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TRACE ALGEBRAS

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Abstract. We give an algebraic unification for those mathematical structures which possess the abstract properties of finite-dimensional vector spaces: scalars, duality theories, trace functions, etc. The unifying concept is the "trace algebra," which is a set with a ternary operation which satisfies certain generalized associativity and identity laws. Every trace algebra induces naturally an object which (even though no additive structure may be available) possesses a summation operator and inner product which obey the Fourier expansion and other familiar properties. We construct the induced object in great detail. The ultimate results of the paper are: a theorem which shows that the induced object of a "well-behaved" trace algebra determines it uniquely; and a theorem which shows that well-behaved trace algebras look, formally, like the trace algebras associated with finite-dimensional vector spaces.

0. **Introduction.** The subject of this paper is an investigation of an algebraic unification for those mathematical structures which act (even though they may have no additive structure) formally like finite-dimensional linear spaces—which is to say, those objects which possess the abstract analogues of scalars, inner products, summation operators, tensor products, homogeneous endomorphisms, trace functions and duality theories on the endomorphisms, and which obey the abstract analogues of the Fourier expansion, Riesz representation, and Parseval relations—in short, all of the familiar behavior one normally associates with finite-dimensional linear spaces.

The focus of our investigation is on the "trace algebra." A trace algebra is a generalized monoid—a set with a ternary operation which satisfies certain generalized associativity and identity laws. As will be shown, every trace algebra induces in a very natural way a mathematical object which exhibits all of the above-mentioned behavior as well as most of the interrelations familiar from the theory of linear spaces (with the notable exception of "Fubini's theorem"). The induced object is induced in a well-behaved manner: its structure is determined by the structure of the trace algebra, and by nothing else. Conversely, if the trace algebra is well behaved, then it is uniquely determined by its induced object, as our first main result, the "uniqueness theorem," will show. This means that when everything is well behaved, then our abstract "linear spaces" are the same thing as trace algebras.

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Our primary model for trace algebras has been the one associated with a given finite-dimensional free module M over a commutative ring K with unity; namely, $\mathrm{Bilin}_K(M)$, the set of all K-bilinear maps $M\times M\to M$. Under the restrictions presented in §2, example (2), there is a ternary product [, ,] defined by

$$(x, y)[f, g, h] = \sum_i ((x, e_i)f, (e_i, y)g)h \tag{$*$}$$

for $f, g, h \in \mathrm{Bilin}_K(M)$ and all $x, y \in M$. Our second main result, the "representation theorem," will show that every well-behaved trace algebra has a defining relation analogous to equation ($*$), with "$\sum$" being the abstract summation operator and f, g, h certain binary maps uniquely associated with the elements of the trace algebra.

Except for examples, everything in the paper is directed ultimately towards proving our two main results. After defining the basic concepts in §1, §3 introduces the "induced semigroup" which is the analogue of the underlying ring K; §§4 and 5 study the "comonoid," which is the abstract analogue of $\hom_K(M; M)$; §§7 and 8 the abstract inner product and duality theory; and §11, the summation operator, trace function, and tensor product. §§6, 9 and 10 deal with a string of characterization theorems which lead up to the uniqueness and representation theorems in §13. §§2 and 12 give examples of trace algebras, in which the Lebesgue integral, real-analytic "supremum" and the set-theoretic union play the part of the abstract summation operator.

The author has been unable to find a precedent in the literature. Other ternary systems, such as the "polyadic groups" of [2], have been studied but have no apparent connection with our work. The author's motivation was primarily his interest in the categorical and formal properties of linear spaces and their generalizations.

Finally, the author wishes to express his gratitude to Professor Saunders Mac Lane for his kind encouragement and many helpful criticisms of this work while the author was an undergraduate at the University of Chicago.

1. **First considerations.**

DEFINITION 1.1. (i) An *S-moduloid* is a set M together with a semigroup with unity S and a map $M\times S\to M$, which we denote by $(\cdot)$, such that $m\cdot(\alpha\beta)=(m\cdot\alpha)\cdot\beta$ and $m\cdot 1=m$ for all $m\in M$ and $\alpha, \beta\in S$, where 1 denotes the unity of S.

(ii) If M, M' are two S-moduloids, then $\pi\colon M\to M'$ is called an *S-homogeneous* map iff $(m\cdot\alpha)\pi=(m\pi)\cdot\alpha$ for all $m\in M$ and $\alpha\in S$. A map $\pi\colon M\times M\to M'$ is called *S-bihomogeneous* if it is S-homogeneous in each argument.

DEFINITION 1.2. A map $\pi\colon A\times B\to C$ between arbitrary sets A, B, C will be called *nondegenerate* if $(a, b_1)\pi=(a, b_2)\pi$ for all $a\in A$ and fixed $b_1, b_2\in B$ implies that $b_1=b_2$ and if $(a_1, b)\pi=(a_2, b)\pi$ for all $b\in B$ and fixed $a_1, a_2\in A$ implies that $a_1=a_2$.

DEFINITION 1.3. Given a set A with a map $\phi: A \times A \times A \to A$, which we denote by $\phi: (x, y, z) \mapsto [x, y, z]$, then, we say that

(i) $b \in A$ is a 1-*atom* if for all $x, y, z, w \in A$,

$$[x, b, [y, z, w]] = [[x, b, y], z, w] \quad \text{and} \quad [[x, y, b], z, w] = [x, [y, z, w], b],$$

(ii) $d \in A$ is a 2-*atom* if for all $x, y, z, w \in A$,

$$[d, x, [y, z, w]] = [y, [d, x, z], w] \quad \text{and} \quad [x, [y, z, d], w] = [[x, y, w], z, d],$$

(iii) $a \in A$ is a 3-*atom* if for all $x, y, z, w \in A$,

$$[[x, y, z], a, w] = [x, y, [z, a, w]] \quad \text{and} \quad [x, y, [a, z, w]] = [a, [x, y, z], w].$$

The equations in (i), (ii), (iii) are called the *partial associativity laws* for the ternary map ϕ. The set of 1-atoms will be denoted by Θ^1, the set of 2-atoms by Θ^2, and the set of 3-atoms either by Θ^3 or just Θ. (The reason for this will be given later.) The sets Θ^i $(i=1, 2, 3)$ will be referred to as the *i-plinths* of ϕ, and Θ more commonly as just the "plinth" of ϕ.

REMARK (*Notation*). (1) For convenience, "a" will always denote a 3-atom, "b" a 1-atom, and "d" a 2-atom. Also, "u" and "v" will always denote atoms, the kind of atom being stipulated in the text as necessary. (2) Because the partial associativity laws are used constantly in what follows, it will be inconvenient to always refer back to Definition 1.3 for a justification of their use. Therefore, we make the following convention: whenever a partial associativity law is invoked, we will write a factorial sign (!) after that atom which is relevant to the law being used. For example, rather than writing "$[[x, y, z], a, w]=[x, y, [z, a, w]]$ because $a \in \Theta$," we write instead, "$[[x, y, z], a!, w]=[x, y, [z, a, w]]$." Similarly,

$$\text{“}[x, y, [z, a!, w]] = [[x, y, z], a, w],\text{”}$$

and

$$\text{“}[[x, b!, y], z, w] = [x, b, [y, z, w]],\text{”} \quad \text{etc.}$$

DEFINITION 1.4. Given a map ϕ as in Definition 1.3, then a *coident pair* for ϕ is a pair of elements $x_1, x_2 \in A$ together with a permutation $\sigma \in S_3$ such that $[x_{1\sigma}, x_{2\sigma}, x_{3\sigma}]=x_3$ for all $x_3 \in A$. An $x \in A$ is called an *ident* if it is a member of a coident pair.

For the following definition, we make the following notational conventions: "$\exists\, a, b, d$:" will mean "there exist $a \in \Theta$, $b \in \Theta^1$ and $d \in \Theta^2$ such that . . .". Further, "$[x, *, y]$" will mean "$[x, z, y]=z$ for all $z \in A$," and similarly for "$[x, y, *]$" and "$[*, x, y]$."

DEFINITION 1.5. A *trace algebra* is a set A with a ternary product $\phi: (x, y, z) \mapsto [x, y, z]$, such that the following four axioms hold:

$$\text{(TA-1)} \qquad \begin{array}{lll} \exists\, a, b, d: & [*, a, d] \text{ and } [a, *, b], \\ \exists\, a, b, d: & [d, b, *] \text{ and } [a, *, b], \\ \exists\, a, b, d: & [d, b, *] \text{ and } [*, a, d]. \end{array}$$

(TA-2) For all a, b, d,

$$[d, b, a] \in \Theta, \qquad [b, a, d] \in \Theta^1, \qquad [a, d, b] \in \Theta^2.$$

(TA-3) For all $x \in A$,

$$\begin{aligned} u, v \in \Theta &\quad \text{implies} \quad [u, v, x] \in \Theta, \\ u, v \in \Theta^1 &\quad \text{implies} \quad [x, u, v] \in \Theta^1, \\ u, v \in \Theta^2 &\quad \text{implies} \quad [u, x, v] \in \Theta^2. \end{aligned}$$

(TA-4) For fixed $x_1, x_2 \in A$,

$$\begin{aligned} [u, v, x_1] &= [u, v, x_2] \text{ for all } u, v \in \Theta \quad \text{implies} \quad x_1 = x_2, \\ [x_1, u, v] &= [x_2, u, v] \text{ for all } u, v \in \Theta^1 \quad \text{implies} \quad x_1 = x_2, \\ [u, x_1, v] &= [u, x_2, v] \text{ for all } u, v \in \Theta^2 \quad \text{implies} \quad x_1 = x_2. \end{aligned}$$

Denote by Θ_0 the set of all members of Θ such that there exists b, d such that $[a, *, b]$ and $[*, a, d]$, and similarly for Θ_0^1 and Θ_0^2. Note that TA-1 tacitly assumes the existence of atoms, coident pairs, and in particular, coident pairs of atoms.

Proposition 1.6 (Principle of duality for trace algebras). *Given a permutation $\sigma \in S_3$, define the ternary product $[\ ,\ ,\ ]_\sigma$ on A by*

$$[x_1, x_2, x_3]_\sigma = [x_{1\sigma}, x_{2\sigma}, x_{3\sigma}]$$

for all $x_1, x_2, x_3 \in A$. Then A is a trace algebra under each of these ternary products, and in this case is denoted by A^σ. In fact, if Θ_σ^i $(i=1, 2, 3)$ denotes the set of i-atoms of A^σ, then $\Theta_\sigma^i = \Theta^{i\sigma}$.

Proof. Follows from the symmetry of our definitions and axioms.

It follows, therefore, that for every statement about trace algebras, there are five more statements corresponding to the five "dual" trace algebras A^σ, and hence if the statement is valid for all trace algebras, then so are the "dual" statements. For this reason, we will usually suppress dual propositions and theorems and their proofs except where necessary for clarity or completeness.

We can now proceed.

Definition 1.7. (i) For $x \in A$ define ϕ_x: $\Theta \times \Theta \to \Theta$ by

$$(u, v)\phi_x = [u, v, x]$$

for all $u, v \in \Theta$. By TA-3, ϕ_x is meaningful, and by TA-4, $\phi_x = \phi_y$ iff $x = y$.

(ii) Similarly, define maps ψ_x: $\Theta^1 \times \Theta^1 \to \Theta^1$ and η_x: $\Theta^2 \times \Theta^2 \to \Theta^2$ by

$$\begin{aligned} (u, v)\psi_x &= [x, v, u] \quad \text{for all } u, v \in \Theta^1, \\ (u, v)\eta_x &= [u, x, v] \quad \text{for all } u, v \in \Theta^2. \end{aligned}$$

The following relations are basic:

Proposition 1.8. (i) *For all $u, v, a \in \Theta$, $x, y \in A$,*

$$(u, v)\phi_{[x,a,y]} = ((u, v)\phi_x, a)\phi_y, \qquad (u, v)\phi_{[a,x,y]} = (a, (u, v)\phi_x)\phi_y.$$

(ii) *For all* $u, v, b \in \Theta^1$, $x, y \in A$,

$$(u, v)\psi_{[x,y,b]} = (b, (u, v)\psi_y)\psi_x, \qquad (u, v)\psi_{[x,b,y]} = ((u, v)\psi_y, b)\psi_x.$$

(iii) *For all* $u, v, d \in \Theta^2$, $x, y \in A$,

$$(u, v)\eta_{[d,x,y]} = (d, (u, v)\eta_y)\eta_x, \qquad (u, v)\eta_{[x,y,d]} = ((u, v)\eta_x, d)\eta_y.$$

Proof. Immediate consequences of the definitions and the partial associativity laws.

2. Examples of trace algebras.

(1) *Trivial trace algebras.* Let S be a commutative semigroup with unity. Define the ternary product $[x, y, z] = t\cdot xyz$ for all $x, y, z \in S$, where t is an element of S which has an inverse. Then S endowed with this product is a trace algebra with $\Theta = \Theta^1 = \Theta^2 = S$.

(2) *Finite-dimensional K-modules.* As was mentioned in the Introduction, this particular class of trace algebras is the basic model for most of our considerations. Let K be a commutative ring with unity, and M be a finite-dimensional free module over K which is self-dual, and let $\mathrm{Bilin}_K(M)$ denote the set of K-bilinear mappings from $M \times M$ to M. Recall that for any $f \in \mathrm{Bilin}_K(M)$ there is an element $fT \in M$ (usually called the *contraction* of f, but which we will refer to as the *trace* of f), defined by $fT = \sum_{j=1} (e_j, e_j)f$ where the summation is over $j = 1, \ldots, \dim M$ and where e_j is a self-dual basis of M, in the sense that $\langle e_i, e_j\rangle = \delta_{ij}$ for all i, j. The trace is independent of the choice of self-dual basis.

Then, if $f, g, h \in \mathrm{Bilin}_K(M)$, define $[f, g, h] \in \mathrm{Bilin}_K(M)$ by

$$(x, y)[f, g, h] = \{(x, y)V\}T$$

for all $x, y \in M$, where $V \in \mathrm{Bilin}_K(M)$ is defined by

$$(z, z')V = ((x, z)f, (z', y)g)h$$

for all $z, z' \in M$. Or, written in more explicit notation,

$$(*) \qquad (x, y)[f, g, h] = \sum_{j=1} ((x, e_j)f, (e_j, y)g)h.$$

The module $\mathrm{Bilin}_K(M)$ is a trace algebra under this ternary product.

In particular, f is a 3-atom iff $(u, v)f = \langle u, v\rangle w$ for all $u, v \in M$ and fixed $w \in M$; f is a 1-atom iff $(u, v)f = \langle u, w\rangle v$ for all $u, v \in M$ and fixed $w \in M$. And finally, f is a 2-atom iff $(u, v)f = \langle v, w\rangle u$ for all u, v.

In the concluding section of this paper, it will be shown that every trace algebra which is well behaved has a defining relation analogous to equation $(*)$, though it need not look anything like $\mathrm{Bilin}_K(M)$, as will be shown when more examples are given in §12.

Throughout this paper, the plinth Θ and the maps ϕ_x will be given precedence over the plinths Θ^1, Θ^2 and maps ψ_x, η_x. The main reason for this is as follows:

if one takes the trace algebra $\mathrm{Bilin}_K(M)$ and constructs ϕ_f, ψ_f, η_f for $f \in \mathrm{Bilin}_K(M)$, one finds that ϕ_f *is isomorphic* (as a bilinear product on M) *to* f, whereas this is *not* true of ψ_f and η_f. Therefore, in an arbitrary trace algebra, we will regard ϕ_x as the "true" representative of a quantity x whose (hidden) status as a binary map is expressed by its membership in the trace algebra.

3. **The induced semigroup.** We show that there is a commutative semigroup with unity, Γ, such that A, and in fact Θ, Θ^1, and Θ^2, are Γ-moduloids and that the ϕ_x, ψ_x, η_x are Γ-bihomogeneous maps.

DEFINITION 3.1. Define maps (a, b), $\{a, d\}$, $[d, b]$: $A \to A$ as follows:

(i) $x(a, b) = [a, x, b]$,

(ii) $x\{a, d\} = [x, a, d]$,

(iii) $x[d, b] = [d, b, x]$,

for all $x \in A$ and for fixed a, b, d.

Denote the sets of these maps as Γ_1, Γ_2, Γ_3, respectively.

PROPOSITION 3.2. (i) *For* $\alpha \in \Gamma_1$,

$$[x\alpha, y, z] = [x, y, z\alpha] = [x, y, z]\alpha,$$

(ii) *for* $\alpha \in \Gamma_2$,

$$[x, y\alpha, z] = [x, y, z\alpha] = [x, y, z]\alpha,$$

(iii) *for* $\alpha \in \Gamma_3$,

$$[x\alpha, y, z] = [x, y\alpha, z] = [x, y, z]\alpha,$$

for all $x, y, z \in A$.

Proof. For example, $[[a, x, b!], y, z] = [a!, [x, y, z], b] = [x, y, [a, z, b]]$, which proves (i).

PROPOSITION 3.3. *Suppose that* $u \in \Theta^i$ *and* $\alpha \in \Gamma_j$ (*for* $i, j = 1, 2, 3$ *fixed*). *Then* $u\alpha \in \Theta^i$.

Proof. For example, let $i = 3$. Then $u(a, b) = [a, u, b] \in \Theta$ and $u\{a, d\} = [u, a, d] \in \Theta$ by TA-3. On the other hand, $u[d, b] = [d, b, u] \in \Theta$ by TA-2.

PROPOSITION 3.4. $\Gamma_1 = \Gamma_2 = \Gamma_3$.

Proof. (i) Let $\alpha \in \Gamma_3$ and $(a, b) = 1 =$ identity map on A (which is possible by TA-1). Then by Propositions 3.2 and 3.3, $\alpha = (a, b) \circ \alpha = (a\alpha, b) \in \Gamma_1$. So $\Gamma_3 \subseteq \Gamma_1$. (ii) Let $\alpha \in \Gamma_1$, and $\{a, d\} = 1$. Then $\alpha = \{a, d\} \circ \alpha = \{a, d\alpha\} \in \Gamma_2$. So $\Gamma_1 \subseteq \Gamma_2$. (iii) Let $\alpha \in \Gamma_2$ and $[d, b] = 1$. Then $\alpha = [d, b] \circ \alpha = [d, b\alpha] \in \Gamma_3$. So $\Gamma_2 \subseteq \Gamma_3$ and we are done.

We therefore denote $\Gamma = \Gamma_1 = \Gamma_2 = \Gamma_3$.

COROLLARY 3.5. *The ternary product* $[\ ,\ ,\ ]$ *is* Γ-*homogeneous in each argument.*

PROPOSITION 3.6. Γ *is a commutative semigroup with unit.*

Proof. We need only demonstrate commutativity. If $\alpha, \beta \in \Gamma$ and $[d, b]=1$, then $x\alpha \circ \beta=[d, b, x]\alpha \circ \beta=[d\alpha, b, x]\beta=[d\alpha, b\beta, x]=[d, b\beta, x]\alpha=[d, b, x]\beta \circ \alpha=x\beta \circ \alpha$ for all $x \in A$.

It follows therefore that A, and Θ, Θ^1, Θ^2, are Γ-moduloids, and that the ϕ_x, ψ_x, η_x are Γ-bihomogeneous maps. Hereafter, we write $\alpha \circ \beta=\alpha\beta$ or $=\alpha\cdot\beta$.

PROPOSITION 3.7. *$(\ , \ )$, $\{ \ , \ \}$, and $[\ , \ ]$ are nondegenerate maps.*

Proof. An application of TA-4, and TA-1.

PROPOSITION 3.8. *Suppose $\alpha, \beta \in \Gamma$ and $u \in \Theta_0^i$ (i fixed). Then $u\alpha=u\beta$ implies that $\alpha=\beta$.*

Proof. For example, suppose $i=3$. Then if we let $(u, b)=1$ for some b, then $x\alpha=[u, x, b]\alpha=[u\alpha, x, b]=[u\beta, x, b]=[u, x, b]\beta=x\beta$ for all $x \in A$.

REMARK. In particular, this shows that the restrictions of any member of the induced semigroup Γ to any one of the Θ^i uniquely determines that member. For this reason we will hereafter freely identify a member of Γ with its restrictions to the various plinths.

We will also denote the set of 1-1 maps in Γ by Γ_0.

DEFINITION 3.9. Given $u, v \in \Theta$, define $\langle u, v\rangle: \Theta \to \Theta$ by

$$a\cdot\langle u, v\rangle = [u, v, a]$$

for all a. Denote the set of these maps by Γ'.

PROPOSITION 3.10. *If Γ is regarded as a set of maps on Θ, then $\Gamma' \subseteq \Gamma$.*

Proof. For $w \in \Theta$, and $(a, b)=1$, we have

$$w\langle u, v\rangle = [u, v, [a, w!, b]] = [[u, v, a], w, b] = w\cdot([u, v, a], b)$$

for all w. Hence we are finished.

We will be in a position later to show that in fact $\Gamma'=\Gamma$ (Corollary 10.8).

REMARK. *Additive trace algebras.* Suppose that the trace algebra A is also an abelian group under $(+)$ in such a way that each argument of the ternary product is additive with respect to addition: $[x+x', y, z]=[x, y, z]+[x', y, z]$, etc., in which case A is called an additive trace algebra. Then note that Γ is a commutative ring with unity (which we will call the "induced ring" of A), that A is a Γ-module, with Θ, Θ^1, Θ^2 as submodules, and that in fact, the maps ϕ_x, ψ_x, η_x are Γ-bilinear maps on Θ, Θ^1, and Θ^2, respectively.

4. **The comonoid.** In this section we define the remainder of those concepts that we will need before we can embark on the proofs of the theorems in the next section. In brief, we construct classes of Γ-homogeneous maps on Θ, Θ^1, Θ^2, which play the same relationship to A as the set $\hom_K(M; M)$ plays to the trace algebra $\mathrm{Bilin}_K(M)$ of example (2).

DEFINITION 4.1. For any maps $g: S\times S\to S$ and $t: S\to S$ on an arbitrary set S, we will define the *left*, *right*, and *rear isotopes* of g by t to be the maps $(t, 1, g)$, $(1, t, g)$ and $g\circ t: S\times S\to S$ defined, respectively, by

$$(x, y)(t, 1, g) = (xt, y)g,$$
$$(x, y)(1, t, g) = (x, yt)g,$$
$$(x, y)(g\circ t) = ((x, y)g)t,$$

for all $x, y\in S$. The isotopes have the obvious properties:

$$(t, 1, (1, t', g)) = (1, t', (t, 1, g)),$$
$$(t, 1, (t', 1, g)) = (t\circ t', 1, g),$$
$$(1, t, g)\circ t' = (1, t, g\circ t'),$$

etc. We hereafter define $(t, t', g)=(t, 1, (1, t', g))$.

DEFINITION 4.2. Let $A\phi$ be the set of all ϕ_x, $A\psi$ the set of all ψ_x, and $A\eta$ the set of all η_x, for $x\in A$.

Then, a map $\theta: \Theta\to\Theta$ is called *compatible* if for every $w\in A$, all three isotopes of ϕ_w by θ are in $A\phi$. That is, there are $x, y, z\in A$ such that $\phi_x=(\theta, 1, \phi_w)$, $\phi_y=(1, \theta, \phi_w)$ and $\phi_z=\phi_w\circ\theta$. If the x, y, z exist, then they are uniquely determined by w and θ, and so we are therefore justified in writing

$$x = (1, \theta, w), \qquad y = (\theta, 1, w), \qquad z = w\circ\theta,$$
$$(\theta, \theta', w) = (1, \theta', (\theta, 1, w)).$$

The properties of isotopes listed above carry over for these formal isotopes.

Denote the set of compatible maps by Δ^3, or just by Δ.

Similarly, we have sets Δ^1, Δ^2 defined as follows: $\delta\in\Delta^1$ iff for all $w\in A$, all three isotopes of ψ_w by δ are in $A\psi$, where, of course, δ is a map $\Theta^1\to\Theta^1$. We denote the corresponding isotopes determined by $(\delta, 1, \psi_w)$, $(1, \delta, \psi_w)$ and $\psi_w\circ\delta$ as $(\delta, 1, w)^1$, $(1, \delta, w)^1$ and $w * \delta$, respectively, and, of course, $(\delta, \delta', w)^1=(\delta, 1, (1, \delta', w)^1)^1$.

Finally, if $\varepsilon: \Theta^2\to\Theta^2$ is such that all three isotopes of η_w by ε are in $A\eta$, for all w, then $\varepsilon\in\Delta^2$, and from $(\varepsilon, 1, \eta_w)$, $(1, \varepsilon, \eta_w)$ and $\eta_w\circ\varepsilon$ we get $(\varepsilon, 1, w)^2$, $(1, \varepsilon, w)^2$ and $w \# \varepsilon$, respectively, and the obvious definition for $(\varepsilon, \varepsilon', w)$. This completes Definition 4.2.

Now, Δ, Δ^1, Δ^2 are obviously monoids, and are called the 3-, 1-, and 2-comonoids of A, respectively. We will usually refer to Δ as just "the comonoid of A."

For ease of expression, we make the following convention: suppose that, for example, for $w\in A$ and $\theta: \Theta\to\Theta$, $(\theta, 1, \phi_x)$ is in $A\phi$. Then we say that "$(\theta, 1, x)$ is defined," or "$(\theta, 1, x)$ exists," or merely "$(\theta, 1, x)\in A$." Similarly for the other isotopes.

DEFINITION 4.3. (i) Suppose that $\theta: \Theta\to\Theta$ is such that $(\theta, 1, b)$ exists and is in Θ^1 for all $b\in\Theta^1$. Then θ is called Θ^1*-consistent.*

(ii) Suppose that $\theta: \Theta\to\Theta$ is such that $(1, \theta, d)$ exists and is in Θ^2 for all $d\in\Theta^2$. Then θ is called Θ^2*-consistent.*

DEFINITION 4.4. For any map $\theta\colon \Theta \to \Theta$ which is Θ^1-consistent (Θ^2-consistent) we may define a map $\theta^1\colon \Theta^1 \to \Theta^1$ ($\theta^2\colon \Theta^2 \to \Theta^2$) by

$$\begin{aligned} b\theta^1 &= (\theta, 1, b) \quad \text{for all } b \in \Theta^1, \\ d\theta^2 &= (1, \theta, d) \quad \text{for all } d \in \Theta^2. \end{aligned}$$

In this case, θ^1 is called the Θ^1-*dual* and θ^2 the Θ^2-*dual* of θ, if they exist.

PROPOSITION 4.5. (i) *Suppose θ is Θ^1-consistent. Then $b * \theta^1$ exists for all b and $b * \theta^1 = b\theta^1$.*

(ii) *Suppose θ is θ^2-consistent. Then $d \# \theta^2$ exists for all d and $d * \theta^2 = d\theta^2$.*

(iii) *For any $\theta\colon \Theta \to \Theta$ and any $a \in \Theta$, $a \circ \theta$ exists and is equal to $a\theta$.*

Proof. Trivial.

PROPOSITION 4.6. (i) *Suppose that $\pi, \tau\colon \Theta \to \Theta$ are Θ^1-consistent. Then $\pi \circ \tau$ is Θ^1-consistent, and $(\pi \circ \tau)^1 = \tau^1 \circ \pi^1$ and $(1 : \Theta)^1 = 1 : \Theta^1$ (where $1{:}S$ means the identity map on the set S). Similarly for Θ^2-consistent maps.*

(ii) *Suppose π, τ are Θ^1-, Θ^2-consistent, respectively. Then*

$$(a\pi, b) = (a, b\pi^1), \qquad \{a\tau, d\} = \{a, d\tau^2\}$$

for all a, b, d.

Proof. Trivial.

In addition we will also need the following:

LEMMA 4.7. *Suppose that $\theta\colon \Theta \to \Theta$. Then*

(i) *$b \circ \theta$ exists iff $(1, \theta, b)$ exists,*

(ii) *$d \circ \theta$ exists iff $(\theta, 1, d)$ exists.*

Proof. For example, $(u, v)\phi = [u, v, b] = v \cdot (u, b)$. The reader may catch a glimpse of what is going to happen in future sections by comparing this last equation to those at the end of example (2).

LEMMA 4.8. *Suppose that for $\theta\colon \Theta \to \Theta$ and $x, y, z \in A$ it is true that $x \circ \theta$, $(\theta, 1, y)$ and $(1, \theta, z)$ exist. Then for arbitrary $w \in A$,*

$$\begin{aligned} [a\theta, w, y] &= [a, w, (\theta, 1, y)], \\ [a, x \circ \theta, z] &= [a, x, (1, \theta, z)], \\ [w, a\theta, z] &= [w, a, (1, \theta, z)], \\ [x \circ \theta, a, y] &= [x, a, (\theta, 1, y)]. \end{aligned}$$

Proof. Application of Proposition 1.8 and the definitions.

5. **The comonoid** (CTD). In this section we prove several theorems concerning Δ. We give a necessary and sufficient condition that a map be in Δ (Theorem 5.7), preparatory to the final and far more elegant version to be proved in §9. It will be useful to know the relations that exist between the ternary product [, ,] and the

three isotopic actions, hence the "exterior action laws" (Theorem 5.9). The remainder of our efforts will be devoted to showing the existence of the Θ^1- and Θ^2-duals of elements of Δ (Lemma 5.2) and to giving a characterization of the maps in Δ (Theorem 5.10).

LEMMA 5.1. (i) *Suppose that* $\theta\colon \Theta \to \Theta$ *is such that* (1) $z \circ \theta \in A$ *and* (2) $b \circ \theta \in A$ ($d \circ \theta \in A$) *for a given* $z \in A$ *and fixed* $b \in \Theta_0^1$ ($d \in \Theta_0^2$). *Then,* $[x, y, z] \circ \theta \in A$ *for all* $x, y \in A$ *and further,* $[x, y, z] \circ \theta = [x, y, z \circ \theta]$.

(ii) *Suppose that* $\delta\colon \Theta^1 \to \Theta^1$ *is such that* (1) $x * \delta \in A$ *and* (2) $d * \delta \in A$ ($a * \delta \in A$) *for some* $d \in \Theta_0^2$ ($a \in \Theta_0$). *Then* $[x, y, z] * \delta \in A$ *for all* $y, z \in A$ *and* $[x, y, z] * \delta = [x * \delta, y, z]$.

(iii) *Suppose that* $\varepsilon\colon \Theta^2 \to \Theta^2$ *is such that* (1) $y \# \varepsilon \in A$ *and* (2) $b \# \varepsilon \in A$ ($a \# \varepsilon \in A$) *for some* $b \in \Theta_0^1$ ($a \in \Theta_0$). *Then* $[x, y, z] \# \varepsilon \in A$ *for all* $x, z \in A$, *and* $[x, y, z] \# \delta = [x, y \# \delta, z]$.

Proof. We will prove (i) for the case $b \circ \theta \in A$. The case for $d \circ \theta \in A$ is similar, and parts (ii), (iii) are dual theorems of (i). Therefore, let $a \in \Theta_0$ be such that $(a, b) = 1$ and then

(1) $[a, z \circ \theta, b] = z \circ \theta = [a, z, b] \circ \theta$.

(2) We show that $[a, z, b \circ \theta] = [a, z, b] \circ \theta$, for

$$\begin{aligned}(u, v)\phi_{[a,z\circ\theta,b]} &= (a, (u, v)\phi_{z\circ\theta})\phi_b = (a, (u, v)\phi_z \circ \theta)\phi_b \\ &= (a, (u, v)\phi_z)\phi_b \circ \theta = (a, (u, v)\phi_z)\phi_{b\circ\theta} = (u, v)\phi_{[a,z,b\circ\theta]}.\end{aligned}$$

$$(3)\qquad \begin{aligned}(u, v)\phi_{[x,y\circ z]} \circ \theta &= (a, (u, v)\phi_{[x,y,z]})\phi_b \circ \theta \quad \text{(remember that } (a, b) = 1) \\ &= (a, (u, v)\phi_{[x,y,z]})\phi_{b\circ\theta} = (u, v)\phi_{[a,[x,y,z],b\circ\theta]}\end{aligned}$$

for all $u, v \in \Theta$. So by definition, $[x, y, z] \circ \theta$ exists for arbitrary $x, y \in A$, and further,

$$[x, y, z] \circ \theta = [a, [x, y, z], b \circ \theta].$$

$$(4)\qquad \begin{aligned}[x, y, z \circ \theta] &= [a!, [x, y, z \circ \theta], b] = [x, y, [a, z \circ \theta, b]] \\ &= [x, y, [a!, z, b \circ \theta]] \qquad \text{(by (2))} \\ &= [a, [x, y, z], b \circ \theta] = [x, y, z] \circ \theta \quad \text{(by (3))}.\end{aligned}$$

LEMMA 5.2. *If* $\theta\colon \Theta \to \Theta$ *is such that* $(\theta, 1, a)$ *exists for some* $a \in \Theta_0$, *then*

(i) θ *is* Θ^1-*consistent,*

(ii) $a * \theta^1$ *is defined and equal to* $(\theta, 1, a)$.

Proof. (i)

$$\begin{aligned}[c, b!, [(\theta, 1, a), u, v]] &= [[c, b, (\theta, 1, a)], u, v] = [[c\theta, b!, a], u, v] \quad \text{(where } c \in \Theta) \\ &= [c\theta, b, [a, u, v]].\end{aligned}$$

So let $u, v \in \Theta^1$, then if we set $E = [(\theta, 1, a), u, v]$, $F = [a, u, v]$, for fixed u, v, then from the above equation we have $[c, b, E] = [c\theta, b, F]$ and so $[x, y, [c!, b, E]] = [x, y, [c\theta!, b, F]]$, hence $[c, [x, y, b], E] = [c\theta, [x, y, b], F]$.

Now let $x \in \Theta^2$, and $[x, b]=1$. Then $[c, y, E]=[c\theta, y, F]$ for all $y \in A$ and $c \in \Theta$. Let $y \in \Theta$, then we have $(c, y)\phi_E=(c\theta, y)\phi_F$ for all $c, y \in \Theta$, hence $(\theta, 1, F)$ exists and is equal to E; that is,

$$(*) \qquad (\theta, 1, [a, u, v]) = [(\theta, 1, a), u, v] = (v, u)\psi_{(\theta,1,a)}.$$

So let $(a, v)=1$, then $(\theta, 1, u)=(v, u)\psi_{(\theta,1,a)}$.

But the right-hand side of this last equation is in Θ^1, so $(\theta, 1, u) \in \Theta^1$ for all $u \in \Theta^1$, so (i) is established.

(ii) Since we have just shown that θ is Θ^1-consistent, then the Θ^1-dual θ^1 is defined. So returning to equation $(*)$ above, we get

$$(\theta, 1, (v, u)\psi_a) = (v, u)\psi_{(\theta,1,a)}$$

and so $(v, u)\psi_a \circ \theta^1=(v, u)\psi_{(\theta,1,a)}$ for all $v, u \in \Theta^1$.

So $a * \theta^1$ is defined and equal to $(\theta, 1, a)$.

LEMMA 5.3. *Suppose that $(\theta, 1, d)$ (or $(\theta, 1, a)$) exists for some $d \in \Theta_0^2$ (or $a \in \Theta_0$). Then*

(i) $(\theta, 1, x) \in A$ *for all* $x \in A$,

(ii) $x * \theta^1 \in A$ *for all* $x \in A$,

(iii) $(\theta, 1, x)=x * \theta^1$ *for all* $x \in A$,

(iv) $[(\theta, 1, x), y, z]=(\theta, 1, [x, y, z])$ *for all* $x, y, z \in A$.

Proof. We prove the theorem for $a \in \Theta$. By Lemma 5.2(ii), we can conclude from the hypothesis that $a * \theta^1$ exists for some $a \in \Theta_0$.

(1) $x * \theta^1$ is defined for all $x \in A$. For since $a * \theta^1$ is defined, let $(a, b)=1$. Then by Lemma 5.1(ii), $[a, x, b] * \theta^1=x * \theta^1$ is also defined, for arbitrary x.

(2) $(\theta^1, 1, a)^1=a\theta$ for all $a \in \Theta$. For

$$\begin{aligned}(u, v)\psi_{a\theta} &= [a\theta, v, u] = [a, v, (\theta, 1, u)] \quad (\text{since } \theta \text{ is } \Theta^1\text{-consistent})\\ &= [a, v, u\theta^1] = (u\theta^1, v)\psi_a = (u, v)(\theta^1, 1, \psi_a),\end{aligned}$$

and so the desired equation results.

(3) For notational convenience, denote $\theta J=\theta^1$ for all $\theta \in \Delta$. Then

$$\begin{aligned}(u, v)\psi_{[a,b,x*\theta J]} &= ((u, v)\psi_{(x*\theta J)}, b)\psi_a = ((u, v)\psi_x \circ \theta J, b)\psi_a\\ &= ((u, v)\psi_x, b)(\theta J, 1, \psi_a).\end{aligned}$$

So let $E=(\theta J, 1, a)^1$, then

$$= ((u, v)\psi_x, b)\psi_E = (u, v)\psi_{[E,b,x]} = (u, v)\psi_{[a\theta,b,x]}$$

(for, $E=a$ by (2)). We have therefore shown that $[a, b, x * \theta^1]=[a\theta, b, x]$.

(4) From this last equation, we get, for $u, v \in \Theta$,

$$[u, v, [a!, b, x * \theta^1]] = [u, v, [a\theta!, b, x]]$$

so $[a, [u, v, b], x * \theta^1]=[a\theta, [u, v, b], x]$. Let $(u, b)=1$. Then $[a, v, x * \theta^1]=[a\theta, v, x]$ for all $a, v \in \Theta$, hence $(a, v)\phi_{x*\theta J}=(a\theta, v)\phi_x=(a, v)(\theta, 1, \phi_x)$. Therefore, $(\theta, 1, x)$ is defined and equal to $x * \theta^1$ for all $x \in A$. This gives (ii) and (iii).

(5) We know from Lemma 5.1 that $[x * \theta^1, y, z] = [x, y, z] * \theta^1$ for all $x, y, z \in A$, on the basis of (1). Therefore, by (4) we get (iv). This completes the proof.

LEMMA 5.4. *If* $(1, \theta, a)$ *is defined for some* $a \in \Theta_0$, *then*

(i) θ *is* Θ^2*-consistent,*

(ii) $a \# \theta^2$ *is defined and equal to* $(1, \theta, a)$.

Proof. This is a dual to Lemma 5.2.

COROLLARY 5.5. *If* $\theta \in \Delta$, *then* θ *is both* Θ^1- *and* Θ^2*-consistent, and hence has* Θ^1- *and* Θ^2*-duals.*

Proof. Lemmas 5.3 and 5.4.

LEMMA 5.6. *Suppose that* $(1, \theta, a)$ *(or* $(1, \theta, b)$*) is defined for some* $a \in \Theta_0$ *(or* $b \in \Theta_0^1$*). Then*

(i) $(1, \theta, y) \in A$ *for all* $y \in A$,

(ii) $y \# \theta^2 \in A$ *for all* $y \in A$,

(iii) $(1, \theta, y) = y \# \theta^2$ *for all* $y \in A$,

(iv) $[x, (1, \theta, y), z] = (1, \theta, [x, y, z])$ *for all* $x, y, z \in A$.

Proof. This, again, is a dual of Lemma 5.3, whose proof is based on Lemma 5.2.

We now give the first version of a necessary and sufficient condition for a map to be in Δ.

THEOREM 5.7. *Suppose* $\theta: \Theta \to \Theta$ *is such that each of the following three statements is valid:*

(i) $b \circ \theta$ *or* $d \circ \theta$ *exists for some* $b \in \Theta_0^1$ *or* $d \in \Theta_0^2$.

(ii) $(\theta, 1, a)$ *or* $d \circ \theta$ *exists for some* $a \in \Theta_0$ *or* $d \in \Theta_0^2$.

(iii) $(1, \theta, a)$ *or* $b \circ \theta$ *exists for some* $a \in \Theta_0$ *or* $b \in \Theta_0^1$.

Then $\theta \in \Delta$. *(The converse is trivial.)*

Proof. We make use of Lemmas 5.1, 5.3, and 5.6. For example, if parts (ii) and (iii) are valid, then by Lemmas 5.3 and 5.6, automatically we can conclude that $(1, \theta, w)$ and $(\theta, 1, w)$ exist for all $w \in A$. Therefore, it only remains to be shown that $w \circ \theta \in A$ for all $w \in A$. For example, suppose that $b \circ \theta$ exists for $b \in \Theta_0^1$. Then by Lemma 5.1, we can conclude that $[a, x, b] \circ \theta$ exists. Since $b \in \Theta_0^1$, let $(a, b) = 1$ and then we get that also $x \circ \theta$ exists. Similarly for the case $d \in \Theta_0^2$. Finally, the proofs of all the other possible combinations of (i), (ii), (iii) are duals of what we have just done.

Therefore, we have shown that one needs only to test the action of the isotopes of a map $\theta: \Theta \to \Theta$ on *at most three* elements of A, rather than on *all* of them, and in fact, the testing elements are permitted to be atoms. We will show in §9 that one needs to use only one testing element.

For the while, however, notice that Theorem 5.7 is actually stronger than is necessary for some cases. For example, from Lemma 4.7, we get the following corollary:

COROLLARY 5.8. *Suppose that for $\theta\colon \Theta \to \Theta$, and for some $a \in \Theta_0$, $b \in \Theta_0^1$, $d \in \Theta_0^2$, at least one of the following two conditions holds*:

(i) *$b \circ \theta \in A$, and $(\theta, 1, a) \in A$ or $d \circ \theta \in A$.*

(ii) *$d \circ \theta \in A$, and $(1, \theta, a) \in A$ or $b \circ \theta \in A$.*

Then $\theta \in \Delta$.

THEOREM 5.9 (EXTERIOR ACTION LAWS). *Suppose that $\theta \in \Delta$. Then for all $x, y, z \in A$,*

$$\begin{aligned}[x, y, z \circ \theta] &= [x, y, z] \circ \theta,\\ [x, (1, \theta, y), z] &= (1, \theta, [x, y, z]),\\ [(\theta, 1, x), y, z] &= (\theta, 1, [x, y, z]).\end{aligned}$$

Proof. Lemmas 5.1, 5.3, 5.6.

Define for fixed $a \in \Theta$, maps $\theta^x, \theta_x\colon \Theta \to \Theta$ by $u\theta_x = (u, a)\phi_x$ and $u\theta^x = (a, u)\phi_x$ for a given x. The maps θ^x, θ_x will be called the *right* and *left a-sections* of x, respectively.

THEOREM 5.10 (CHARACTERIZATION OF Δ). *A map $\theta\colon \Theta \to \Theta$ is in Δ iff it is a section of some x in A.*

Proof. (1) Given a section θ_x, which for convenience we denote by π. If $b \in \Theta_0^1$, then $(u, v)\phi_b \circ \pi = ((u, v)\phi_b, a)\phi_x = (u, v)\phi_{[b,a,x]}$. Hence, $b \circ \pi \in A$ for some $b \in \Theta_0^1$.

Similarly, take $d \in \Theta_0^2$. Then

$$\begin{aligned}(u, v)\phi_d \circ \pi &= (u\pi, v)\phi_d = ((u, a)\phi_x, v)\phi_d = [[u, a, x], v, d\,!]\\ &= [u, [a, v, d], x] = (u, (a, v)\phi_d)\phi_x = (u, vt)\phi_x\end{aligned}$$

where $vt = (a, v)\phi_d$ for all $v \in \Theta$. But if we show that $(1, t, z) \in A$ for all $z \in A$, then we are finished, for then $(u\pi, v)\phi_d = (u, v)\phi_{(1,t,x)}$ and hence $(\pi, 1, d) \in A$, so that we can then apply Corollary 5.8.

Now, $(u, v)\phi_z \circ t = (a, (u, v)\phi_z)\phi_d = [a!, [u, v, z], d] = [u, v, [a, z, d]] = (u, v)\phi_{[a,z,d]}$. Hence, $z \circ t$ exists for all $z \in A$. On the other hand, $(u, v)(1, t, \phi_z) = (u, vt)\phi_z = (u, (a, v)\phi_d)\phi_z = (u, a\cdot\{v, d\})\phi_z = (u \cdot \{v, d\}, a)\phi_z = ((u, v)\phi_d, a)\phi_z = (u, v)\phi_{[d,a,z]}$. Therefore, $(1, t, z)$ exists for all $z \in A$. But we now have enough information to apply Corollary 5.8, and so $t \in \Delta$, and so we are done.

(2) Conversely, suppose that $\pi \in \Delta$. Let $b \in \Theta_0^1$, with $(a, b) = 1$ for some a. Then certainly $b \circ \pi \in A$, and we have, if $t\colon \Theta \to \Theta$ denotes the left a-section of $b \circ \pi$,

$$ut = (a, u)\phi_b \circ \pi = u\pi$$

where we are making use of the observation in the proof of Lemma 4.7.

6. **Characterization of Θ^1 and Θ^2.** In this section we prove the "center lemma," which is a characterization of the induced semigroup Γ, and we use it to prove necessary and sufficient conditions that maps be in Θ^1 or Θ^2, by means of testing their isotopic properties with the members of Δ.

LEMMA 6.1 ("CENTER LEMMA"). *Suppose that* $k\colon \Theta \to \Theta$ *is such that* $\pi \circ k = k \circ \pi$ *for all* $\pi \in \Delta$. *Then* $k \in \Gamma$.

Proof. Note immediately that k is Γ-homogeneous, since in particular, $\lambda \circ k = k \circ \lambda$ for all $\lambda \in \Gamma$. Then

(1) Suppose that $k \notin \Gamma$. Then we first show that there exists an $a_0 \in \Theta_0$ such that $a_0 k \neq \lambda a_0$ for all $\lambda \in \Gamma$. For assume the contrary. Then for all $a \in \Theta_0$, there exists a $\lambda_a \in \Gamma$ such that $ak = \lambda_a \cdot a$. Now, for a fixed $a \in \Theta_0$, define θ_u by $v\theta_u = (v, b)\cdot u$, where $b \in \Theta_0^1$ is such that $(a, b) = 1$. Then by Theorem 5.10, $\theta_u \in \Delta$ for all u; and further, $a\theta_u = u$. Hence, $uk = a\theta_u \circ k = ak \circ \theta_u = \lambda_a \cdot a\theta_u = \lambda_a \cdot u$ for all $u \in \Theta$. Hence, $k \in \Gamma$, contrary to assumption. Hence, there exists an $a_0 \in \Theta_0$ such that $a_0 k \neq \lambda a_0$ for all $\lambda \in \Gamma$.

(2) Define $c_0 = a_0 k$ (hence $c_0 \neq \lambda a_0$ for all $\lambda \in \Gamma$). Then we define as before $u\pi = (u, b)\cdot a_0$ where b is such that $(a_0, b) = 1$. Then note that $a_0 \pi = a_0$ and $c_0 \pi = \lambda a_0$ for some $\lambda \in \Gamma$. Then, on the one hand, $a_0(k \circ \pi) = c_0 \pi = \lambda a_0$. But on the other hand, $a_0(\pi \circ k) = a_0 k = c_0$. Since $k \circ \pi = \pi \circ k$, we get $c_0 = \lambda a_0$, a contradiction. Therefore, we must conclude that our original assumption that $k \notin \Gamma$ is false.

COROLLARY 6.2. Γ *is the center of* Δ.

THEOREM 6.3 (CHARACTERIZATION OF Θ^1 AND Θ^2). *Given a map* $\phi\colon \Theta \times \Theta \to \Theta$, *then*

(i) *If* $(1, \theta, \phi) = \phi \circ \theta$ *for all* $\theta \in \Delta$, *and if the right a-section of* ϕ *is in* Δ *for some* $a \in \Theta_0$, *then* $\phi = \phi_b$ *for some* $b \in \Theta^1$.

(ii) *If* $(\theta, 1, \phi) = \phi \circ \theta$ *for all* $\theta \in \Delta$, *and if the left a-section of* ϕ *is in* Δ *for some* $a \in \Theta_0$, *then* $\phi = \phi_d$ *for some* $d \in \Theta^2$.

Proof. Note that in using the term "a-section" we are extending in the obvious manner our definition given just before the statement of Theorem 5.10. The converse of this theorem is contained in Lemma 4.7 and Theorem 5.10. We prove (i). Define $\pi\colon u \mapsto (u, a)\phi$ as the a-section whose existence is hypothesized in (i), and for arbitrary $v \in \Theta$, let $\tau\colon u \mapsto (v, u)\phi$. Then given $\theta \in \Delta$, we have $u(\tau \circ \theta) = (v, u)\phi \circ \theta = (v, u\theta)\phi = u(\theta \circ \tau)$. Hence $\theta \circ \tau = \tau \circ \theta$ for all $\theta \in \Delta$, and so by the center lemma, $\tau \in \Gamma$. Hence, for each $v \in \Theta$, there is a $\lambda_v \in \Gamma$ such that $(v, u)\phi = \lambda_v \cdot u$ for all $u, v \in \Theta$.

Now, let a be as stated and let $(a, b') = 1$ for some b'. Then $(v, u)\phi = \lambda_v \cdot u = (\lambda_v \cdot a, b')\cdot u = ((v, a)\phi, b')\cdot u = (v\pi, b')\cdot u$. But by assumption, $\pi \in \Delta$, and so by Corollary 5.5 the Θ^1-dual of π exists, and hence $(v\pi, b')\cdot u = (v, b'\pi^1)\cdot u$ by Proposition 4.6(ii). Then if we set $b = b'\pi^1$ then $\phi = \phi_b$. The proof for (ii) is similar.

COROLLARY 6.4. *Let* $x \in A$. *Then*

(i) $x \in \Theta^1$ *iff* $(1, \theta, x) = x \circ \theta$ *for all* $\theta \in \Delta$,

(ii) $x \in \Theta^2$ *iff* $(\theta, 1, x) = x \circ \theta$ *for all* $\theta \in \Delta$.

Proof. Theorem 5.10 applied to Theorem 6.3.

7. **Ions and the dual pairing.** In this section we develop certain concepts needed in later sections to prove the existence of a duality theory on Δ, to characterize Θ, as well as several other things. We define an "inner product" and the functor (*), and the section concludes with what amounts to a "Riesz representation theorem" (Theorem 7.9).

DEFINITION 7.1. Given $x \in A$, the Θ-, Θ^1-, Θ^2-*ions* of x, if they exist, are elements x^-, x_+ and x^+, respectively, which obey the following conditions:

$$\begin{aligned} [u, v, x] &= [v, u, x^-] \quad \text{for all } u, v \in \Theta, \\ [x, u, v] &= [x_+, v, u] \quad \text{for all } u, v \in \Theta^1, \\ [u, x, v] &= [v, x^+, u] \quad \text{for all } u, v \in \Theta^2. \end{aligned}$$

If an ion exists, then it is uniquely determined by x. Assuming that everything exists, the ions are involutions: $x^{--}=x$, $x_{++}=x$, $x^{++}=x$.

PROPOSITION 7.2. (i) *For every $b \in \Theta^1$, b^- exists and is in Θ^2.*

(ii) *For every $d \in \Theta^2$, d^- exists and is in Θ^1.*

Proof. Define ϕ: $\Theta \times \Theta \to \Theta$ by $(u, v)\phi = (v, b)\cdot u$ for all $u, v \in \Theta$ and fixed b. Then ϕ satisfies the requirements for Theorem 6.3(ii), hence there is a unique $d \in \Theta^2$ such that $\phi = \phi_d$, i.e., such that $[u, v, d] = [v, u, b]$ for all $u, v \in \Theta$. Hence b^- exists and is equal to d, which is in Θ^2. Similarly for the second part.

The dual propositions of 7.2 are:

PROPOSITION 7.3. (i) *For every $a \in \Theta$, a_+ exists and is in Θ^2. For every $d \in \Theta^2$, d_+ exists and is in Θ.*

(ii) *For every $a \in \Theta$, a^+ exists and is in Θ^1. For every $b \in \Theta^1$, b^+ exists and is in Θ.*

PROPOSITION 7.4. (i) *Suppose that x^-, y^- exist; then $[a, x, y]^-$ and $[x, a, y]^-$ exist for every $a \in \Theta$, and $[a, x, y]^- = [x^-, a, y^-]$ and $[x, a, y]^- = [a, x^-, y^-]$.*

(ii) *Suppose that x_+, y_+ exist; then $[x, b, y]_+$ and $[x, y, b]_+$ exist and $[x, b, y]_+ = [x_+, y_+, b]$ and $[x, y, b]_+ = [x_+, b, y_+]$.*

(iii) *Suppose that x^+, y^+ exist; then $[d, x, y]^+$ and $[x, y, d]^+$ exist and $[d, x, y]^+ = [y^+, x^+, d]$ and $[x, y, d]^+ = [d, y^+, x^+]$.*

Proof. Simple.

COROLLARY 7.5. *For all $a \in \Theta$, $b \in \Theta^1$, $d \in \Theta^2$,*

(i) $(a, b) = \{a, b^-\}$,

(ii) $(a, b) = [a_+, b]$,

(iii) $\{a, d\} = [d, a^+]$.

Proof. Easy consequence of Proposition 7.4.

COROLLARY 7.6. (i) *There exists d such that $[*, a, d]$ iff there exists b such that $[a, *, b]$.*

(ii) *There exists d such that $[d, b, *]$ iff there exists a such that $[a, *, b]$.*

(iii) *There exists a such that $[*, a, d]$ iff there exists b such that $[d, b, *]$.*

Proof. Corollary 7.5.

This last corollary, when compared to axiom TA-1, shows the interconnection between coident pairs involving atoms.

DEFINITION 7.7. Define:

(i) $\langle\!\langle u, v\rangle\!\rangle=\{u, v_+\}$ for all $u, v \in \Theta$.

(ii) For $\theta \in \Delta$, $\theta^*\colon \Theta \to \Theta$ by $u\theta^*=(u_+\theta^2)_+$ for all $u \in \Theta$.

PROPOSITION 7.8. (i) $\langle\!\langle u\theta, v\rangle\!\rangle=\langle\!\langle u, v\theta^*\rangle\!\rangle$ *for all* $u, v \in \Theta$ *and* $\theta \in \Delta$.

(ii) $(\pi \circ \tau)^*=\tau^* \circ \pi^*$ *for all* $\pi, \tau \in \Delta$.

(iii) $1^*=1$.

(iv) $(\alpha\theta)^*=\alpha\cdot\theta^*$ *for all* $\alpha \in \Gamma$, $\theta \in \Delta$.

Proof. Simple application of Corollary 7.5.

Therefore, provided that θ^* is in Δ for every $\theta \in \Delta$, then (*) is a contravariant functor on Δ. We intend to show in the next section that (*) is a duality theory for Δ. The map $\langle\!\langle\ ,\ \rangle\!\rangle$ is called the *dual pairing* on Θ. Aside from its suggestive relation to the functor (*), it also has the following familiar property:

THEOREM 7.9 ("RIESZ REPRESENTATION THEOREM"). *Let L be the set of maps $\pi\colon \Theta \to \Gamma$ such that for some $a \in \Theta_0$, the map t defined by $t\colon u \mapsto u\pi\cdot a$ is in Δ. Then for each $\pi \in L$, there is a unique $a \in \Theta$ such that $u\pi=\langle\!\langle u, a\rangle\!\rangle$ for all $u \in \Theta$.*

Proof. Define $\phi\colon \Theta\times\Theta \to \Theta$ by $(u, v)\phi=v\pi\cdot u$. Then ϕ satisfies the requirements of Theorem 6.3(ii), and hence there exists a unique $d \in \Theta^2$ such that $\phi=\phi_d$, i.e., such that $v\pi\cdot u=\{v, d\}u=\langle\!\langle v, d_+\rangle\!\rangle\cdot u$, which gives us the desired equation.

8. The duality theory on Δ.

In this section, we show that (*) is a duality theory on Δ, that Θ, Θ^1, Θ^2 are naturally equivalent with respect to the functors $(^1)$, $(^2)$, (*), and that Δ, Δ^1, Δ^2 are isomorphic as monoids.

LEMMA 8.1. *For each $\theta \in \Delta$, $\theta^1 \in \Delta^1$ and $\theta^2 \in \Delta^2$.*

Proof. We will assume the duals of Corollary 5.8; in particular, to prove the first assertion, we assume that if $a * \theta^1$ and $(\theta^1, 1, d)^1 \in A$ for some $a \in \Theta_0$, $d \in \Theta_0^2$, then $\theta^1 \in \Delta^1$. Then we need only the two following simple facts:

(i) $d \circ \theta \in A$ implies $(\theta^1, 1, d)^1 \in A$,

(ii) $(\theta, 1, a) \in A$ implies $a * \theta^1 \in A$.

For (i),

$$\begin{aligned}[(\theta^1, 1, d)^1, v, u] &= (u\theta^1, v)\psi_d = [d, v, u\theta^1] = [d, v]\cdot(\theta, 1, u)\\ &= (\theta, 1, [d, v]\cdot u) = (\theta, 1, [d, v, u])\\ &= [d\circ\theta, v, u] = (u, v)\psi_{d\circ\theta},\end{aligned}$$

and so $(\theta^1, 1, d)^1$ indeed exists. The second relation (ii) is a dual of (i). (i) and (ii) together with our initial remarks, prove the lemma.

REMARK. Therefore, if we define for $\delta\colon \Theta^1 \to \Theta^1$ and $\varepsilon\colon \Theta^2 \to \Theta^2$ the maps $\delta_1\colon \Theta \to \Theta$ and $\varepsilon_3\colon \Theta^1 \to \Theta^1$ by

$$a\delta_1 = (\delta, 1, a)^1, \qquad b\varepsilon_3 = (\varepsilon, 1, b)^2$$

then by the duals of Lemma 5.2, δ_1 and ε_3 are indeed maps on Θ and Θ^1, respectively, and by the duals of Lemma 8.1, we can therefore conclude that $(_1)$, $(_3)$ are maps $\Delta^1 \to \Delta$ and $\Delta^2 \to \Delta$, respectively.

LEMMA 8.2. $(_1)=(^1)^{-1}$.

Proof. The fact that $a\theta^1{}_1=(\theta^1, 1, a)^1=a\theta$ has already been shown in part (2) of the proof of Lemma 5.3. The fact that also $b\delta_1^1=b\delta$ is a dual statement, but because of the importance of this lemma, we prove it.

$$(1)\qquad \begin{aligned}[u, v, b\delta_1{}^1] &= [u, v, (\delta_1, 1, b)] = (u\delta_1, v)\phi_b\\ &= (u\delta_1, b)\cdot v = ((\delta, 1, u)^1, b)\cdot v.\end{aligned}$$

(2) But for $u, v \in \Theta^1$,

$$\begin{aligned}[(\delta, 1, a)^1, v, u] &= (u\delta, v)\psi_a = [a, v, u\delta] = (a, u\delta)\cdot v = (a\delta_1, u)\cdot v\\ &= [a\delta_1, v, u].\end{aligned}$$

So $(\delta, 1, a)^1=a\delta_1$. Hence from (1),

$$(3)\qquad ((\delta, 1, u)^1, b)\cdot v = (u\delta_1, b)\cdot v = (u, v)\phi_{b\delta}$$

and so $b\delta_1{}^1=b\delta$ for all b as required.

The following is a consequence of Lemma 8.2 and its duals, since we have that $(^1)\colon \Delta \to \Delta^1$ and $(_3)\colon \Delta^2 \to \Delta$ and $(_3)\circ(^1)\colon \Delta^2 \to \Delta^1$ is an isomorphism of monoids.

THEOREM 8.3. *Δ, Δ^1, Δ^2 are isomorphic as monoids.*

THEOREM 8.4. *For every $\theta \in \Delta$, $\theta^* \in \Delta$.*

Proof. For $\theta \in \Delta$, we can conclude that $\theta^2{}_3 \in \Delta^1$. And since $(^1)$ is 1-1 and onto by Lemma 8.2, then there is a unique $\theta' \in \Delta$ such that $(\theta')^1=\theta^2{}_3$ for each $\theta \in \Delta$. In fact, $\theta'=\theta^*$. For it may be noted that $[d\theta^2, b]=[d, b\theta^2{}_3]$ for all d, b (this is a dual of Proposition 4.6(ii)). Hence, $[d\theta^2, b]=[d, b\theta'^1]$ and so $[u_+\theta^2, b]=[u_+, b\theta'^1]$ and so $((u_+\theta^2)_+, b)=(u, b\theta'^1)$. Then $(u\theta^*, b)=(u\theta', b)$ and so at last, $u\theta^*=u\theta'$ for all $u \in \Theta$, hence $\theta'=\theta^*$.

THEOREM 8.5. *$\langle\!\langle u, v\rangle\!\rangle=\zeta\langle\!\langle v, u\rangle\!\rangle$ for all $u, v \in \Theta$, where ζ is an element of Γ such that $\zeta\cdot\zeta=1$.*

Proof. (1) Define $a^\sim=a_+{}^{-+}$, $d^\cdot=d^{+-}{}_+$ and $b^{\prime}=b^+{}_+{}^-$. Then by repeated application of Proposition 7.4, we get

$$\begin{aligned}[d, a^{\sim\sim}, b] &= [d, a^\sim{}_+{}^{-+}, b] = [b^+, a^\sim{}_+{}^-, d]^+ = [a^\sim{}_+, b^+, d^-]^{-+}\\ &= [a^\sim, d^-, b^+{}_+]^\sim = [a_+{}^{-+}, d^-, b^+{}_+]^\sim = [b^+{}_+, d^{-+}, a_+{}^-]^{+\sim}\\ &= [d^{-+}, b^+{}_+{}^-, a_+]^{-+\sim} = [d^{-+}{}_+, a, b^+{}_+{}^-]^{\sim\sim}\\ &= [d^\cdot, a, b^{\prime}]^{\sim\sim}.\end{aligned}$$

(2) However, $(u, v)\psi_{[d,a,b]}=(b\,(u, v)\psi_a)\psi_d=(u, v)\psi_a \circ \delta$ where $u\delta=(b, u)\psi_d$ for all $u \in \Theta^1$. (Note that by a dual of Theorem 5.10, $\delta \in \Delta^1$.) But by Lemma 8.2, there is a $\theta \in \Delta$ with $\theta^1=\delta$, and so $[d, a, b]=a * \theta^2=(\theta, 1, a)$ where note that θ is independent of a. Now, it is an easy matter to prove that if x_+, y^-, z^+ exist, then so do $(\theta, 1, x)_+$, $(\theta, 1, y)^-$ and $(1, \theta, z)^+$ for any $\theta \in \Delta$, and that they are equal, respectively, to $(\theta, 1, x_+)$, $(1, \theta, y^-)$ and $(1, \theta, z^+)$. It follows therefore that $(\theta, 1, a)^{\sim\sim}=(\theta, 1, a^{\sim\sim})$, and hence that from (1), $[d^{\cdot}, a, b']^{\sim\sim}=[d, a^{\sim\sim}, b]=[d, a, b]^{\sim\sim}$ and then since $(^{\sim})$ is 1-1, we get $[d^{\cdot}, a, b']=[d, a, b]$.

(3) $[a', a_0, [d, a!, b]]=[a', a_0, [d^{\cdot}, a!, b']]$ and so

$$[[a', a_0, d], a, b] = [[a', a_0, d^{\cdot}], a, b'].$$

Let $\{a_0, d\}=1$, then we get $[a', a, b]=\zeta\cdot[a', a, b']$ where $\zeta=\{a_0, d^{\cdot}\}$. This last equation is for all a', a, and so applying TA-4 we get at last, $b=\zeta b'$. Therefore, applying the definition of $(')$, $b=\zeta b^{+}{}_{+}{}^{-}$ and so $b^-=\zeta b^{+}{}_{+}$ and so substituting $b \mapsto a^+$ then $a^{+-}=\zeta a^{++}{}_{+}=\zeta a_+$.

(4) So $\langle\!\langle u, v\rangle\!\rangle=\{u, v_+\}=[v_+, u^+]=(v, u^+)=\{v, u^{+-}\}=\zeta\{v, u_+\}=\zeta\langle\!\langle v, u\rangle\!\rangle$. And since $\langle\!\langle u, v\rangle\!\rangle=\zeta\langle\!\langle v, u\rangle\!\rangle=\zeta\cdot\zeta\langle\!\langle u, v\rangle\!\rangle$ for all u, v, it follows that $\zeta\cdot\zeta=1$.

PROPOSITION 8.6. *$\theta^{**}=\theta$ for all $\theta \in \Delta$.*

Proof. Trivial.

PROPOSITION 8.7 (NATURAL EQUIVALENCE OF Θ, Θ^1, Θ^2). *Remember the maps $(^+)$: $\Theta \to \Theta^1$, $(_+)$: $\Theta \to \Theta^2$, and $(^-)$: $\Theta^1 \to \Theta^2$, i.e., the ions. Then $(^+)$, $(_+)$, $(^-)$ are natural bijections with respect to the functors $(*)$, $(^1)$, $(^2)$:*

(i) $\theta^* \circ (^+)=(^+) \circ \theta^1$,

(ii) $\theta^* \circ (_+)=(_+) \circ \theta^2$,

(iii) $\theta^1 \circ (^-)=(^-) \circ \theta^2$.

Proof. (ii) follows from Theorem 8.4 and the definition of $(*)$. For (i), we have $(a\theta, b)=(a, b\theta^1)=\{a, (b\theta^1)^-\}$ on the one hand, and $(a\theta, b)=\{a\theta, b^-\}=\{a, (b^-)\theta^2\}$ on the other. For (i), use the fact that $(u, v^+)=\{u, v^{+-}\}=\langle\!\langle u, v^{+-}{}_{+}\rangle\!\rangle=\zeta\langle\!\langle u, v\rangle\!\rangle$ (from the proof of Theorem 8.5) $=\langle\!\langle v, u\rangle\!\rangle$.

9. **Characterization of Θ.** In this section we give a characterization for $\langle\!\langle\ ,\ \rangle\!\rangle$ and the elements of Θ, give the final version of the necessary and sufficient condition for a map to be in Δ, state the "interior action laws," and show the existence of a^-, b_+, d^+.

LEMMA 9.1. *If $\theta \in \Delta$, then for all $d \in \Theta^2$ and $x, y \in A$,*

$$[d\theta^2, x, y] = [d, (\theta^2, 1, x)^2, y].$$

Proof. Let $E=(\theta^2, 1, x)^2$ and $\theta I=\theta^2$ for all $\theta \in \Delta$. Then

$$(u, v)\eta_{[d,E,y]} = (d, (u, v)\eta_y)\eta_E = (d\theta^2, (u, v)\eta_y)\eta_x = (u, v)\eta_{[dI,x,y]}.$$

LEMMA 9.2. $(\theta^2, 1, x)^2 = (\theta^*, 1, x)$ *for all* $\theta \in \Delta$, $x \in A$.

Proof. The proof is essentially analogous to the proof of Lemma 5.1.

(1) $(\theta^*, 1, b) = (\theta^2, 1, b)^2$ for all $b \in \Theta^1$. For $(\theta^*, 1, b) = b\theta^{*1} = b\theta^2{}_3$ (cf. Theorem 8.4), hence for $u \in \Theta^2$,

$$u\cdot[d, b\theta^2{}_3] = u\cdot[d\theta^2, b] = [d\theta^2, b, u] = [(1, \theta, d), b, u] = [d, (\theta^2, 1, b)^2, u]$$

(by Lemma 9.1), hence $[d, b\theta^2{}_3, u] = [d, (\theta^2, 1, b)^2, u]$ for all $d, u \in \Theta^2$ and so $b\theta^2{}_3 = (\theta^2, 1, b)^2$.

(2) $(1, \theta^{*1}, d)^1 = d\theta^2$ for all d; $\theta \in \Delta$. For

$$\begin{aligned}[(1, \theta^{*1}, d)^1, v, u] &= (u, v\theta^{*1})\psi_d = [d, v\theta^{*1}, u] = [d, (\theta^*, 1, v), u] \\ &= [d, (\theta^2, 1, v)^2, u] \text{ (by (1))} = [(1, \theta, d), v, u] \\ &= [d\theta^2, v, u] = (u, v)\psi_{d\theta J},\end{aligned}$$

for $\theta J = \theta^2$ for all $\theta \in \Delta$.

(3) Again for notational convenience, set $\theta K = \theta^{*1}$ for all $\theta \in \Delta$. $[d, x * \theta K, b] = [d\theta^2, x, b]$ for all d, b, θ, x is what we want to establish. But letting $F = (1, \theta^{*1}, d)^1$, we have

$$\begin{aligned}(u, v)\psi_{[d, x * \theta K, b]} &= (b, (u, v)\psi_{x*\theta K})\psi_d = (b, (u, v)\psi_x \circ \theta K)\psi_d = (b, (u, v)\psi_x)\psi_F \\ &= (u, v)\psi_{[F, x, b]} = (u, v)\psi_{[d(\theta I), x, b]} \quad \text{(by (2))}.\end{aligned}$$

(4) From (3), we obtain $[u, [d!, x * \theta^{*1}, b], v] = [u, [d\theta^2!, x, b], v]$ and so $[d, x * \theta^{*1}, [u, b, v]] = [d\theta^2, x, [u, b, v]]$. Let $u, v \in \Theta^2$ and let $(u, b) = 1$. Then we have $[d, x * \theta^{*1}, v] = [d\theta^2, x, v]$ for all $d, v \in \Theta^2$, hence $(d, v)\eta_{x*\theta K} = (d\theta^2, v)\eta_x = (d, v)\eta_E$ where $E = (\theta^2, 1, x)^2$. But as is already known (Lemma 5.3) $x * \theta^{*1} = (\theta^*, 1, x)$. This completes the proof.

It will prove useful to give in a tabular form the relations between the various isotopes.

PROPOSITION 9.3.

(i) $\quad (\theta^2, 1, x)^2 = (\theta^*, 1, x), \quad (1, \theta^2, x)^2 = x \circ \theta, \quad x \# \theta^2 = (1, \theta, x).$

(ii) $\quad (\theta^1, 1, x)^1 = x \circ \theta, \quad (1, \theta^1, x)^1 = (1, \theta^*, x), \quad x * \theta^1 = (\theta, 1, x).$

Proof. The relations for $x \# \theta^2$, $x * \theta^1$ are from Lemmas 5.3 and 5.6. The relation for $(\theta^2, 1, x)^2$ is from Lemma 9.2 and the relation for $(1, \theta^1, x)^1$ is a dual statement of it. The relations for $(\theta^1, 1, x)^1$ and $(1, \theta^2, x)^2$ are duals of Lemmas 5.3 and 5.6.

THEOREM 9.4. *If* $a \in \Theta$, *then for all* $\theta \in \Delta$, $(\theta^*, 1, a) = (1, \theta, a)$.

Proof. For $u, v \in \Theta^2$,

$$\begin{aligned}[u, (\theta^2, 1, a)^2, v] &= (u\theta^2, v)\eta_a = [u\theta^2, a, v] = u\theta^2\cdot\{a, v\} = (u\cdot\{a, v\})\theta^2 \\ &= [u, a, v]\theta^2 = [u, a, v] \# \theta^2 = [u, a \# \theta^2, v]\end{aligned}$$

and hence $(\theta^2, 1, a)^2 = a \# \theta^2$. But by Proposition 9.3, we immediately get from this that $(\theta^*, 1, a) = (1, \theta, a)$.

THEOREM 9.5 (CHARACTERIZATION OF $\langle\!\langle\ ,\ \rangle\!\rangle$). *Suppose that* $\phi\colon \Theta\times\Theta\to\Theta$ *is such that* $(u\theta, v)\phi=(u, v\theta^*)\phi$ *for all* $u, v\in\Theta$ *and* $\theta\in\Delta$. *Then there exists an* $a\in\Theta$ *such that* $(u, v)\phi=\langle\!\langle u, v\rangle\!\rangle\cdot a$ *for all* $u, v\in\Theta$.

Proof. Define $v\theta_u=\{v, d\}\cdot u$ for $d\in\Theta_0^2$ and $\{a, d\}=1$ for some a. Then of course, $a\theta_u=u$. Now, for $w\in\Theta$, $w(\theta_u)^*=\zeta d_+\cdot\langle\!\langle w, u\rangle\!\rangle$ for all w. To see this, note that $\langle\!\langle w(\theta_u)^*, v\rangle\!\rangle=\langle\!\langle w, v\theta_u\rangle\!\rangle=\{v, d\}\langle\!\langle w, u\rangle\!\rangle=\langle\!\langle v, d_+\rangle\!\rangle\langle\!\langle w, u\rangle\!\rangle=\zeta\langle\!\langle d_+, v\rangle\!\rangle\langle\!\langle w, u\rangle\!\rangle$ (by Theorem 8.5) and so we get the desired equality.

To complete the theorem, note that on the one hand, $(w(\theta_u)^*, a)\phi=(w, a\theta_u)\phi=(w, a\theta_u)\phi=(w, u)\phi$, whereas on the other hand, we also get $(w(\theta_u)^*, a)\phi=\zeta\langle\!\langle w, u\rangle\!\rangle(d_+, a)\phi$ and so setting $a_0=\zeta(d_+, a)\phi$, we get $(u, v)\phi=\langle\!\langle u, v\rangle\!\rangle\cdot a_0$ as required.

COROLLARY 9.6. *There exists* $\mu\in\Gamma_0$ *such that* $\langle u, v\rangle=\mu\cdot\langle\!\langle u, v\rangle\!\rangle$ *for all* $u, v\in\Theta$.

Proof. By Theorem 9.4 and Theorem 9.5, we know that $\langle u, v\rangle a=\langle\!\langle u, v\rangle\!\rangle a_0$ where $a_0=\zeta\langle d_+, a'\rangle a$ for some d, a' and arbitrary a. Hence let $a\in\Theta_0$ and set $\mu=\zeta\langle d_+, a'\rangle$ and we get the desired equality. To show that $\mu\in\Gamma_0$, note that $a\mu=a'\mu$ hence $\langle\!\langle u, v\rangle\!\rangle\cdot a\cdot\mu=\langle\!\langle u, v\rangle\!\rangle\cdot a'\cdot\mu$ and so $\langle u, v\rangle a=\langle u, v\rangle a'$, hence $(u, v)\phi_a=(u, v)\phi_{a'}$ and so $a=a'$.

COROLLARY 9.7. $\langle u, v\rangle=\zeta\langle v, u\rangle$ *for all* $u, v\in\Theta$.

COROLLARY 9.8. *For all* $a\in\Theta$, $b\in\Theta^1$, $d\in\Theta^2$, *then* a^-, b_+, *and* d^+ *exist and are equal to, respectively,* ζa, ζb, *and* ζd.

Proof. That $a^-=\zeta a$ follows from Corollary 9.7. Dually, it is true that d^+ exists and $d^+=\zeta' d$ where $d_+{}^{+-}=\zeta' d$. But from $a_+{}^{-+}=\zeta a$, substituting $a=d_+$, we get $d^{-+}=\zeta d_+$ and $d\zeta=d_+{}^{+-}$ and so $\zeta=\zeta'$. Similarly for the remaining case.

THEOREM 9.9 (CHARACTERIZATION OF Θ). *If* $x\in A$ *is such that* $(\theta^*, 1, x)=(1, \theta, x)$ *for all* $\theta\in\Delta$, *then* $x\in\Theta$.

Proof. It follows from Theorem 9.5 that $(u, v)\phi_x=\langle\!\langle u, v\rangle\!\rangle a$ for some $a\in\Theta$, for all $u, v\in\Theta$. Therefore, $x\cdot\mu=a$. Then it may be shown that $x\in\Theta$ by using Definition 1.3 to show that x satisfies the proper partial associativity laws, using of course the fact that μ is 1-1.

REMARK. Note that if we knew that μ has an inverse, then Theorem 9.5 would be a characterization theorem for 3-atoms analogous to Theorem 6.3. In §10 we show that indeed μ has an inverse.

THEOREM 9.10 (RESTRICTED INTERIOR ACTION LAWS). *Suppose that at least one of the following six relations is true for the triple* (x, y, z), $x, y, z\in A : x\in\Theta$, $y\in\Theta$, $y\in\Theta^1$, $z\in\Theta^1$, $z\in\Theta^2$, $x\in\Theta^2$. *Then for all* $\theta\in\Delta$,

$$\begin{aligned}
[x\circ\theta, y, z] &= [x, y, (\theta, 1, z)],\\
[x, y\circ\theta, z] &= [x, y, (1, \theta, z)],\\
[(1, \theta, x), y, z] &= [x, (\theta^*, 1, y), z].
\end{aligned}$$

Proof. Except for the cases $x, y \in \Theta$, the proofs involve easy calculations using the equalities of Proposition 9.3. On the other hand, suppose $x=a$. Then

$$\begin{aligned}[d, b, [a!, (\theta^*, 1, y), z]] &= [a, [d, b, (\theta^*, 1, y)], z] = [a!, [d\theta^*, b, y], z] \\ &= [d\theta^*, b, [a, y, z]] = [d, b, (\theta^*, 1, [a, y, z])] \\ &= [d, b, [(1, \theta, a), y, z]]\end{aligned}$$

and so if $[d, b]=1$ then we get the desired result.

THEOREM 9.11. (i) *If x^- exists then so do $(\theta, 1, x)^-$, $(1, \theta, x)^-$ and $(x \circ \theta)^-$ for all $\theta \in \Delta$, and $(\theta, 1, x)^- = (1, \theta, x^-)$, $(1, \theta, x)^- = (\theta, 1, x^-)$ and $(x \circ \theta)^- = x^- \circ \theta$.*

(ii) *If x_+ exists then so do $(\theta, 1, x)_+$, $(1, \theta, x)_+$ and $(x \circ \theta)_+$ for all $\theta \in \Delta$, and $(\theta, 1, x)_+ = (\theta, 1, x_+)$, $(1, \theta, x)_+ = x_+ \circ \theta^*$ and $(x \circ \theta)_+ = (1, \theta^*, x_+)$.*

(iii) *If x^+ exists then so do $(\theta, 1, x)^+$, $(1, \theta, x)^+$ and $(x \circ \theta)^+$, for all $\theta \in \Delta$, and $(\theta, 1, x)^+ = x^+ \circ \theta^*$, $(1, \theta, x)^+ = (1, \theta, x^+)$, and $(x \circ \theta)^+ = (\theta^*, 1, x^+)$.*

Proof. Simple calculations involving the relations of Proposition 9.3.

THEOREM 9.12 (NECESSARY AND SUFFICIENT CONDITION FOR Δ). *Suppose that for θ: $\Theta \to \Theta$ at least one of the following conditions holds*:

(i) *$(\theta, 1, a)$ exists for some $a \in \Theta_0$.*

(ii) *$d \circ \theta$ exists for some $d \in \Theta_0^2$.*

(iii) *$b \circ \theta$ exists for some $b \in \Theta_0^1$.*

Then $\theta \in \Delta$.

Proof. We use Theorem 9.11 and Theorem 5.7. Note that because of Corollary 7.5, the maps $(^+)$, $(_+)$, $(^-)$ preserve idents which are also atoms; e.g., if $a \in \Theta_0$ then $a_+ \in \Theta_0^2$ and $a^+ \in \Theta_0^1$. Therefore, suppose that $(\theta, 1, a)$ exists for some $a \in \Theta_0$. Then by Theorem 9.11, $(\theta, 1, a_+)$ exists for $a_+ \in \Theta_0^2$, and hence $a_+ \circ \theta$ exists (Lemma 4.7). So by Corollary 5.8, $\theta \in \Delta$. (ii) and (iii) are dual statements.

10. **Characterization of the ternary product.**

In this section we give a characterization of the isotopes and the ternary product, a generalization of the center lemma, and we show that $\mu = \zeta$.

LEMMA 10.1. *For appropriate $\theta \in \Delta$,*

(i) $[x, a, y] = x \circ \theta$, $[a, x, y] = x \circ \theta$;

(ii) $[x, y, b] = (\theta, 1, y)$, $[x, b, y] = (\theta, 1, y)$;

(iii) $[d, x, y] = (1, \theta, y)$, $[x, y, d] = (1, \theta, x)$.

We are *not* claiming that the θ in the six cases are the same maps, but rather, that for each case there exists a $\theta \in \Delta$ so the given equality holds.

Proof. Easy consequences of Proposition 1.8 and Theorem 5.10 and its duals.

THEOREM 10.2 (CHARACTERIZATION OF ISOTOPES). (i) *Suppose F: $A \to A$ is such that $(\tau, 1, x \circ \pi)F = (\tau, 1, xF) \circ \pi$ for all $\tau, \pi \in \Delta$. Then there exists a $\theta \in \Delta$ such that $xF = (1, \theta, x)$ for all $x \in A$.*

(ii) *Suppose* $F: A \to A$ *is such that* $(1, \tau, x \circ \pi)F = (1, \tau, xF) \circ \pi$ *for all* $\tau, \pi \in \Delta$. *Then there exists a* $\theta \in \Delta$ *such that* $xF = (\theta, 1, x)$ *for all* $x \in A$.

(iii) *Suppose* $F: A \to A$ *is such that* $(\tau, \pi, x)F = (\tau, \pi, xF)$ *for all* $\tau, \pi \in \Delta$. *Then there exists a* $\theta \in \Delta$ *such that* $xF = x \circ \theta$ *for all* $x \in A$.

Proof. We show (i). The proofs for (ii) and (iii) follow in a similar manner.

(1) By the lemma, $[x, a, y]F = [xF, a, y]$ and $[x, y, b]F = [x, yF, b]$.

(2) Since $(\theta, 1, dF) = (dF) \circ \theta$ for all $\theta \in \Delta$, it follows from Theorem 6.3 that $dF \in \Theta^2$ for all d.

(3) $[dF, a, b] = [d, a, b]F = [d, aF, b]$ by (1). Hence for all $u, v \in \Theta^1$,

$$\begin{aligned}(v, u)\psi_{[dF,a,b]} &= [[dF, a, b!], u, v] = [dF, [a, u, v], b] = (a, v)\cdot[dF, u, b] \\ &= (a, v)\cdot[dF, u]\cdot b.\end{aligned}$$

On the other hand,

$$\begin{aligned}(v, u)\psi_{[d,aF,b]} &= [[d, aF, b!], u, v] = [d, [aF, u, v], b] = [d, (v, u)\psi_{aF}, b] \\ &= [d, (v, u)\psi_{aF}]\cdot b.\end{aligned}$$

Let $(a, v) = 1$. Then if $u\delta = (v, u)\psi_{aF}$ for all $u \in \Theta^1$, then $\delta \in \Delta^1$ and hence $b\delta = (\theta^*, 1, b)$ for some $\theta^* \in \Delta$. Hence, $[d, (v, u)\psi_{aF}] = [d, u\delta] = [d, (\theta^*, 1, u)] = [(1, \theta, d), u]$ for all $u \in \Theta^1$. But $[dF, u] = [d, (v, u)\psi_{aF}]$ for all u, hence $dF = (1, \theta, d)$ for all d. Therefore, returning to the equation $(a, v)\cdot[dF, u] = [d, (v, u)\psi_{aF}]$ we get $[dF, u] = [d, (\theta^*, 1, u)]$ for all d. Hence $(v, u)\psi_{aF} = (a, v)\cdot(\theta^*, 1, u) = [a, (\theta^*, 1, u), v] = [(1, \theta, a), u, v]$ for all $u, v \in \Theta^1$. This gives at last, $aF = (1, \theta, a)$ for all a.

(4) $[xF, a, b] = [x, a, b]F = [x, aF, b] = [x, (1, \theta, a), b] = [x, (\theta^*, 1, a), b] = [(1, \theta, x), a, b]$ for all a, b. This gives $xF = (1, \theta, x)$ for all $x \in A$, as required.

COROLLARY 10.3 (GENERALIZED CENTER LEMMA). *Suppose that* $F: A \to A$ *is a function such that* $(\tau, \pi, x \circ \sigma)F = (\tau, \pi, xF) \circ \sigma$ *for all* $\pi, \tau, \sigma \in \Delta$. *Then there exists a* $\lambda \in \Gamma$ *such that* $xF = \lambda \cdot x$ *for all* $x \in A$.

THEOREM 10.4 (CHARACTERIZATION OF THE TERNARY PRODUCT). *Assume that we are given a trace algebra* A *with ternary product* $[\ ,\ ,\ ]$. *Let* $\phi: (x, y, z) \mapsto \{x, y, z\}$ *be another map* $A \times A \times A \to A$ *which satisfies the exterior action laws, the restricted interior action laws, and which has coident pairs of atoms. That is,*

(i) *for all* $\pi, \tau, \sigma \in \Delta$, *and* $x, y, z \in A$,

$$(\pi, \tau, \{x, y, z\} \circ \sigma) = \{(\pi, 1, x), (1, \tau, y), z \circ \sigma\};$$

(ii) *whenever* $x \in \Theta$, $y \in \Theta$, $y \in \Theta^1$, $z \in \Theta^1$, $z \in \Theta^2$ *or* $x \in \Theta^2$, *then for all* $\pi, \tau, \sigma \in \Delta$,

$$\{(1, \pi, x), y \circ \tau, (\sigma, 1, z)\} = \{x \circ \sigma, (\pi^*, 1, y), (1, \tau, z)\};$$

(iii) *if as usual* a, b, d *denote atoms for* $[\ ,\ ,\ ]$,

$$\exists\, a, b\colon\ \{a, *, b\}, \qquad \exists\, a, d\colon\ \{*, a, d\}, \qquad \exists\, d, b\colon\ \{d, b, *\}.$$

Then, the set A endowed with the product $\{\ ,\ ,\ \}$ *is also a trace algebra, and further, if* $\tilde{\Theta}^i$ *denotes the set of i-atoms of* $\{\ ,\ ,\ \}$, $\tilde{\Gamma}$ *the induced semigroup and* $\tilde{\Delta}$ *the comonoid, then*

(1) $\tilde{\Theta}^i=\Theta^i$, $\tilde{\Gamma}=\Gamma$, $\tilde{\Delta}=\Delta$.

(2) *If* $x\in\Theta$, $y\in\Theta$, $y\in\Theta^1$, $z\in\Theta^1$, $z\in\Theta^2$ *or* $x\in\Theta^2$, *then* $\{x,y,z\}=\lambda\cdot[x,y,z]$ *where* λ *is a fixed invertible element of* Γ.

Proof. (a) We show that $\tilde{\Theta}^i=\Theta^i$. Let $a\in\Theta$, and define $xF=\{x,a,y\}$ for fixed a,y. Then

$$\begin{aligned}(\pi,\tau,x)F &= \{(\pi,\tau,x),a,y\} = (\pi,1,\{x,(\tau^*,1,a),y\})\\ &= (\pi,1,\{x,(1,\tau,a),y\}) = (\pi,\tau,\{x,a,y\}) = (\pi,\tau,xF).\end{aligned}$$

So by Theorem 10.2, $\{x,a,y\}=x\circ\theta$ for some $\theta\in\Delta$ which depends on a and y. Similarly, we may deduce equalities of the form $\{a,x,y\}=x\circ\theta$, $\{x,y,b\}=(\theta,1,y)$, etc. analogous to those of Lemma 10.1. This means that, for example, $\{\{x,y,z\},a,w\}=\{x,y,z\}\circ\theta=\{x,y,z\circ\theta\}=\{x,y,\{z,a,w\}\}$. Similarly the other partial associativity laws may be gotten in this manner. Hence, $\tilde{\Theta}^i\supseteq\Theta^i$.

Suppose on the other hand that $\{\{x,y,z\},\bar{a},w\}=\{x,y,\{z,\bar{a},w\}\}$ for all $x,y,z,w\in A$ and some $\bar{a}\in A$. Pick $y\in\Theta^2$, $w\in\Theta^1$, and then we get

$$\begin{aligned}\{x,y,\{z,(1,\theta,\bar{a}),w\}\} &= \{x,y,(1,\theta,\{z,\bar{a},w\})\} = \{x,y\circ\theta,\{z,\bar{a},w\}\}\\ &= \{\{x,y\circ\theta,z\},\bar{a},w\} = \{\{x,y,(1,\theta,z)\},\bar{a},w\}\\ &= \{x,y,\{(1,\theta,z),\bar{a},w\}\} = \{x,y,\{z,(\theta^*,1,\bar{a}),w\}\}.\end{aligned}$$

Then using (iii) we get $(\theta^*,1,\bar{a})=(1,\theta,\bar{a})$ for all $\theta\in\Delta$, and hence $\bar{a}\in\Theta$. Hence $\tilde{\Theta}^3=\Theta$. Similarly it may be shown that the other two cases follow.

(b) TA-3. For example,

$$\begin{aligned}(\theta,1,\{a,a',x\}) &= \{(\theta,1,a),a',x\} = \{(1,\theta^*,a),a',x\} = \{a,(\theta,1,a'),x\}\\ &= \{a,(1,\theta^*,a'),x\} = (1,\theta^*,\{a,a',x\})\end{aligned}$$

for all $\theta\in\Delta$, and hence $\{a,a',x\}\in\Theta$.

(c) TA-2. Define $xF=\{d,b,x\}$ for fixed d,b. Then trivially we get that $(\pi,\tau,x\circ\sigma)F=(\pi,\tau,xF)\circ\sigma$ for all $\pi,\tau,\sigma\in\Delta$, hence $xF=\lambda\cdot x$ for some $\lambda\in\Gamma$, and so in particular, $\{d,b,a\}\in\Theta$ for all d,b,a. The other cases are the same.

(d) TA-1. As in (c), we find that the maps $x\mapsto\{d,b,x\}$, $x\mapsto\{a,x,b\}$, $x\mapsto\{x,a,d\}$ are maps in Γ. Define $(u,v)\phi\cdot x=\{x,u,v_+\}$ for all $u,v\in\Theta$, so that $\phi\colon\Theta\times\Theta\to\Gamma$. Then

$$\begin{aligned}(u\theta,v)\phi\cdot x &= \{x,u\theta,v_+\} = \{x,u,(\theta,1,v_+)\} = \{x,u,v_+\theta^2\}\\ &= \{x,u,(v\theta^*)_+\} = (u,v\theta^*)\phi\cdot x\end{aligned}$$

for all $u,v\in\Theta$, $x\in A$, $\theta\in\Delta$. Hence $(u,v)\phi=\lambda\cdot\langle\!\langle u,v\rangle\!\rangle$ for all u,v, for some $\lambda\in\Gamma$. Because of (i), λ is invertible. So we have that $\{x,a,d\}=\lambda[x,a,d]$. Similarly we get that $\{d,b,x\}=\lambda'\cdot[d,b,x]$ and $\{a,x,b\}=\lambda''\cdot[a,x,b]$ for some invertible $\lambda',\lambda''\in\Gamma$. Hence suppose that a is such that there exist d,b such that $[a,*,b]$ and $[*,a,d]$.

Then immediately we get that $\{a, x, b\cdot(\lambda'')^{-1}\}=x$ for all x and $\{x, a, d\cdot(\lambda)^{-1}\}=x$ for all x. This establishes one of the three parts of TA-1; the others are similar. It follows then that $\tilde{\Gamma}=\Gamma$.

(e) TA-4. Suppose that $\{x, b_1, b_2\}=\{x', b_1, b_2\}$ for all b_1, b_2. If we define $\theta_b\colon u\mapsto(u, a)\phi_b$, for some b, it follows that $\{x, b_1, b_2(\theta_b)^1\}=\{x', b_1, b_2(\theta_b)^1\}$ for all b_1, b_2, b. But

$$\begin{aligned}\{[x, a, b], b_1, b_2\} &= \{x\circ\theta_b, b_1, b_2\} = \{x, b_1, (\theta_b, 1, b_2)\} = \{x, b_1, b_2(\theta_b)^1\}\\ &= \{x', b_1, b_2(\theta_b)^1\} = \{x', b_1, (\theta_b, 1, b_2)\} = \{x'\circ\theta_b, b_1, b_2\}\\ &= \{[x', a, b], b_1, b_2\}.\end{aligned}$$

On the other hand, since $[x, y, b]=(\theta, 1, y)$ for some θ, for all y, and each fixed x, b, it follows that

$$\begin{aligned}\{[x, a, b], b_1, b_2\} &= \{(\theta, 1, a), b_1, b_2\} = (\theta, 1, \{a, b_1, b_2\}) = [x, \{a, b_1, b_2\}, b]\\ &= \lambda''\cdot(a, b_2)\cdot[x, b_1, b].\end{aligned}$$

Similarly,

$$\begin{aligned}\{[x', a, b], b_1, b_2\} &= \{(\theta', 1, a), b_1, b_2\} = (\theta', 1, \{a, b_1, b_2\}) = [x', \{a, b_1, b_2\}, b]\\ &= \lambda''\cdot(a, b_2)\cdot[x', b_1, b].\end{aligned}$$

Hence $\lambda''\cdot(a, b_2)\cdot[x, b_1, b]=\lambda''\cdot(a, b_2)\cdot[x', b_1, b]$ for all b_1, b_2, b. So since λ'' is invertible and since we can pick $(a, b_2)=1$, we apply TA-4 for $[\ ,\ ,\ ]$ to get $x=x'$. The other cases are proved similarly.

(f) From (e) we found that $\{x, b_1, b_2(\theta_b)^1\}=\lambda\cdot(a, b_2)\cdot[x, b_1, b]$. But $b_2(\theta_b)^1=(a, b_2)\cdot b$ as may easily be checked. Hence, $(a, b_2)\cdot\{x, b_1, b\}=\lambda''\cdot(a, b_2)\cdot[x, b_1, b]$ and so $\{x, b_1, b\}=\lambda''\cdot[x, b_1, b]$ for all b_1, b. From this it follows that the $(^+)$-ions with respect to $\{\ ,\ ,\ \}$ and with respect to $[\ ,\ ,\ ]$ coincide. Similarly, the other two types of ions coincide for these two products. From this, using Corollary 7.5, it follows that $\lambda=\lambda'=\lambda''$.

However, this now gives us $\{x, b_1, b\}=\lambda\cdot[x, b_1, b]$. From this immediately follows that $\{x, y, b\}=\lambda\cdot[x, y, b]$ for all x, y, b. The other relations of equation (2) above are established in a similar manner. This completes the theorem.

LEMMA 10.5. *For $\pi\in\Delta$, there exist $\theta_1, \theta_2, \theta_3\in\Delta$ such that*

$$\begin{aligned}[x, d\circ\pi, y] &= d\circ\theta_1 && \textit{for all } d,\\ [b\circ\pi, x, y] &= b\circ\theta_2 && \textit{for all } b,\\ [x, y, (1, \pi, a)] &= (1, \theta_3, a) && \textit{for all } a,\end{aligned}$$

where $x, y\in A$ are fixed.

Proof. Define $(u, v)\phi=(a_0, u)\phi_{[x,(v'\circ\pi),y]}$ for all $u, v\in\Theta$ and fixed $a_0\in\Theta$, and where for notational convenience we set $v'=v_+$. Then if $E=[x, v_+\circ\pi, y]$ then $(u\theta, v)\phi=(a_0, u\theta)\phi_E=(a_0, u)\phi_{(1,\theta,E)}$ and then

$$(1, \theta, E) = [x, (1, \theta, v_+)\circ\pi, y] = [x, (v_+\theta^2)\circ\pi, y] = [x, (v\theta^*)_+\circ\pi, y]$$

which then implies that $(a_0, u)\phi_{(1,\theta,E)}=(u, v\theta^*)\phi$. Hence by Theorem 9.5, there exists $a_{x,y,w,\pi}\in\Theta$ such that $(w, u)\phi_{[x,d\circ\pi,y]}=\langle\!\langle u, d_+\rangle\!\rangle\cdot a_{x,y,w,\pi}$ for all $w, u\in\Theta$. Define $\theta'\in\Delta$ by $w\mapsto a_{x,y,w,\pi}$. It follows from the above equation that indeed θ' is in Δ, and hence we get $(w, u)\phi_{[x,d\circ\pi,y]}=(w, u)\phi_d\circ\theta'$ or that $[x, d\circ\pi, y]=d\circ\theta'$ as required. The other cases are similar.

LEMMA 10.6. *Let A_Δ be the set of all elements $(\pi, \tau, x\circ\sigma)$ in A such that x is an atom and π, τ, σ are in Δ. Then A_Δ is a trace algebra under the ternary product in A such that the atoms of A_Δ are exactly the atoms of A.*

Proof. It is easily seen that all that needs to be shown is that A_Δ is closed under the ternary product (cf. Proposition 12.1). But this follows immediately from Lemmas 10.1 and 10.5.

THEOREM 10.7. $\mu=\zeta$.

Proof. Consider the product on $A_\Delta\phi$ (i.e., the set of all ϕ_x with $x\in A_\Delta$) defined by $[\phi_x, \phi_y, \phi_z]=\phi_{[x,y,z]}$. Then $A_\Delta\phi$ endowed with this product becomes a trace algebra. Now, if ϕ^-, ϕ^+, ϕ_+ denote the functions $\Theta\times\Theta\to\Theta$ defined by $(u, v)\phi^-=(v, u)\phi$, $\langle(u, v)\phi, w\rangle=\langle u, (w, v)\phi^+\rangle=\langle v, (u, w)\phi_+\rangle$ for $\phi\in A_\Delta\phi$, it is easily seen that $\phi^-, \phi^+, \phi_+\in A_\Delta\phi$, and that these maps $(^-)$, $(^+)$, $(_+)$ obey isotopic action laws analogous to those presented for the ions in Theorem 9.11. Therefore, define $F\colon A_\Delta\phi\to A_\Delta\phi$ by $\phi_xF=(\phi_y)^+$ where $y=x^+$. It easily follows that $(\pi, \tau, \phi_x\circ\sigma)F=(\pi, \tau, \phi_xF)\circ\sigma$ for all $\pi, \tau, \sigma\in\Delta$ and hence by the generalized center lemma (Corollary 10.3), $\phi_y=\lambda\cdot(\phi_x)^+$ for some $\lambda\in\Gamma$. But note that for $x\in\Theta^2$, $\phi_y=\zeta\phi_x$ and $(\phi_x)^+=\phi_x$. It follows therefore that $\lambda=\zeta$. But on the other hand, note that for $x\in\Theta$, we have $(\phi_x)^+=\mu\cdot\phi_y$, and hence $\mu=\zeta$.

COROLLARY 10.8. $\Gamma'=\Gamma$.

11. **Trace, summation, and tensor product.** In this section we introduce the "trace" of elements in A and in Δ, as promised. The trace of an element of Δ is, in the case of example (2), the usual trace of an endomorphism as encountered in linear algebra. We then use the trace to define a "summation operator" which obeys some of the familiar properties associated with summation in the theory of vector spaces and Hilbert spaces. Finally, we introduce a "tensor product" which also obeys the familiar properties.

DEFINITION 11.1. Given $x\in A$, suppose that there is an $x'\in A$ such that $[b, d, x]=[d, x', b]$ for all d, b. If x' exists then by Proposition 11.3 below it is uniquely determined by x, and we then call it the *trace* of x and denote it by xT.

PROPOSITION 11.2. *For $x_1, x_2\in A$, if $[d, x_1, b]=[d, x_2, b]$ for all d, b then $x_1=x_2$.*

Proof. Trivial.

PROPOSITION 11.3. *xT is uniquely determined by x.*

Proof. Proposition 11.2.

THEOREM 11.4 (EXISTENCE OF THE TRACE). *For every* $x \in A$, xT *exists and is in* Θ.

Proof. As in the proof of Lemma 10.5, if we define $(u, v)\phi = (w, u)\phi_{[b,v_+,x]}$ for fixed w we find that there exist $a_{w,b,x}$ in Θ such that $(w, u)\phi_{[b,d,x]} = \{u, d\} \cdot a_{w,b,x}$. Similarly, let $(u, v)\phi' = a_{u,k,x}$ where $k = v^+$. Then

$$(w\theta, v)\phi' \cdot \{u, d\} = (w\theta, u)\phi_{[k,d,x]} = (w, u)\phi_{[j,d,x]}$$

(where $j = (v\theta^*)^+$), $= (w, v\theta^*)\phi' \cdot \{u, d\}$. Hence again, this means that there exists $(x)a \in \Theta$ such that $(w, v)\phi' = (w, v^+) \cdot (x)a$ and hence that

$$(w, u)\phi_{[b,d,x]} = (w, b) \cdot \{u, d\}(x)a = (w, u)\phi_{[d,(x)a,b]}$$

and so $[b, d, x] = [d, (x)a, b]$ for all d, b, and we are finished.

The following are easy consequences of Definition 11.1.

PROPOSITION 11.5. (i) *If* $\theta \in \Delta$ *then* $(x \circ \theta)T = (xT)\theta$.

(ii) $[a, x, y]T = [a, xT, y]$, $[x, a, y]T = [xT, a, y]$.

(iii) $[b, d, x] = [d, xT, b] = [xT, b, d]$.

REMARK. In a similar manner it is possible to show that existence of maps $Q: A \to \Theta^1$ and $P: A \to \Theta^2$ such that

$$[x, d, a] = [a, xQ, d] = [d, a, xQ],$$
$$[b, x, a] = [xP, a, b] = [a, b, xP]$$

with the properties $(1, \theta, x)P = (1, \theta, xP)$ and $(\theta, 1, x)Q = (\theta, 1, xQ)$.

PROPOSITION 11.6. *There exists* $\xi \in \Gamma$ *such that for all* d, b, $dT = \xi \cdot d_+$ *and* $bT = \xi \cdot b^+$.

Proof. (1)

$$\begin{aligned}\langle\!\langle u, v_+T\rangle\!\rangle &= w \cdot \zeta\{v_+T, u_+\} = w \cdot \{v_+T, u^{+-}\} = w \cdot (v_+T, u^+) = [v_+T, w, u^+] \\ &= [v_+, w, u^+]T = [v, u^+, w_+]_+T = [w_+, v, u^+]_+T\end{aligned}$$

(that $[a, b, d] = [d, a, b]$ for all d, a, b is the statement of Proposition 11.5(iii)) $= [w, u^+, v_+]T = [w, u^+T, v_+] = w \cdot \langle\!\langle u^+T, v\rangle\!\rangle$, hence we have shown that $\langle\!\langle u, v_+T\rangle\!\rangle = \langle\!\langle u^+T, v\rangle\!\rangle$ for all $u, v \in \Theta$.

(2) For $u \in \Theta$, $\theta \in \Delta$, $(\theta^*, 1, u^+)T = (1, \theta, u^+)T$. This follows from applying the interior action laws (Theorem 9.10) to the definition of the trace.

(3) Define $k: u \mapsto u^+T$. Then

$$\begin{aligned}u(\theta \circ k) &= (u\theta)^+T = (\theta^*, 1, u^+)T = (1, \theta, u^+)T = (u^+ \circ \theta)T \\ &= (u^+T)\theta = u(k \circ \theta),\end{aligned}$$

for all $\theta \in \Delta$. So by the center lemma, $u^+T = \xi \cdot u$ for some $\xi \in \Gamma$. Utilizing (1), it also follows that $u_+T = \xi \cdot u$.

REMARK. For the remainder we assume that we are working with a trace algebra for which ξ has an inverse, ξ^{-1}, which (by the center lemma) is necessarily in Γ. The last proposition in this section will show that this assumption is unnecessary, i.e., ξ always has an inverse, and in fact, $\xi = \zeta$.

PROPOSITION 11.7. *If* $\theta \in \Delta$, *then there exists a* (*unique*) $\lambda_\theta \in \Gamma$ *such that* $(1, \theta, u)T = \lambda_\theta \cdot u$ *for all* $u \in \Theta$.

Proof. Define $k\colon \Theta \to \Theta$ by $k\colon u \mapsto (1, \theta, u)T$. Then $u(k \circ \pi) = ((1, \theta, u)T)\pi = (1, \theta, u\pi)T = u(\pi \circ k)$ for all $\pi \in \Delta$. Hence by the center lemma, $k \in \Gamma$.

DEFINITION 11.8. We define the number $\xi^{-1} \cdot \lambda_\theta$ to be the *trace* of θ, and we denote it by θT. So $T\colon \Delta \to \Gamma$.

REMARK. We choose a useful (and highly suggestive) notation for the trace of an element in $A\phi$. Define, for $\phi_x \in A\phi$,

$$\sum_s (s, s^*)\phi_x = \xi^{-1} \cdot xT.$$

It follows therefore that

$$\sum_s \langle s, s^*\theta\rangle \cdot a = \theta T \cdot a$$

and this leads us to define, for functions in $A\phi$ of the form $(u, v) \mapsto (u, v)\phi \cdot a$ where $\phi\colon \Theta \times \Theta \to \Gamma$ and $a \in \Theta_0$, that

$$\left(\sum_s (s, s^*)\phi\right) \cdot a = \sum_s (s, s^*)\phi \cdot a.$$

$\sum_s (s, s^*)\phi$ is, then, a quantity in Γ, and is independent of the choice of $a \in \Theta_0$.

We can now prove:

THEOREM 11.9. (i) ("*Fourier expansion*".) *For all* $a \in \Theta$,

$$\sum_s \langle\!\langle s^*, a\rangle\!\rangle s = \sum_s \langle\!\langle a, s\rangle\!\rangle s^* = a.$$

(ii) ("*Linearity*".) *Let* $\theta \in \Delta$, $\phi \in A\phi$, *and* $\pi, \tau \in L$ *where* L *is as defined in Theorem* 7.9. *Then*

$$\sum_s \{(s, s^*)\phi\}\theta = \left(\sum_s (s, s^*)\phi\right)\theta,$$

$$\zeta \sum_s s\pi \cdot s^*\theta = \sum_s s^*\pi \cdot s\theta = \left(\sum_s s^*\pi \cdot s\right)\theta,$$

$$\zeta \sum_s s\pi \cdot s^*\tau = \sum_s s^*\pi \cdot s\tau = \left(\sum_s s^*\pi \cdot s\right)\tau.$$

(iii) ("*Parseval's identity*".) *For all* $u, v \in \Theta$,

$$\sum_s \langle\!\langle u, s\rangle\!\rangle \langle\!\langle s^*, v\rangle\!\rangle = \langle\!\langle u, v\rangle\!\rangle.$$

Proof. (i) $\sum_s \langle\!\langle s^*, a\rangle\!\rangle s = \sum_s (s, s^*)\phi_w$ where $w = a_+$, and so $= \xi^{-1}\cdot a_+ T = a$. Similarly, for the other case.

(ii)

$$\left(\sum_s (s, s^*)\phi_x\right)\theta = (\xi^{-1}xT)\theta = \xi^{-1}\cdot(x \circ \theta)T = \sum_s (s, s^*)\phi_x \circ \theta = \sum_s \{(s, s^*)\phi_x\}\theta. \tag{1}$$

(2) If $\pi \in L$, then $u\pi = \langle\!\langle u, a\rangle\!\rangle$ for some $a \in \Theta$, by Theorem 7.9. Hence

$$\begin{aligned}\left(\sum_s s^*\pi\cdot s\right)\theta &= a\theta = \xi^{-1}\cdot\xi\cdot a_{++}\theta = \xi^{-1}\cdot(a_+T)\theta \\ &= \xi^{-1}\cdot(a_+ \circ \theta)T = \sum_s (s, s^*)\phi_w \circ \theta \quad (\text{for } w = a_+) \\ &= \sum_s \langle\!\langle s^*, a\rangle\!\rangle s\theta = \sum_s s^*\pi\cdot s\theta.\end{aligned}$$

Similarly for the other case.

(3) This follows from the fact that the map $u \mapsto u\pi\cdot a_0$ for $a_0 \in \Theta_0$, and $\pi \in L$, is a map in Δ, hence the result follows from (2).

(iii) $\sum_s \langle\!\langle u, s\rangle\!\rangle\langle\!\langle s^*, v\rangle\!\rangle = \zeta \sum_s s^*\pi\cdot s\tau$ where $s^*\pi = \langle\!\langle s^*, v\rangle\!\rangle$ and $s\tau = \langle\!\langle s, u\rangle\!\rangle$, so $\pi, \tau \in L$, hence by case (2), $= \langle\!\langle u, v\rangle\!\rangle$.

Definition 11.10. (i) Define for each $x \in A$ the function $\Phi_x\colon \Delta \to \Theta$ by

$$(\theta)\Phi_x = \xi^{-1}(1, \theta, x)T = \sum_s (s, s^*\theta)\phi_x$$

and let the set of all Φ_x be denoted by $A\Phi$.

(ii) Define the map $u \otimes v \in \Delta$, for $u, v \in \Theta$, by

$$w(u \otimes v) = \langle\!\langle w, u\rangle\!\rangle v$$

for all $w \in \Theta$. This map is clearly in Δ, and is called the *tensor product* of u ana v, for reasons suggested by Proposition 11.11 below.

Proposition 11.11. (i) $(u \otimes v)\Phi_x = (u, v)\phi_x$ *for all* $u, v \in \Theta$, *and further*, $\Phi_x = \Phi_y$ *iff* $x = y$.

(ii) $(u \otimes v)T = \langle u, v\rangle$ *for all* $u, v \in \Theta$.

Proof. Trivial.

Theorem 11.12. $\xi = \zeta$.

Proof. (1) The trace algebra A_Δ was defined at the end of §10. In A_Δ, define a new ternary product as follows: $\{x, y, z\} = [z^+, y^+, x^+]^+$ for all $x, y, z \in A_\Delta$. It is easily verified that $\{\ ,\ ,\ \}$ satisfies the requirements of Theorem 10.4, and hence we can conclude that

$$[x, y, z]^+ = \lambda\cdot[z^+, y^+, x^+]$$

for some invertible $\lambda \in \Gamma$ and for suitable x, y, z as stipulated in Theorem 10.4. It is easily shown that $\lambda = \zeta$. Therefore, in particular, we find that

$$[b_1, d, b_2]^+ = \zeta[b_2^+, d^+, b_1^+] = [b_2^+, d, b_1^+].$$

Now, from this we find that $aQ = (a^+T)^+$. For, if $c \in \Theta$, then

$$[c, (a^+T)^+, d] = [d, a^+T, c^+]^+ = [c^+, d, a^+]^+ = [a, d, c] = [c, aQ, d]$$

for all c, d from which we get our result.

(2) By (1), we now have $aQ = (a^+T)^+ = \xi \cdot a^+$. We show that $\xi \in \Gamma_0$. For suppose $a_1Q = a_2Q$. Then $[a, a_1Q, d] = [a, a_2Q, d]$ for all a, d, and hence $[a_1, d, a] = [a_2, d, a]$ and from this it immediately follows that $a_1 = a_2$.

(3) By definition, $[a, a_0Q, d]T = [a_0, d, a]T$. The left-hand side gives

$$= [a, (a_0Q)T, d] = \xi \cdot [a, (a_0Q)^+, d].$$

The right-hand side gives

$$= [a_0, dT, a] = \xi \cdot [a_0, d_+, a]$$

and so cancelling ξ between these two equations we get $[a, (a_0Q)^+, d] = [a_0, d_+, a]$. But

$$[a, (a_0Q)^+, d] = [d, a_0Q, a^+]^+ = [d, a_0Q] \cdot a = \{(a_0Q)^+, d\} \cdot a.$$

On the other hand,

$$\begin{aligned}[a_0, d_+, a] &= \langle a_0, d_+\rangle \cdot a = \zeta \langle\!\langle a_0, d_+\rangle\!\rangle \cdot a \quad \text{(Theorem 10.7)}\\ &= \zeta\{a_0, d\} \cdot a.\end{aligned}$$

Since this is for all d, a, we get that $a_0Q = \zeta a_0^+$ and hence that $\zeta = \xi$.

12. **More examples.**

(3) *Bounded bilinear operators.* Let $\mathscr{H}$ be the set of harmonic regular functions $f\colon D \to \boldsymbol{R}$ on an open domain D of the complex plane such that the Lebesgue integral $\int_D |(z)f|^2\, dz$ is bounded. Then $\mathscr{H}$ is a Hilbert space under the symmetric inner product

$$\langle f, g\rangle = \int_D (z)f \cdot (z)g\, dz.$$

Also, $\mathscr{H}$ is a reproducing kernel space with kernel function $(w)K_z = (w, z)K$ (cf. Yosida [3] and Aronszajn [1]), i.e., $\langle f, K_z\rangle = (z)f$ for all $f \in \mathscr{H}$, $z \in D$.

Denote the set of bounded bilinear operators $\mathscr{H} \times \mathscr{H} \to \mathscr{H}$ as A. Consider the map

$$(z, w)M = (w)((f, K_z)F, (K_z, g)G)H$$

for $f, g \in \mathscr{H}$, $F, G, H \in A$, $z, w \in D$. M is certainly measureable, and so if we define

$$(w)h = \int_D (z, w)M\, dz$$

then by an application of Tonelli's theorem we get

$$(*)\qquad \int_D |(w)h|^2\,dw \leqq \int_D \|((f, K_z)F, (K_z, g)G)H\|^2\,dz$$
$$\leqq \|f\|^2\cdot\|g\|^2\cdot\|F\|^2\cdot\|G\|^2\cdot\|H\|^2\cdot N^2$$

where $0<N^2=\int_D (z, z)K^2\,dz<\infty$. Hence, h is in $\mathscr{H}$, provided that it is regular harmonic; we can differentiate under the integral sign to prove regularity and harmonicity. Therefore, define $(f, g)[F, G, H]=(1/N)\cdot h$ on D. Equation $(*)$ above then implies that $\|[F, G, H]\|\leqq\|F\|\cdot\|G\|\cdot\|H\|$ and A endowed with this ternary product is a trace algebra with the obvious atoms. The induced ring is $\boldsymbol{R}$, and the comonoid is the set of bounded endomorphisms of $\mathscr{H}$.

(4) *The* "sup" *trace algebras*. For X a set, and $S=[0, 1]$ (the closed interval in $\boldsymbol{R}$), define $\mathscr{H}$ to be the set of all maps $X\to S$. Then define $\langle\ ,\ \rangle: \mathscr{H}\times\mathscr{H}\to S$, $K: X\times X\to S$ by

$$\langle f, g\rangle = \sup\,(xf\cdot xg)$$

(the sup being taken over all x in X) and

$$\begin{aligned}(x, y)K &= 0 \qquad (x\neq y),\\ &= 1 \qquad (x=y).\end{aligned}$$

In addition, let $\mathscr{H}_2$ denote the set of all maps $X\times X\to S$, and A the set of maps $F: \mathscr{H}\times\mathscr{H}\to\mathscr{H}$ such that

(1) $(\alpha f, \beta g)F=\alpha\beta\cdot(f, g)F,\ \alpha, \beta\in S$,

(2) $\sup_x (H_x, g)F=(\sup_x H_x, g)F,\ \sup_x (f, H_x)F=(f, \sup_x H_x)F$

for all $f, g\in\mathscr{H}$, $H\in\mathscr{H}_2$ (where $yH_x=(y, x)H$). Then the map $[F, G, H]\in A$ for $F, G, H\in A$ is defined by

$$(f, g)[F, G, H] = \sup_x\,((f, K_x)F, (K_x, g)G)H.$$

Then A becomes a trace algebra with induced semigroup S.

(5) *Ternary relations*. Let X be an arbitrary set, and let $\mathscr{A}$ be the set of all subsets of $X\times X\times X$, i.e., the set of all "ternary relations." Then for $U, V, W\in\mathscr{A}$, define $[U, V, W]$ as the set of all (x, y, z) such that there are $r, s, t\in X$ with $(x, r, s)\in U$, $(r, y, t)\in V$, and $(s, t, z)\in W$. Or, equivalently,

$$[U, V, W] = \bigcup_{r,s,t} U^{*rs}\times V^{r*t}\times W^{st*}$$

where $U^{*rs}=\{x\in X : (x, r, s)\in U\}$, and so on. (The union is over all r, s, t in X.) Then, $\mathscr{A}$ endowed with this ternary product is a trace algebra. The atoms are sets of the form

$$\begin{aligned}D_A &= \{(x, x, a) : x\in X, a\in A\},\\ D_A^1 &= \{(a, x, x) : x\in X, a\in A\},\\ D_A^2 &= \{(x, a, x) : x\in X, a\in A\}\end{aligned}$$

for A a subset of X.

The induced semigroup is the multiplicative part of the field of two elements, with the action defined by $0\cdot S=\varnothing$ and $1\cdot S=S$ for any subset S of X.

For a given $U\in\mathscr{A}$, the trace UT of U is the set D_A where $A=\bigcup_s U^{ss*}$.

It is possible to show that the comonoid of $\mathscr{A}$ may be identified with the set of all relations on the set X (recall that the *composition* $R\circ S$ of relations R, S on X is the set of all (x, y) such that there is an $r\in X$ with $(x, r)\in R$ and $(r, y)\in S$). In fact, define the set function θ_R on the power set of X (for R a relation on X) by

$$A\theta_R = \{y\in X : \exists\, x\in A \text{ with } (x, y)\in R\}.$$

Then, the correspondence $R\mapsto\theta_R$ from the set of relations of X to the comonoid of $\mathscr{A}$ may be shown to be an isomorphism of monoids.

(6) *Finitely-generated projective modules.* Example (1) may be extended in the following manner: it may be shown that any finitely-generated projective R-module (R commutative with unit) may be regarded as a module of functions $m: S\to R$ for S some finite set, which possesses a symmetric R-bilinear form $\langle\ ,\ \rangle: M\times M\to R$ and a function $K: S\times S\to R$, such that

(1) $\langle m, K_s\rangle=(s)m$,

(2) $\sum_s (s)K_t\cdot K_s=K_t$ (the proof of this is relatively simple and is left to the reader), where $(s)K_t=(s, t)K$ and where the summation is over all s in S. In this case, the set of all R-bilinear functions on M become a trace algebra via the ternary product

$$(m, n)[f, g, h] = \sum_s ((m, K_s)f, (K_s, n)g)h.$$

(7) *A counterexample.* In example (1), let $\theta: M\to M$ be a nonsingular skew-symmetric map: $\theta^*=-\theta$, and define (1) an "inner product" on M by $\langle u, v\rangle=\langle u, v\theta\rangle_0$ (for $\langle\ ,\ \rangle_0$ the inner product of example (1), and (2) a "summation operator" on A by

$$\sum_s (s, s^*)f = \sum_j (e_j, e_j\theta^{-1})f.$$

Then, this makes A into a new trace algebra in such a way that $\zeta=-1$. It is also an example of a nonregular trace algebra which has a nonregular trace and is also representable (see §13).

(8) *Trace subalgebras.* A trace subalgebra A' of a trace algebra A is a subset of A which contains Θ, Θ^1, Θ^2 and which is a trace algebra under the restriction of the ternary product defined on A. It follows that the atoms of A' are exactly the atoms of A.

The following are offered without proof:

PROPOSITION 12.1. *A subset $A'\subseteq A$ is a subalgebra of A iff it contains Θ, Θ^1, and Θ^2, and is closed under the ternary product of A.*

PROPOSITION 12.2. *If $A_j\subseteq A$ is an arbitrary indexed family of subalgebras of A, then $\bigcap_j A_j$ is a subalgebra of A.*

The trace algebra A_Δ introduced in §10 is therefore a subalgebra of A and is in fact the smallest subalgebra of A which has Δ as its comonoid.

PROPOSITION 12.3. *Proposition* 12.2 *implies the existence of minimal subalgebras. One shows that the minimal subalgebra A_0 of A consists of all those elements of the form $(\pi, \tau, x \circ \sigma)$ where x is an atom, and $\pi, \tau, \sigma \in \Delta_0$, where Δ_0 is the set consisting of the elements of Γ and maps of the form $u \otimes v$ and $(u \otimes v)^*$.*

(9) *Matrix trace algebras.* For A additive, let $C_n(A)$ be the set of all $n \times n \times n$ cubic matrices with entries in A. If $X=(x_{ijk})$, $Y=(y_{ijk})$ and $Z=(z_{ijk})$ are in $C_n(A)$, then define

$$[X, Y, Z]_{ijk} = \sum_{r,s,t} [x_{irs}, y_{rjt}, z_{stk}]$$

where the summation is over $r, s, t=1, \ldots, n$. Then $C_n(A)$ endowed with this ternary product is an additive trace algebra. In fact, the 3-atoms are of the form $a_{ijk}=\delta_{ij}a_k$ ($a_k \in \Theta$), the 1-atoms are of the form $b_{ijk}=b_i\delta_{jk}$ ($b_i \in \Theta^1$) and the 2-atoms are of the form $d_{ijk}=\delta_{ik}\, d_j$ ($d_j \in \Theta^2$). If then $A=(a_{ijk})$ is a 3-atom and $B=(b_{ijk})$ is a 1-atom then $(A, B)=\sum_i (a_i, b_i)$ and hence, the induced ring of $C_n(A)$ is the induced ring of A. The comonoid of $C_n(A)$ is isomorphic to the matrix ring $M_n(\Delta)$.

Finally, if R' denotes the trivial trace algebra of the ring R, then note that $C_n(R')$ are, up to an identification, the trace algebras $\mathrm{Bilin}_R\,(M)$ (where of course R is commutative with unity), of example (2).

13. **Regular trace algebras.** In this concluding section, we determine a class of well-behaved trace algebras, prove a uniqueness theorem for them, and then prove our main theorem (Theorem 13.6) which shows that these well-behaved trace algebras can be regarded as being constructed in terms of a "summation operator" acting on a class of binary maps, as was the case in our examples (2), (3), (4).

DEFINITION 13.1. A trace algebra will be called *regular* if the interior action laws of Theorem 9.10 hold without restriction, that is, for all $x, y, z \in A$ and $\pi, \tau, \sigma \in \Delta$,

$$[(1, \pi, x), y \circ \tau, (\sigma, 1, z)] = [x \circ \sigma, (\pi^*, 1, y), (1, \tau, z)].$$

The three equations encompassed in this relation will be referred to as the *interior action laws.*

THEOREM 13.2 (UNIQUENESS THEOREM). *Suppose that A, $[\ ,\ ,\ ]$ is a regular trace algebra whose plinths and comonoid are Θ^i and Δ, respectively. Let A, $\{\ ,\ ,\ \}$ be another trace algebra which also has Θ^i as i-plinths and which satisfies the interior and exterior action laws with respect to the elements of the comonoid Δ of $[\ ,\ ,\ ]$. Let T denote the trace for $[\ ,\ ,\ ]$ and T' the trace for $\{\ ,\ ,\ \}$. Then: If $T'=\lambda \cdot T$ for some invertible $\lambda \in \Gamma$, then also there exists an invertible $\lambda' \in \Gamma$ such that $\{x, y, z\} =\lambda' \cdot [x, y, z]$ for all $x, y, z \in A$. Or, stated in words: a regular trace algebra is uniquely determined by its comonoid and trace function.*

Proof. Since the plinths of $\{\ ,\ ,\ \}$ are Θ^i, it follows, for example, that there are $a \in \Theta$ and $b \in \Theta^1$ with $\{a, x, b\} = x$ for all $x \in A$. Combining this observation with the fact that $\{\ ,\ ,\ \}$ obeys the interior and exterior action laws of Theorem 5.9 and Definition 13.1 with respect to Δ, then it follows by the characterization theorem (Theorem 10.4) that there is an invertible $\lambda'' \in \Gamma$ such that $\{x, y, z\} = \lambda'' \cdot [x, y, z]$ whenever $x \in \Theta$, $y \in \Theta$, $y \in \Theta^1$, $z \in \Theta^1$, $z \in \Theta^2$ or $x \in \Theta^2$. In particular, Theorem 10.4 shows that the comonoids of $[\ ,\ ,\ ]$ and $\{\ ,\ ,\ \}$ coincide. Then:

$$
\begin{aligned}
\{a, b, \{x, d, z\}\} &= \{\{a, b, x\}, d, z\} = \{b \circ \theta, d, z\} = \{b, d, (\theta, 1, z)\} \\
&= \{d, (\theta, 1, z)T', b\} = \lambda'[d, (\theta, 1, z)T, b] \quad (\text{where } \lambda' = \lambda\lambda'') \\
&= \lambda'[b, d, (\theta, 1, z)] = \lambda'[b \circ \theta, d, z] = \lambda'[\{a, b, x\}, d, z] \\
&= \lambda'\lambda''[[a, b, x], d, z] = \lambda'\lambda''[a, b, [x, d, z]] = \lambda'\{a, b, [x, d, z]\}
\end{aligned} \tag{1}
$$

for all a, b, and hence $\{x, d, z\} = \lambda'[x, d, z]$ for all $x, z \in A$, $d \in \Theta^2$.

$$
\begin{aligned}
\{d, a, \{x, y, z\}\} &= \{x, \{d, a, y\}, z\} = \{x, d \circ \theta, z\} = \{x, d, (1, \theta, z)\} \\
&= \lambda'[x, d, (1, \theta, z)] = \lambda'[x, d \circ \theta, z] = \lambda'\lambda''[x, [d, a, y], z] \\
&= \lambda'\lambda''[d, a, [x, y, z]] = \lambda'\{d, a, [x, y, z]\}
\end{aligned} \tag{2}
$$

for all d, a and hence $\{x, y, z\} = \lambda'[x, y, z]$, as required.

As an immediate application of the uniqueness theorem, the following theorem shows that whenever all the ions exist, a regular trace algebra is "isomorphic" to its dual algebras.

THEOREM 13.3 (SELF-DUALITY). *Suppose that A is a regular trace algebra such that for all $x \in A$, x^-, x^+, x_+ exist. Then for all $x, y, z \in A$,*

$$
[x, y, z]^- = \zeta \cdot [y^-, x^-, z^-], \quad [x, y, z]^+ = \zeta \cdot [z^+, y^+, x^+],
$$
$$
[x, y, z]_+ = \zeta \cdot [x_+, z_+, y_+].
$$

Proof. We prove the first equation. The other cases are similar. Define $\{\ ,\ ,\ \}$ in A by $\{x, y, z\} = [y^-, x^-, z^-]^-$ for all $x, y, z \in A$. It is easily seen that A endowed with this new ternary product is also a trace algebra, whose i-plinths are precisely the Θ^i and whose comonoid is Δ. Then by Theorem 10.4, $\{x, y, z\} = \lambda[x, y, z]$ for some invertible λ whenever $x \in \Theta$, $y \in \Theta$, etc. (in fact, $\lambda = \zeta$). Hence, if T' denotes the trace function for $\{\ ,\ ,\ \}$, then

$$
\begin{aligned}
\zeta[d, xT', b] &= \{d, xT', b\} = \{b, d, x\} = [d^-, b^-, x^-]^- = [b^-, x^-T, d^-]^- \\
&= [x^-T, b, d] = [d, x^-T, b]
\end{aligned}
$$

for all d, b and hence, $\zeta \cdot xT' = x^-T$. But by the same argument used to prove Propostion 13.4, $x^-T = \zeta \cdot xT$. This shows that $T' = T$. It follows, then, that since $[\ ,\ ,\ ]$ is regular, then $\{\ ,\ ,\ \}$ is also regular, and we can then apply Theorem 13.2.

PROPOSITION 13.4. *Suppose A is a regular trace algebra. Then*

$$
(\theta, 1, x)T = (1, \theta^*, x)T, \quad (\pi \circ \tau)T = (\tau \circ \pi)T, \quad \theta^*T = \zeta \cdot \theta T.
$$

Proof. Follows immediately from regularity.

DEFINITION 13.5. Given $x, y, z \in A$, define the map $[\phi_x, \phi_y, \phi_z]\colon \Theta \times \Theta \to \Theta$ as follows:

$$(u, v)[\phi_x, \phi_y, \phi_z] = \sum_s ((u, s)\phi_x, (s^*, v)\phi_y)\phi_z = \zeta(\theta^u, \theta_v, z)T$$

where $x\theta^u = (u, s)\phi_x$ and $s^*\theta_v = (s^*, v)\phi_y$. It is not clear whether, in general, $[\phi_x, \phi_y, \phi_z]$ is a map in $A\phi$ or not. We will call a trace algebra *representable* if $[\phi_x, \phi_y, \phi_z]$ is in $A\phi$ for all x, y, z and if $[\phi_x, \phi_y, \phi_z] = \zeta\phi_{[x,y,z]}$ for all x, y, z.

THEOREM 13.6 (REPRESENTATION THEOREM). *Any regular trace algebra is representable, i.e., for all* $x, y, z \in A$, $u, v \in \Theta$,

$$(u, v)\phi_{[x,y,z]} = \zeta \sum_s ((u, s)\phi_x, (s^*, v)\phi_y)\phi_z.$$

Proof. We imitate the proof of Theorem 13.2 with a few modifications. First note the following facts:

(1) For x or $y \in \Theta$, $[\phi_x, \phi_y, \phi_z] = \zeta\phi_{[x,y,z]}$.

(2) For all $x, y, z \in A$ and $\pi, \tau, \sigma \in \Delta$,

$$(\pi, \tau, [\phi_x, \phi_y, \phi_z]) \circ \sigma = [(\pi, 1, \phi_x), (1, \tau, \phi_y), x \circ \sigma],$$

$$[(1, \pi, \phi_x), \phi_y \circ \tau, (\sigma, 1, \phi_z)] = [\phi_x \circ \sigma, (\pi^*, 1, \phi_y), (1, \tau, \phi_z)].$$

The first equation is obvious from Definition 13.5. In the second relation, the cases for τ and σ are also obvious. The case for π follows from Proposition 13.4, or rather, in another form, the fact that $\sum_s (s\theta, s^*)\phi_x = \sum_s (s, s^*\theta^*)\phi_x$ for all x. Then

(3) For all d, b there exists a unique $\phi_x T$ for every $x \in A$ such that $[\phi_b, \phi_d, \phi_x] = [\phi_d, \phi_x T, \phi_b]$ for all d, b. In fact, it is easily verified from Definition 13.5 that $\phi_x T = \phi_{xT}$.

(4) Consider the map in Δ defined by $\theta_{ab}\colon u \mapsto (u, v)\cdot a$ for fixed b and a. Then note that $(\theta_{ab}, 1, \phi_x) = \phi_b \circ \pi_{xa}$ for all x, b, where $\pi_{xb}\colon u \mapsto (a, u)\phi_x$ (Lemma 10.1). Then, we find that

$$\begin{aligned}(\theta_{ab}, 1, [\phi_x, \phi_d, \phi_z]) &= [(\theta_{ab}, 1, \phi_x), \phi_d, \phi_z] = [\phi_b \circ \pi_{xa}, \phi_d, \phi_z]\\ &= [\phi_b, \phi_d, (\pi_{xa}, 1, \phi_z)] = [\phi_b, \phi_d, \phi_E]\end{aligned}$$

where

$$\begin{aligned}E &= (\pi_{xa}, 1, z) = [\phi_d, \phi_E T, \phi_b] = [\phi_d, \phi_{ET}, \phi_b] = \zeta\phi_{[d,ET,b]}\\ &= \zeta\phi_{[b,d,E]} = \zeta\phi_{[b\circ\pi,d,z]} \quad (\text{for } \pi = \pi_{xa})\\ &= \zeta\phi_{[F,d,z]} \quad (F = (\theta_{ab}, 1, x))\\ &= \zeta(\theta_{ab}, 1, \phi_{[x,d,z]}).\end{aligned}$$

Hence,

$$\zeta(u\theta_{ab}, v)\phi_{[x,d,z]} = (a, v)[\phi_x, \phi_d, \phi_z]$$

for all a, b, u, v, which implies $\zeta\phi_{[x,d,z]} = [\phi_x, \phi_d, \phi_z]$.

(5) Let θ_{da} be the map in Δ defined by $v \mapsto \{v, d\}a$. Then note that for all y, d, a, $(1, \theta_{da}, \phi_y)=\phi_d \circ \pi_{ya}$ for $\pi_{ya}: u \mapsto (u, a)\phi_y$, again from Lemma 10.1. Then,

$$\begin{aligned}(1, \theta_{da}, [\phi_x, \phi_y, \phi_z]) &= [\phi_x, (1, \theta_{da}, \phi_y), \phi_z] = [\phi_x, \phi_d \circ \pi_{ya}, \phi_z] \\ &= [\phi_x, \phi_d, (1, \pi_{ya}, \phi_z)] = [\phi_x, \phi_d, \phi_E]\end{aligned}$$

$(E=(1, \pi_{xa}, z))$ and by part (4), $=\zeta\phi_{[x,d,E]}$. But as usual, $[x, d, E]=[x, d \circ \pi_{ya}, z]$ $=[x, (1, \theta_{da}, y), z]$, and so $\zeta\phi_{[x,d,E]}=\zeta(1, \theta_{da}, \phi_{[x,y,z]})$ so that

$$(u, v\theta_{da})[\phi_x, \phi_y, \phi_z] = \zeta(u, v\theta_{da})\phi_{[x,y,z]}$$

for all u, v, d, a from which follows

$$(u, a)[\phi_x, \phi_y, \phi_z] = \zeta(u, a)\phi_{[x,y,z]}$$

for all u, a. This implies that $[\phi_x, \phi_y, \phi_z]$ is in $A\phi$ and is equal to $\zeta\phi_{[x,y,z]}$. This completes the proof of the theorem.

REMARK. A trace T will be called *regular* if $(1, \theta, x)T=(\theta^*, 1, x)T$ for all $x \in A$ and $\theta \in \Delta$.

COROLLARY 13.7. *A trace algebra is regular iff it is representable and has a regular trace.*

Proof. The one direction is Theorem 13.6. The converse follows from our last remark in (2) of the proof of that theorem.

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