alpha_n(s) f_n(\cdot) \right\|_{(k,2)} \leqq \tfrac{1}{2}(k+1)(k+2)\|h\|_{(k,2)}\|\alpha_n\|\ \|f\|_{(k,1)}$$

so that

$$\|S_n h\|_{(k,2)} \leqq \tfrac{1}{2}(k+1)(k+2) \sum_{i=1}^{n} \|\alpha_i\|\ \|f_i\|_{(k,1)}\|h\|_{(k,2)}.$$

Since S_n is a projection on $C^k(\boldsymbol{T})$, it follows that as an operator on $C^k(\boldsymbol{T}^2)$ it will also be a projection. The proof for T_n follows by a symmetrical argument. Q.E.D.

LEMMA 3.3. *If $h \in C^k(\boldsymbol{T}^2)$ then $\|h - S_n h\|_{(k,2)} \to 0$ and $\|h - T_n h\|_{(k,2)} \to 0$ as $n \to \infty$.*

Proof. It suffices to show that $\|D^{(i,j)}(h - S_n h)\|_\infty \to 0$ as $n \to \infty$ for each pair (i,j) with $i+j \leqq k$. Now,

$$\begin{aligned} \|D^{(i,j)}(h-S_n h)\|_\infty &= \|D^{(i,0)}(D^{(0,j)}h - S_n D^{(0,j)}h)\|_\infty \\ &= \|D^{(i,0)}(f - S_n f)\|_\infty, \quad \text{where } f(x,y) = D^{(0,j)}h(x,y). \end{aligned}$$

We consider the family of functions $\{f(\cdot, y); y \in \boldsymbol{T}\}$ as a subset of $C^i(\boldsymbol{T})$. For each fixed y, the map $\tilde{y} \to \|f(\cdot,\tilde{y}) - f(\cdot, y)\|_{(i,1)}$ is a real valued function of $\tilde{y}$. The continuity of this map follows from the uniform continuity of the functions f, $D^{(1,0)}f, \ldots, D^{(i,0)}f$. Thus, if we let $\varepsilon > 0$ be given, then

$$(3.4) \qquad U_y = \{\tilde{y} \in T : \|f(\cdot, y) - f(\cdot, \tilde{y})\|_{(i,1)} < \varepsilon\}$$

is an open set for each y in $\boldsymbol{T}$. (In (3.4), $f(x,y) - f(x,\tilde{y})$ is considered as a function of x alone, for each $y, \tilde{y} \in \boldsymbol{T}$.) Since $\{U_y; y \in \boldsymbol{T}\}$ is an open cover of $\boldsymbol{T}$, there exists a finite subcover $\{U_{y_1}, U_{y_2}, \ldots, U_{y_p}\}$. Now for $y \in U_{y_m}$, we have

$$\begin{aligned} &|D^{(i,0)}[f(x,y) - S_n f(x,y)]| \\ &\quad \leqq |D^{(i,0)}[f(x,y) - f(x,y_m)]| + |D^{(i,0)}[f(x,y_m) - S_n f(x,y_m)]| \\ &\qquad + |D^{(i,0)}[S_n(f(x,y_m) - f(x,y))]| \\ &\quad \leqq \|f(\cdot,y) - f(\cdot,y_m)\|_{(i,1)} + \|f(\cdot,y_m) - S_n f(\cdot,y_m)\|_{(i,1)} \\ &\qquad + L_i\|f(\cdot,y) - f(\cdot,y_m)\|_{(i,1)} \end{aligned}$$

where L_i is the constant given in inequality (3.3). Since $f(\cdot, y_m) \in C^i(\boldsymbol{T})$, there exists N_m such that, for $n \geqq N_m$, $\|f(\cdot,y_m) - S_n f(\cdot,y_m)\|_{(i,1)} < \varepsilon$, from which follows

$$\|D^{(i,j)}(h - S_n h)\|_\infty = \|D^{(i,0)}(f - S_n f)\|_\infty < (2 + L_i)\varepsilon,$$

for $n \geqq N = \max\{N_1, N_2, \ldots, N_p\}$ and hence $\|h - S_n h\|_{(k,2)} \to 0$ as $n \to \infty$. The result for T_n is obtained similarly. Q.E.D.

LEMMA 3.4. *There exist constants L and M such that for any $h \in C^k(\boldsymbol{T}^2)$ we have*

$$(3.5) \qquad \begin{aligned} \|S_n h\|_{(k,2)} &\leqq L\|h\|_{(k,2)}, \\ \|T_n h\|_{(k,2)} &\leqq M\|h\|_{(k,2)}, \qquad n = 1, 2, \ldots. \end{aligned}$$

Proof. The proof follows from Lemma 3.2, Lemma 3.3, and the uniform boundedness principle. Q.E.D.

We now give the proof of Theorem 3.1.

Let $\{f_m; \alpha_m\}$ and $\{g_n; \beta_n\}$ be simultaneous bases for $C^k(\boldsymbol{T})$, let S_n and T_n be defined as in (3.4), and let the sequence $\{h_p\}$ be defined as follows:

$$\{h_p\} = \{f_1g_1, f_1g_2, f_2g_1, f_2g_2, \ldots, f_ng_n, f_1g_{n+1}, f_2g_{n+1}, \ldots, f_ng_{n+1}, \\ f_{n+1}g_1, f_{n+1}g_2, \ldots, f_{n+1}g_{n+1}, \ldots\}.$$

In view of Lemma 3.3 and Lemma 3.4, we see that for any $h \in C^k(\boldsymbol{T}^2)$ we have

$$\begin{aligned} \|h-S_mT_nh\|_{(k,2)} &\leqq \|h-S_mh\|_{(k,2)} + \|S_m(h-T_nh)\|_{(k,2)} \\ &\leqq \|h-S_mh\|_{(k,2)} + L\|h-T_nh\|_{(k,2)} \to 0 \quad \text{as } (m,n) \to (\infty,\infty). \end{aligned}$$

Therefore, the set of functions of the form $h=\sum_{i=1}^n a_ih_i$ is dense in the space $C^k(\boldsymbol{T}^2)$.

We next show that for any scalars a_i and any natural numbers p and r if $h=\sum_{i=1}^{p+r} a_ih_i$ then we have

$$\left\| \sum_{i=1}^{p} a_ih_i \right\|_{(k,2)} \leqq 3LM\|h\|_{(k,2)} \tag{3.6}$$

where L and M are the constants given by Lemma 3.4. Inequality (3.6) is established in the same way as inequality (9) of [5]. Thus $\{h_p\}$ is a basis for $C^k(\boldsymbol{T}^2)$.

Since $\{f_n\}$ and $\{g_n\}$ are simultaneous bases for $C^m(\boldsymbol{T})$, $m=0, 1, 2, \ldots, k$, it follows that $\{h_p\}$ is a basis for $C^m(\boldsymbol{T}^2)$, $m=0, 1, \ldots, k$. Q.E.D.

We remark that in order to use this method to obtain a basis for $C^k(\boldsymbol{T}^2)$ it is essential that the bases $\{f_m\}$ and $\{g_n\}$ be simultaneous bases for $C^k(\boldsymbol{T})$. In particular, the existence of the constants L_i, M_i of inequality (3.3) was needed in the proof of Lemma 3.3.

COROLLARY 3.5. *For any natural numbers k and q, the space $C^k(\boldsymbol{T}^q)$ has a simultaneous basis which is an interpolating basis for $C(\boldsymbol{T}^q)$.*

Proof. If $q=1$, the result is given by Theorem 2.1. The general result follows by induction on q. In the preceding sequence of lemmas, the expected generalizations hold if $\{f_n\}$ is a simultaneous basis for $C^k(\boldsymbol{T}^{q-1})$ and $\{g_n\}$ is a simultaneous basis for $C^k(\boldsymbol{T})$. This permits one to establish the existence of a simultaneous basis for $C^k(\boldsymbol{T}^q)$ from the existence of a simultaneous basis for $C^k(\boldsymbol{T}^{q-1})$. That the resulting product basis is interpolating when $\{f_n\}$ and $\{g_n\}$ are interpolating follows from the work of Semadeni [8]. Q.E.D.

4. **Remarks.** Boris Mitjagin [5, Lemma 12] has proved that the following spaces are isomorphic: $C^k(\boldsymbol{T}^q)$, $C^k(\boldsymbol{I}^q)$, $C^k(\boldsymbol{M})$ (where $\boldsymbol{M}$ is a q-dimensional compact C^k-manifold) and $C^k(\boldsymbol{D})$ (where $\boldsymbol{D}$ is a domain in $\boldsymbol{R}^q$ with boundary such that there exists a linear extension operator L: $C^k(\partial\boldsymbol{D}) \to C^k(\boldsymbol{D})$). These isomorphisms together with the present work establish the existence of a Schauder basis for each of these spaces.

BIBLIOGRAPHY

1. J. H. Ahlberg, E. N. Nilson and J. L. Walsh, *Best approximation and convergence properties of higher-order spline approximations*, J. Math. Mech. **14** (1965), 231–243. MR **35** #5823.

2. ———, *The theory of splines and their applications*, Academic Press, New York, 1967. MR **39** #684.

3. Z. Ciesielski, *A construction of basis in* $C^{(1)}(I^2)$, Studia Math. **33** (1969), 243–247. MR **40** #1759.

4. Z. Ciesielski and J. Domsta, *Construction of an orthonormal basis in* $C^m(I^d)$ *and* $W_p^m(I^d)$, Studia Math. (to appear).

5. B. S. Mitjagin, *Homotopic structures of linear groups of Banach spaces*, Uspehi Mat. Nauk **25** (1970), 63–109. (Russian).

6. J. Radecki, *Orthonormal basis in the space* $C_1[0, 1]$, Studia Math. **35** (1970), 123–163.

7. S. Schonefeld, *Schauder bases in spaces of differentiable functions*, Bull. Amer. Math. Soc. **75** (1969), 586–590. MR **39** #6067.

8. Z. Semadeni, *Product Schauder bases and approximation with nodes in spaces of continuous functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 387–391. MR **27** #4068.

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