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SETS OF UNIQUENESS ON THE 2-TORUS

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Abstract. $H^{(J)}$-sets are defined on the 2-torus and the following results are established: (1) $H^{(J)}$-sets are sets of uniqueness both for Abel summability and circular convergence of double trigonometric series; (2) a countable union of closed sets of uniqueness of type (A) (i.e., Abel summability) is also a set of uniqueness of type (A).

1. **Introduction.** Using the notation $X=(x, y)$, $M=(m, n)$, $(M, X)=mx+ny$, $|x|^2=(X, X)$, and letting M designate an integral lattice point, we shall say that the double trigonometric series $\sum a_M e^{i(M,X)}$ is circularly convergent to zero at the point X_0 if

$$\lim_{R\to\infty} \sum_{|M|\leqq R} a_M \exp(i(M, X_0)) = 0.$$

Designating the 2-torus T_2 by $T_2=\{X: -\pi\leqq x<\pi, -\pi\leqq y<\pi\}$, we shall say that a set $E\subset T_2$ is a set of uniqueness of type (C) on the 2-torus provided the following holds:

If a double trigonometric series $\sum a_M e^{i(M,X)}$ is circularly convergent to zero in T_2-E, then $a_M=0$ for every integral lattice point M.

Next, adopting the notation in [11, p. 346], we let $\{V_k^J\}_{k=1}^\infty$ be a sequence of J-tuples with positive integral entries, i.e. $V_k^J=(v_k^1, \ldots, v_k^J)$ with v_k^j a positive integer for $j=1, \ldots, J$ and $k=1, 2, \ldots$. We call $\{V_k^J\}_{k=1}^\infty$ a normal sequence provided the following holds:

If $B^J=(b^1, \ldots, b^J)$ is a J-tuple with each entry an integer and at least one entry different from zero, then

$$\lim_{k\to\infty} |b^1 v_k^1+\cdots+b^J v_k^J| = +\infty.$$

We call the sequence $\{V_k^J, W_k^J\}_{k=1}^\infty$ a sequence of normal pairs if each of the sequences $\{V_k^J\}_{k=1}^\infty$ and $\{W_k^J\}_{k=1}^\infty$ are normal.

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Designating the $2J$-torus by T_{2J}, i.e. $T_{2J}=\{(x^1,\ldots,x^{2J}): -\pi\leqq x^j<\pi, j=1,\ldots,2J\}$, we say a set $E\subset T_2$ is a set of type $H^{(J)}$ provided the following holds:

There is a sequence of normal pairs $\{V_k^J, W_k^J\}_{k=1}^{\infty}$ and there is a nonempty domain $\mathscr{D}\subset T_{2J}$ such that if $X=(x,y)$ is in E, the $2J$-tuple $(xv_k^1, yw_k^1,\ldots,xv_k^J, yw_k^J)$ is in $T_{2J}-\mathscr{D}$ mod 2π in each entry for $k=1,2,\ldots$. To be quite explicit about this last statement, we mean if we replace each entry xv_k^j or yw_k^j by the number in the interval $[-\pi,\pi)$ which is congruent to it mod 2π, then this new $2J$-tuple lies in $T_{2J}-\mathscr{D}$.

We intend to establish the following result:

THEOREM 1. *Sets $H^{(J)}$ are sets of uniqueness of type* (C) *on the* 2-*torus.*

On the 1-torus, the analogue of the above result is well known [11, p. 346]. On the higher dimensional tori and in particular on the 3-torus the analogous result is an open question.

We shall prove Theorem 1 by showing that it is an immediate corollary to Theorem 2 which concerns itself with sets of uniqueness for Abel summability (sets of uniqueness of type (A)). We refer the reader to §3 below for the full statement of Theorem 2 and the definitions involved.

In §3, we also establish the fact that a countable union of closed sets of uniqueness of type (A) on the 2-torus is a set of uniqueness of type (A).

Both the analogues of this last result and of Theorem 2 are open questions on the 3-torus in particular and on the N-torus in general, $N\geqq 3$.

2. **Formal multiplication.** The main new difficulty that arises in establishing Theorems 1, 2 and 3 lies in the area of formal multiplication of double trigonometric series. An examination of the proof of the 1-dimensional version of Theorem 1 [11, pp. 345–346] shows that it depends heavily on the formal product theorem [11, 4.9, p. 331]. The direct analogue of this theorem in 2-dimensions (that is, when we assume $a_M=o(1)$ as $|M|\to\infty$) is easily seen to be false (see §4 below where this matter is discussed in greater detail).

Let $S_k(X)=\sum a_M^k e^{i(M,X)}$, $k=1,2$, be two double trigonometric series. If, for each integral lattice point M, the series $\sum_P |a_P^1 a_{M-P}^2|<\infty$ (where the sum is taken over all integral lattice points P), we say that the formal product of $S_1(X)$ and $S_2(X)$ exists. We designate this formal product by $S_3(X)=S_1(X)S_2(X)$ where $S_3(X)$ is the trigonometric series defined as follows:

$$S_3(X) = \sum a_M^3 e^{i(M,X)}, \tag{2.1}$$

$$a_M^3 = \sum_P a_P^1 a_{M-P}^2. \tag{2.2}$$

We shall say a double series $\sum a_M$ is Bochner-Riesz summable of order β, $\beta\geqq 0$, henceforth designated by $(B\text{-}R,\beta)$, to ξ if

$$\lim_{R\to\infty} \sum_{|M|\leqq R} a_M(1-|M|^2/R^2)^\beta = \xi. \tag{2.3}$$

In order to obtain our main result on formal multiplication of double trigonometric series, we need the following lemma concerning Bochner-Riesz summability.

LEMMA 1. *Let* $\sum a_M$ *be a double series enjoying the following two properties*:

(2.4) $\sum_{R \leqq |M| \leqq R+1} |a_M| = o(R)$ *as* $R \to \infty$,

(2.5) $\sum m a_M$ *and* $\sum n a_M$ *are respectively* (*B-R*, 2) *summable to the finite values* ξ_1 *and* ξ_2.

Suppose also that $\sum \alpha_M$ *is a double trigonometric series enjoying the following two properties*:

(2.6) $|\alpha_M| = O(|M|^{-15})$ *as* $|M| \to \infty$,

(2.7) $\sum_{|M| \leqq R} \alpha_M \to 0$ *as* $R \to \infty$.

Set $A_M = \sum_P a_P \alpha_{M-P}$. *Then the double series* $\sum A_M$ *is* (*B-R*, 3) *summable to zero.*

The techniques used in establishing this result are in part similar to those in [1] and [8]. This lemma, however, has a somewhat different twist to it.

Before starting the proof, we observe as in [1, (3), p. 326] that

(2.8) $\beta_M = O(|M|^{-\theta})$, $\theta > 2$, implies that $\sum_{R \leqq |M|} |\beta_M| = O(R^{-\theta+2})$ as $R \to \infty$.

To prove the lemma we observe first from (2.4) and (2.6) that

(2.9) $\sum_P |a_P \alpha_{M-P}| < \infty$ for every M.

Consequently A_M is well defined.

It follows from the definition of A_M, (2.3), and (2.9) that the lemma will be established once we show that, as $R \to \infty$,

$$\sum_P a_P \left[\sum_{|M| \leqq R} \alpha_{M-P} (R^2 - |M|^2)^3 \right] = o(R^6). \tag{2.10}$$

In the rest of the proof of the lemma, R will be understood to be $\geqq 10$.

Observing that, for $|P| \geqq 2R$, $\sum_{|M| \leqq R} |\alpha_{M-P}| \leqq \sum_{|P|-R \leqq |M|} |\alpha_M|$, we conclude from (2.6) and (2.8) that there is a constant C_1, such that

$$\sum_{|M| \leqq R} |\alpha_{M-P}| \leqq C_1 |P|^{-13} \quad \text{for } |P| \geqq 2R. \tag{2.11}$$

From (2.4) and (2.11), we obtain that

$$\sum_{2R \leqq |P|} |a_P| \sum_{|M| \leqq R} |\alpha_{M-P}| = o(R^{-11}).$$

It consequently follows from this last fact that (2.10) will be established once we show

$$\sum_{|P| \leqq 2R} a_P \left[\sum_{|M| \leqq R} \alpha_{M-P} (R^2 - |M|^2)^3 \right] = o(R^6). \tag{2.12}$$

To establish (2.12), we start out by observing that the following two identities are valid:

$$(R^2 - |M|^2)^3 = \sum_{k=0}^{3} \binom{3}{k} (R^2 - |P|^2)^{3-k} (|P|^2 - |M|^2)^k, \tag{2.13}$$

$$|M|^2 - |P|^2 = (m-p)^2 + (n-q)^2 + 2p(m-p) + 2q(n-q). \tag{2.14}$$

Next, we observe from (2.6) and (2.8) that there is a constant C_2 such that

$$\sum_{|M|\leqq R} |\alpha_{M-P}|\,|M-P|^6 \leqq \sum_{|P|-R\leqq|M|} |\alpha_{M-P}|\,|M-P|^6 \leqq C_2[|P|-R]^{-7} \quad \text{for } |P| \geqq R+2. \tag{2.15}$$

It consequently follows from (2.4), (2.13), (2.14), and (2.15) that there is a constant C_3 such that

$$\begin{aligned} &\sum_{R+2\leqq|P|\leqq 2R} |a_P| \left| \sum_{|M|\leqq R} \alpha_{M-P}(R^2-|M|^2)^3 \right| \\ &\qquad \leqq C_2 \sum_{R+2\leqq|P|\leqq R} \{|a_P|[R^3(|P|-R)^3+R^2(|P|-R)^2|P| \\ &\qquad\qquad +R(|P|-R)|P|^2+|P|^3]\} \\ &\qquad \sum_{|M|\leqq R} |\alpha_{M-P}|\,|P-M|^6 \\ &\qquad \leqq 15R^3C_3C_2 \sum_{R+2\leqq|P|\leqq R} |a_P|[|P|-R]^{-4} \\ &\qquad \leqq 15R^3C_3C_2o(R) \sum_{2\leqq|P|} |P|^{-4} \leqq o(R^4). \end{aligned} \tag{2.16}$$

Likewise, with C_3 as in (2.16), it follows that there is a constant C_4 such that

$$\begin{aligned} &\sum_{R-2\leqq|P|\leqq R+2} |a_P| \left| \sum_{|M|\leqq R} \alpha_{M-P}(R^2-|M|^2)^3 \right| \\ &\qquad \leqq C_3 \sum_{R-2\leqq|P|\leqq R+2} \{|a_P|[R^3|\,|P|-R|^3+R^2(|P|-R)^2|P| \\ &\qquad\qquad +R|\,|P|-R|\,|P|+|P|^3]\} \\ &\qquad \sum_{|M|\leqq R} |\alpha_{M-P}|\,|P-M|^6 \\ &\qquad \leqq C_3C_4R^3 \sum_{j=-2}^{1} \left[\sum_{R+j\leqq|P|\leqq R+j+1} |a_P| \right] \leqq o(R^4). \end{aligned} \tag{2.17}$$

We consequently conclude from (2.16) and (2.17) that (2.12) will follow once we show

$$\sum_{|P|\leqq R-2} a_P \left[\sum_{|M|\leqq R} \alpha_{M-P}(R^2-|M|^2)^3 \right] = o(R^6). \tag{2.18}$$

To establish (2.18), we first observe from (2.6) and (2.7) that $\sum_{|M|\leqq R} \alpha_{M-P} = \sum_{R<|M|} \alpha_{M-P}$ and furthermore that, for $|P| \leqq R-2$, $\sum_{R<|M|} |\alpha_{M-P}| \leqq \sum_{R-|P|\leqq|M|} |\alpha_M|$. We consequently have from (2.6) and (2.8) that there is a constant C_5 such that

$$\left| \sum_{|M|\leqq R} \alpha_{M-P} \right| \leqq C_5(R-|P|)^{-13} \quad \text{for } |P| \leqq R-2. \tag{2.19}$$

From (2.19), we obtain that

$$\begin{aligned}\left|\sum_{|P|\leqq R-2} a_P\left[\sum_{|M|\leqq R} \alpha_{M-P}(R^2-|P|^2)^3\right]\right| &\\ \leqq \sum_{|P|\leqq R-2} |a_P|(R^2-|P|^2)^3 C_5(R-|P|)^{-13} &\\ \leqq C_5(2R)^3 \sum_{|P|\leqq R-2} |a_P|(R-|P|)^{-10} \leqq o(R^4). &\end{aligned} \tag{2.20}$$

Next, we observe that, for $|P|\leqq R-2$,

$$\sum_{|M|\leqq R} |\alpha_{M-P}|(R^2-|P|^2)|P|^2|M-P|^4 = o(R^4) \tag{2.21}$$

and

$$\sum_{|M|\leqq R} |\alpha_{M-P}|\ |P|^3|M-P|^6 = o(R^3). \tag{2.22}$$

Since $\sum_{|P|\leqq R-2} |a_P| = o(R^2)$, we conclude from (2.13), (2.14), (2.20), (2.21), and (2.22) that (2.18) will follow once we show

$$\sum_{|P|\leqq R-2} a_P \sum_{|M|\leqq R} \alpha_{M-P}(R^2-|P|)^2(|M|^2-|P|^2) = o(R^6). \tag{2.23}$$

Since $\sum_{|M|\leqq R} |\alpha_{M-P}|\ |M-P|^2 = O(1)$ and $\sum_{|P|\leqq R-2} |a_P|(R^2-|P|^2)^2 = o(R^6)$, it follows from (2.14) that (2.23) will follow once we show

$$|\mathrm{I}_R| = o(R^6) \quad \text{and} \quad |\mathrm{II}_R| = o(R^6) \tag{2.24}$$

where

$$\begin{aligned}\mathrm{I}_R &= \sum_{|P|\leqq R-2} pa_p(R^2-|P|^2)^2\left[\sum_{|M|\leqq R} \alpha_{M-P}(m-p)\right] \quad \text{and}\\ \mathrm{II}_R &= \sum_{|P|\leqq R} qa_p(R^2-|P|^2) \sum_{|M|\leqq R} \alpha_{M-P}(n-q).\end{aligned} \tag{2.25}$$

We now show that $\mathrm{I}_R = o(R^6)$; a similar proof will prevail to show that $\mathrm{II}_R = o(R^6)$.

We begin by observing from (2.6) that there is a finite constant C_6 such that $\lim_{R\to\infty} \sum_{|M|\leqq R} \alpha_{M-P}(m-p) = C_6$. Consequently,

$$\sum_{|M|\leqq R} \alpha_{M-P}(m-p) = C_6 - \sum_{R<|M|} \alpha_{M-P}(m-p). \tag{2.27}$$

But then from (2.25) and (2.27) we see that

$$\begin{aligned}\mathrm{I}_R = C_6 &\sum_{|P|\leqq R-2} pa_p(R^2-|P|^2)^2\\ &- \sum_{|P|\leqq R-2} pa_p(R^2-|P|^2)^2\left[\sum_{R<|M|} \alpha_{M-P}(m-p)\right].\end{aligned} \tag{2.28}$$

Now

$$\begin{aligned}\sum_{|P|\leqq R-2} &pa_p(R^2-|P|^2)^2\\ &= \sum_{|P|\leqq R} pa_p(R^2-|P|^2)^2 - \sum_{R-2<|P|\leqq R} pa_p(R^2-|P|^2)^2.\end{aligned} \tag{2.29}$$

From (2.5), it follows that

$$\left|\sum_{|P|\leqq R} pa_p(R^2-|P|^2)^2\right| = O(R^4). \tag{2.30}$$

On the other hand, from (2.4) it follows that

$$\sum_{R-2<|P|\leqq R} |p|\,|a_p|(R^2-|P|^2)^2 \leqq (4R^2)^2R \sum_{R-2<|P|\leqq R} |a_p| \leqq o(R^6). \tag{2.31}$$

We conclude from (2.30) and (2.31) that

$$\left|\sum_{|P|\leqq R-2} pa_p(R^2-|P|^2)^2\right| = o(R^6). \tag{2.32}$$

Next, we observe from (2.6) and (2.8) that there is a constant C_7, such that, for $|P|\leqq R-2$,

$$\left|\sum_{R<|M|} \alpha_{M-P}(m-p)\right| \leqq \sum_{R-|P|\leqq|M|} |\alpha_{M-P}|\,|M-P| \leqq C_7(R-|P|)^{-12}.$$

Consequently, we obtain from this last fact and (2.4) that there is a constant C_8 such that

$$\begin{aligned}\sum_{|P|\leqq R-2} &|p|\,|a_p|(R^2-|P|^2)^2\left|\sum_{R<|M|} \alpha_{M-P}(m-p)\right| \\ &\leqq 2C_7R^3 \sum_{|P|\leqq R-2} |a_p|(R-|P|)^{-10} \leqq 2C_7C_8R^3o(R) \leqq o(R^4).\end{aligned} \tag{2.33}$$

We consequently obtain from (2.28), (2.32) and (2.33) that

$$|\mathrm{I}_R| = o(R^6). \tag{2.34}$$

A similar proof shows that

$$|\mathrm{II}_R| = o(R^6). \tag{2.35}$$

(2.34) and (2.35) together give (2.24), and the proof of the lemma is complete.

Next, we introduce the notion of Abel summability. We say the double trigonometric series $\sum a_M e^{i(M,X)}$ with coefficients $|a_M|=O(|M|^\theta)$ for some $\theta\geqq 0$ is Abel summable at the point X_0 to the finite value s if

$$\lim_{t\to 0+} \sum_M a_M \exp(i(M, X_0)-|M|t) = s. \tag{2.36}$$

It follows from the Cauchy criterion (see [6, pp. 67–68] or [7, p. 468]) that if $|\sum_M a_M \exp(i(M, X_0)-|M|t)|<C<\infty$ for $0<t<1$, then

$$\lim_{t\to 0+} \sum_{M\neq 0} a_M|M|^{-2} \exp(i(M, X_0)-|M|t)$$

exists and is finite. (In the sequel, we shall drop the $+$ in $\lim_{t\to 0+}$.)

With $B(X, h)$ designating the open 2-ball (=open disc) with center X and radius h, the next lemma that we establish is the following:

LEMMA 2. *Let* $\sum a_M e^{i(M,X)}$ *be a double trigonometric series having the following properties. Suppose that*

(2.37) $a_M = \bar{a}_M$ *for all* M;

(2.38) $a_M = o(1)$ *as* $|M| \to \infty$;

(2.39) $|\sum_M a_M e^{i(M,X)-|M|t}| \leqq C < \infty$ *for* $0 < t < 1$ *and* X *in* $B(X_0, h_0)$, $h_0 > 0$;

(2.40) $\lim_{t\to 0} \sum_M a_M e^{i(M,X)-|M|t} = 0$ *almost everywhere in* $B(X_0, h_0)$.

Set

(2.41) $F(X) = a_0|X|^2/4 - \lim_{t\to 0} \sum_{M\neq 0} a_M |M|^{-2} e^{i(M,X)-|M|t}$ *for* X *in* $B(X_0, h_0)$.

Then

(2.42) $F(X)$ *is harmonic in* $B(X_0, h_0)$

and both

(2.43) $\sum m a_M \exp(i(M, X_0))$ *and* $\sum n a_M \exp(i(M, X_0))$ *are each* (*B-R*, 2) *summable to zero.*

To prove the lemma, we set, for $t > 0$,

(2.44) $F(X, t) = -\sum_{M\neq 0} a_M |M|^{-2} e^{i(M,X)-|M|t}$

and observe from (2.39) and (2.44) that

(2.45) $|\partial^2 F(X, t)/\partial^2 t| < C + |a_0|$ for $0 < t < 1$ and X in $B(X_0, h_0)$.

It consequently follows from the Cauchy criterion and (2.41) that $F(X, t) \to F(X) - a_0|X|^2/4$ uniformly for x in $B(X_0, h_0)$ as $t \to 0$. Consequently

(2.46) $F(X)$ is continuous in $B(X_0, h_0)$.

Letting $\bar{Z}$ designate the closure of Z, we set, for $h > 0$ and $\bar{B}(X, h) \subset B(X_0, h_0)$,

$$F_h(X) = (\pi h^2)^{-1} \int_{B(X,h)} F(y)\, dy \tag{2.47}$$

and

$$\begin{aligned} \Delta^* F(X) &= 8 \limsup_{h\to 0} [F_h(X) - F(X)]/h^2, \\ \Delta_* F(X) &= 8 \liminf_{h\to 0} [F_h(X) - F(X)]/h^2. \end{aligned} \tag{2.48}$$

From (2.39), (2.40), (2.41) and [6, Lemma 7, p. 66] we have

$$\begin{aligned} &\Delta^* F(X) \geqq -C \quad \text{for } X \text{ in } B(X_0, h_0) \quad \text{and} \\ &\Delta^* F(X) \geqq 0 \quad \text{almost everywhere in } B(X_0, h_0), \end{aligned} \tag{2.49}$$

and

$$\begin{aligned} &\Delta_* F(X) \leqq C \quad \text{for } X \text{ in } B(X_0, h_0) \quad \text{and} \\ &\Delta_* F(X) \leqq 0 \quad \text{almost everywhere in } B(x_0, h_0). \end{aligned} \tag{2.50}$$

Using [10, Lemma 5, p. 91], we have from (2.46), (2.49), and (2.50) that

(2.51) $F(X)$ is harmonic in $B(X_0, h_0)$,

and (2.42) is therefore established.

(Incidentally, in the proof of [10, Lemma 5, p. 91], "$0 < r_0 < 1$" should read "$0 < r_0 < \frac{1}{2}$.")

To establish (2.43) we shall confine ourselves to showing that $\sum ma_M \exp(i(M, X_0))$ is (B-R, 2) summable to zero. A similar proof will show that $\sum na_M \exp(i(M, X_0))$ is (B-R, 2) summable to zero.

Using (2.38), we set

$$G(X) = a_0|X|^4/4^3 + \sum_{M \neq 0} a_M|M|^{-4}e^{i(M,X)} \tag{2.52}$$

and observe that

(2.53) $G(X)$ is a continuous function in the plane, for the series in (2.52) converges absolutely.

It follows, therefore, from this last remark, (2.41) and [5, Lemma 2, p. 66] that

(2.54) $\Delta_* G(X) \leqq F(X) \leqq \Delta^* G(X)$ in $B(X_0, h_0)$.

Next, we set

$$G_1(X) = -(2\pi)^{-1} \int_{B(X_0, h_0/2)} \log|X-Y|^{-1} F(Y)\, dY \tag{2.55}$$

and observe from (2.51) that

$$\begin{aligned} &G_1(X) \text{ is in class } C^\infty[B(X_0, h_0/2)] \quad \text{and} \\ &\Delta G_1(X) = F(X) \quad \text{in } B(X_0, h_0/2) \end{aligned} \tag{2.56}$$

where Δ is the Laplace operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

Consequently, we obtain from (2.54) that

$$\Delta_*[G(X) - G_1(X)] \leqq 0 \leqq \Delta^*[G(X) - G_1(X)] \tag{2.57}$$

and we conclude from (2.53), (2.56), (2.57) and [4, p. 14] that

$$G(X) - G_1(X) \text{ is harmonic in } B(X_0, h_0/2).$$

We therefore obtain from (2.56) that $G(X)$ is in class $C^\infty[B(X_0, h_0/2)]$, that $\Delta G(X) = F(X)$ in $B(X_0, h_0/2)$ and consequently that

$$\Delta^2 G(X) = 0 \quad \text{in } B(X_0, h_0/2). \tag{2.58}$$

Next, we set

$$g(X) = a_0[\partial|X|^4/\partial x]4^{-3} + \sum_{M \neq 0} ima_M|M|^{-4}e^{i(M,X)} \tag{2.59}$$

where we see that the series in (2.59) converges absolutely because of (2.38). It follows from (2.52), (2.59), and [9, Lemma 2] that

$$\partial G(X)/\partial x = g(X) \quad \text{in } B(X_0, h_0/2). \tag{2.60}$$

Therefore

$$\begin{aligned} &g(X) \text{ is in } C^\infty[B(X_0, h_0/2)] \quad \text{and} \\ &\Delta^2 g(X) = 0 \quad \text{in } B(X_0, h_0/2). \end{aligned} \tag{2.61}$$

Setting

$$g'(X) = \sum_{M \neq 0} ima_M |M|^{-4} e^{i(M,X)}, \tag{2.62}$$

we obtain consequently from (2.59), (2.61), and (2.62) that

$$\begin{aligned} &g'(X) \text{ is in } C^\infty[B(X_0, h_0/2)] \quad \text{and} \\ &\Delta^2 g'(X) = 0 \quad \text{in } B(X_0, h_0/2). \end{aligned} \tag{2.63}$$

Observing that $|ima_M| = o(|M|)$ as $M \to \infty$, we conclude from (2.62), (2.63) and [1, Theorem 4, p. 341] that $\sum ima_M e^{i(M,X)}$ is (B-R, 2) summable to zero uniformly in $B(X_0, h_0/4)$. The first part of (2.43) is therefore established. A similar proof establishes the second part. The proof of the lemma is therefore complete.

Next, we establish the following lemma:

LEMMA 3. *Let $\sum a_M e^{i(M,X)}$ be a double trigonometric series having the following properties: $a_M = \bar{a}_{-M}$ for all M and $a_M = o(1)$ as $|M| \to \infty$. Suppose furthermore that* $\limsup_{t\to 0} |\sum_M a_M e^{i(M,X)-|M|t}| < \infty$ *for X in $B(X_0, h_1)$, $h_1 > 0$, and that*

$$\lim_{t\to 0} \sum_M a_M e^{i(M,X)-|M|t} = 0$$

almost everywhere in $B(X_0, h_1)$. Then

$$\lim_{t\to 0} \sum_M a_M e^{i(M,X)-|M|t} = 0 \quad \textit{uniformly for } X \textit{ in } B(X_0, h_1/2).$$

To establish the lemma, we set

$$f(X, t) = \sum_M a_M e^{i(M,X)-|M|t} \quad \text{for } t > 0 \tag{2.64}$$

and

$$\limsup_{t\to 0} f(X, t) = f^*(X), \qquad \liminf_{t\to 0} f(X, t) = f_*(X) \tag{2.65}$$

and observe from the start that with no loss in generality we can assume that $X_0 = 0$. Consequently, we obtain from the hypothesis of the lemma that

(2.66) $f_*(X) = f^*(X) = 0$ almost everywhere in $B(0, h_1)$, and

(2.67) $f_*(X)$ and $f^*(X)$ are finite for X in $B(0, h_1)$.

From (2.67) and [7, Lemma 1], we obtain that for X in $B(0, h_1)$

$$F(X) = a_0|X|/4 - \lim_{t\to 0} \sum_{M\neq 0} a_M |M|^2 e^{i(M,X)-|M|t} \tag{2.68}$$

is well defined and finite. Also from [7, Lemma 6] we have that

(2.69) $\Delta_* F(X) \leqq f^*(X)$ and $f_*(X) \leqq \Delta^* F(X)$ for X in $B(0, h_1)$.

We set

$$Z = \{X : X \text{ in } B(0, h_1) \text{ and } F \text{ is not continuous at } X\}.$$

If we show

(2.70) Z is empty,

then it will follow from (2.66), (2.67), (2.69), and [10, Lemma 5] that

(2.71) F is harmonic in $B(0, h_1)$.

But then [9, Lemma 5] in conjunction with (2.71) establishes the conclusion to this lemma.

We now establish (2.70). First using the fact that $f(X, t)$, defined in (2.65), is continuous for $t>0$ and periodic of period 2π in the x and y variables, we select a sequence

$$1 = t_1 > t_2 > \cdots > t_k > \cdots \to 0$$

such that

$$\sup_{X \text{ in } B(0,h_1)} \sup_{t_{k+1} \leqq t \leqq t_k} |f(X, t) - f(X, t_k)| \leqq 1. \tag{2.72}$$

Given $\bar{B}(X_2, h_2) \subset B(0, h_1)$ with $h_2 > 0$, it follows from (2.65) and the Baire category theory (see [11, p. 29]) that there exists a subball $B(X_3, h_3) \subset \bar{B}(X_2, h_2)$ with $h_3 > 0$ and a finite constant C such that

(2.73) $|f(X, t_k)| \leqq C$ for X in $B(X_3, h_3)$ and $k = 1, 2, \ldots$.

But then it follows from (2.66), (2.72), and (2.73) that conditions (2.39) and (2.40) in the hypothesis of Lemma 2 are met in $B(X_3, h_3)$. Since (2.37) and (2.38) also hold, we conclude from Lemma 2 that $F(X)$ is harmonic in $B(X_3, h_3)$ and therefore that

(2.74) Z is nowhere dense in $B(0, h_1)$.

If Z is nonempty in $B(0, h_1)$ then there exists h_4 with $0 < h_4 < h_1$ such that $Z \cap B(0, h_4) = Z_1$ is nonempty. Then $\bar{Z}_1$ is a closed set contained in $B(0, h_1)$ and we obtain from (2.66), (2.72), and the Baire category theory that there is an X^* in Z, an $h_5 > 0$, and a constant C_1 such that

(2.75) $\bar{B}(X^*, h_5) \subset B(0, h_1)$

and, for $0 < t \leqq 1$,

(2.76) $|f(X, t)| \leqq C_1$ for $B(X^*, h_5) \cap \bar{Z}$.

We shall establish (2.70) by showing that

(2.77) X^* in Z is false.

To establish (2.77), we observe from (2.76), [7, Lemma 2], and [7, Lemma 4] that

$$\lim_{X \to X^*, X \text{ in } Z} F(X) = F(X^*) \tag{2.78}$$

and

$$\lim_{h \to 0} F_h(X) = F(X) \quad \text{uniformly for } X \text{ in } \bar{Z} \cap B(X^*, h_5). \tag{2.79}$$

Also, we observe from [7, Lemma 5] that

$$\lim_{h \to 0} \left[\sup_{|W| \leqq h, X \text{ in } B(0,h_1)} |F_h(X+W) - F_h(X)| \right] = 0. \tag{2.80}$$

Given $\varepsilon>0$, we use (2.78), (2.79) and (2.80) and choose h_6 such that, for $0<h_6<h_5/10$,

$$(2.81)\qquad \begin{aligned} |F(X)-F(X^*)| &< \varepsilon \quad \text{for } X \text{ in } \bar{Z}\cap B(X^*, h_6),\\ |F_h(X)-F(X)| &< \varepsilon \quad \text{for } X \text{ in } \bar{Z}\cap B(X^*, h_6) \text{ and } 0<h<h_6, \quad \text{and}\\ |F_h(X+W)-F_h(X)| &< \varepsilon \quad \text{for } X \text{ in } B(0, h_1),\ |W|\leqq h, \text{ and } 0<h\leqq h_6. \end{aligned}$$

We see from (2.81) that (2.77) will follow if we show

$$(2.82)\qquad |F(X)-F(X^*)| < 3\varepsilon \quad \text{for } X \text{ in } B(X^*, h_6/2) \text{ and } X \text{ not in } \bar{Z}.$$

To establish (2.82), we fix X in $B(X^*, h_6/2)$ and assume that X is not in $\bar{Z}$. Let ρ designate the distance from X to $\bar{Z}$. Then $0<\rho<h_6/2$, and we obtain from (2.75) that F is harmonic in $B(X, \rho)$. Therefore $F(X)=F_\rho(X)$. Let X' be in $\bar{Z}$ and such that $|X-X'|=\rho$. We consequently have from (2.81) that

$$(2.83)\qquad |F(X)-F_\rho(X')| < \varepsilon, \quad |F(X')-F_\rho(X')| < \varepsilon, \quad |F(X')-F(X^*)| < \varepsilon.$$

We conclude from (2.83) that $|F(X)-F(X^*)|<3\varepsilon$ and the proof of the lemma is complete.

Let $\lambda(X)$ be a function in $L^1(T_2)$. Then we shall designate the Fourier coefficients of $\lambda(X)$ as follows:

$$\hat{\lambda}(M) = (4\pi^2)^{-1}\int_{T_2} \lambda(X)e^{-i(M,X)}\,dX.$$

Next, we establish the following lemma:

LEMMA 4. *Let $\lambda(X)$ be a continuous periodic function in the plane of period 2π in each variable with $\hat{\lambda}(M)=O(|M|^{-15})$ as $|M|\to\infty$. Suppose that $\sum a_M e^{i(M,X)}$ is a double trigonometric series having the following properties: $\bar{a}_M=a_{-M}$ and $a_M=o(1)$ as $|M|\to\infty$. Suppose furthermore that $\lim_{t\to 0}\sum_M a_M e^{i(M,X)-|M|t}=0$ for X in $B(X_0, h_1)$, $h_1>0$. Set $A_M=\sum_P a_P\hat{\lambda}(M-P)$. Then*

$$(2.84)\qquad \lim_{R\to\infty}\sum_{|M|\leqq R} [A_M-\lambda(X_0)a_M]\exp(i(M, X_0))(1-|M|^2R^{-2})^3 = 0.$$

We first observe from Lemma 3 that $\lim_{t\to 0}\sum_M a_M e^{i(M,X)-|M|t}=0$ uniformly for X in $B(X_0, h_1/2)$. Consequently with $h_0=h_1/2$, we see that (2.39) and (2.40) hold. Since (2.37) and (2.38) hold by hypothesis, we conclude from Lemma 2 that (2.43) holds. In particular, if we set

(2.85) $a_M^*=a_M\exp(i(M, X_0))$

then we see that the following facts obtain:

(2.86) $\sum ma_M^*$ and $\sum na_M^*$ are both $(B\text{-}R, 2)$ summable to zero,

(2.87) $\sum_{R\leqq|M|\leqq R+1}|a_M^*|=o(R)$ as $R\to\infty$.

Next, we set

(2.88) $\alpha_M=\hat{\lambda}(M)\exp(i(M, X_0))$ for $M\neq 0$ and $\alpha_0=\hat{\lambda}(0)-\lambda(X_0)$

and observe that

(2.89) $\alpha_M = O(|M|^{-15})$ as $|M| \to \infty$ and

(2.90) $\sum_{|M| \leqq R} \alpha_M \to 0$ as $R \to \infty$.

Consequently, on setting

(2.91) $A_M^* = \sum_P a_P^* \alpha_{M-P}$,

we obtain from (2.87), (2.88), (2.89), (2.90), and Lemma 1 that

(2.92) $\lim_{R\to\infty} \sum_{|M| \leqq R} A_M^*(1-|M|^2R^{-2})^3 = 0$.

We observe from (2.74) that

$$A_M^* = A_M \exp(i(M, X_0)) - \lambda(X_0) a_M \exp(i(M, X_0))$$

and (2.84) follows immediately from this fact and (2.92). The proof of the lemma is therefore complete.

3. **Sets of uniqueness of type** (A). We shall say that a set $E \subset T_2$ is a set of uniqueness of type (A) on the 2-torus provided the following holds:

If a double trigonometric series $\sum a_M e^{i(M,X)}$ with $a_M = o(1)$ as $|M| \to \infty$ is Abel summable to zero in $T_2 - E$, then $a_M = 0$ for every integral lattice point M.

In this section, we shall prove the following two theorems concerning sets of uniqueness of type (A).

THEOREM 2. *Sets $H^{(J)}$ are sets of uniqueness of type* (A) *on the* 2-*torus.*

THEOREM 3. *Suppose that $\{E_k\}_{k=1}^\infty$ is a sequence of sets of uniqueness of type* (A) *on the* 2-*torus. Suppose, furthermore, that for each k, E_k is closed in the torus sense, i.e. $E_k^* = \bigcup_M [E_k + 2\pi M]$ is closed in the plane. Then $E = \bigcup_{k=1}^\infty E_k$ is a set of uniqueness of type* (A) *on the* 2-*torus.*

We need the following lemma.

LEMMA 5. *Let $E \subset T_2$ be closed in the torus sense, i.e. $E^* = \bigcup_M [E + 2\pi M]$ is a closed set in the plane. Suppose there exists a sequence, $\{\lambda_k(X)\}_{k=1}^\infty$, of continuous periodic functions of period 2π in each variable having the following properties*:

(3.1) *for each k, there exists a set D_k^* open in the plane such that $E^* \subset D_k^*$ and λ_k vanishes on D_k^**;

(3.2) *for each k, $\hat{\lambda}_k(M) = O(|M|^{-15})$ as $|M| \to \infty$*;

(3.3) *there is a finite constant C such that $\sum_M |\hat{\lambda}_k(M)| \leqq C$ for $k = 1, 2, \ldots$*;

(3.4) $\lim_{k\to\infty} \hat{\lambda}_k(M) = 0$ *for* $M \neq 0$;

(3.5) $\lim_{k\to\infty} \hat{\lambda}_k(0) = 1$.

Then E is a set of uniqueness of type (A) *on the* 2-*torus.*

Of course this lemma is motivated by [11, p. 345]. We establish it, however, for Abel summability and do not assume to start with that E is of measure zero.

To establish the lemma, suppose that $\sum a_M e^{i(M,X)}$ is a double trigonometric series such that

(3.6) $a_M = o(1)$ as $|M| \to \infty$, and

(3.7) $\lim_{t\to 0} \sum_M a_M e^{i(M,X) - |M|t} = 0$ for X in $T_2 - E$.

The lemma will be established if we show

(3.8) $a_M = 0$ for every integral lattice point M.

To establish (3.8), with no loss in generality we can suppose from the start that $\bar{a}_M = a_{-M}$ (see [6, p. 67]). Next, fix k and set

(3.9) $A_M^k = \sum_P a_P \hat{\lambda}_k(M-P)$.

From the properties of E^* and D_k^* set forth in the hypothesis of the lemma, and from (3.7), we see that for a given X_0 in T_2 one (or both) of the following two situations prevail:

(3.10) there exists an $h_1 > 0$ such that, for X in $B(X_0, h_1)$,

$$\lim_{t \to 0} \sum a_M e^{i(M,X) - |M|t} = 0,$$

or

(3.11) there exists an $h_1 > 0$ such that, for X in $B(X_0, h_1)$, $\lambda_k(X) = 0$.

If (3.10) prevails, we conclude from Lemma 4 that the double series

$$\sum [A_M^k - \lambda_k(X_0) a_M] \exp(i(M, X_0)) \tag{3.12}$$

is (B-R, 3) summable to zero. But a double series which is (B-R, 3) summable to zero is Abel summable to zero. We have therefore that the double series in (3.12) is Abel summable to zero. However, from (3.10) we have that

$$\sum \lambda_k(X_0) a_M \exp(i(M, X_0))$$

is Abel summable to zero, also. We conclude that

$$\lim_{t \to 0} \sum_M A_M^k e^{i(M,X_0) - |M|t} = 0. \tag{3.13}$$

If the second situation prevails for X_0, namely (3.11), we have from (3.2) that

(3.14) $\lambda_k(X)$ and all its partial derivatives of orders 1 and 2 vanish in $B(X_0, h_1)$.

We conclude from [1, Theorem 2, p. 330] that $\sum A_M^k \exp(i(M, X_0))$ is (B-R, 1) summable to zero. But this in turn implies that (3.13) holds in this case also.

We conclude that for fixed k, the double trigonometric series $\sum A_M^k e^{i(M,X)}$ is Abel summable to zero for every X in T_2. From (3.2), (3.6), (3.9), and [1, p. 325], it follows that, for fixed k, $A_M^k = o(1)$ as $|M| \to \infty$. We have therefore from [6, Theorem 7, p. 65] that

(3.15) $A_M^k = 0$ for every M and $k = 1, 2, \ldots$.

From (3.5), (3.9), and (3.15), we consequently have that for every M

(3.16) $\hat{\lambda}_k(0) a_M = -\sum_{P \neq 0} a_{M-P} \hat{\lambda}_k(P)$, $k = 1, 2, \ldots$.

Fix M, and let $\varepsilon > 0$ be given. Then it follows from (3.6) that we can find $R_0 > 1$ such that $|a_{M-P}| < \varepsilon C^{-1}$ for $|P| \geqq R_0$. We conclude from (3.3) and (3.16) that, for every k,

$$|\hat{\lambda}_k(0) a_M| \leqq \varepsilon + \sum_{1 \leqq |P| \leqq R_0} |a_{M-P}| \, |\hat{\lambda}_k(P)|. \tag{3.17}$$

Leaving $k \to \infty$, we obtain from (3.4), (3.5), and (3.17) that $|a_M| \leqq \varepsilon$. But this implies that $a_M = 0$. Therefore (3.8) is established, and the proof of the lemma is complete.

We now prove Theorem 2. Since the closure in the torus sense of an $H^{(J)}$-set is an $H^{(J)}$-set, we can assume from the start that $E \subset T_2$ is an $H^{(J)}$-set which is closed in the torus sense. (To be specific, we assume that the set $E^* = \bigcup_M [E + 2\pi M]$ is a closed set in the plane.) Let $\{V_k^J, W_k^J\}$ be the sequence of normal pairs and $\mathfrak{D} \subset T_{2J}$ be the nonempty domain used in the definition that E is an $H^{(J)}$-set where $V_k^J = (v_k^1, \ldots, v_k^J)$ and $W_k^J = (w_k^1, \ldots, w_k^J)$. Then there are numbers $-\pi < \alpha_j < \beta_j < \pi$, $j = 1, \ldots, 2J$, such that the following prevails:

(3.18) $Q_j = [\alpha_j, \beta_j]$, $j = 1, \ldots, 2J$, and $Q_1 \times \cdots \times Q_{2J} \subset \mathfrak{D}$.

Consequently, on setting

(3.19) $Q_j^* = \bigcup_{m=-\infty}^{\infty} [Q_j + 2\pi m]$, $j = 1, \ldots, 2J$,

we have that

(3.20) $(xv_k^1, yw_k^1, \ldots, xv_k^J, yw_k^J)$ is not in $Q_1^* \times \cdots \times Q_{2J}^*$ for (x, y) in E and $k = 1, 2, \ldots$.

Next, we select numbers α_j', α_j'', β_j', β_j'' such that

$$-\pi < \alpha_j < \alpha_j' < \alpha_j'' < \beta_j'' < \beta_j' < \beta_j < \pi, \qquad j = 1, \ldots, 2J, \tag{3.21}$$

and define functions of one real variable $\eta_j(x)$ which are of class $C^{(\infty)}$ on the real line and periodic of period 2π such that for $j = 1, \ldots, 2J$

$$\begin{aligned} \eta_j(x) &= 1 \quad \text{in } [\alpha_j'', \beta_j''], \\ &= 0 \quad \text{in } [-\pi, \pi] - [\alpha_j', \beta_j'], \end{aligned} \tag{3.22}$$

and

$$\hat{\eta}_j(0) = 1 \qquad \text{where } (2\pi)^{-1} \int_{-\pi}^{\pi} \eta_j(x) e^{-imx} = \hat{\eta}_j(m), \quad m = 0, \pm 1, \pm 2, \ldots. \tag{3.22'}$$

For future reference, we observe in particular from the fact that $\eta_j(x)$ is in class $C^{(\infty)}$ on the real line and periodic of period 2π that

(3.23) $\sum_{m=-\infty}^{\infty} |\hat{\eta}_j(m)| = c_j < \infty$, $j = 1, \ldots, 2J$.

Next, we set

(3.24) $\lambda_k(x, y) = \eta_1(v_k^1 x)\eta_2(w_k^1 y) \cdots \eta_{2J-1}(v_k^J x)\eta_{2J}(w_k^J y)$, $k = 1, 2, \ldots$.

Then for each k,

(3.25) $\lambda_k(x, y)$ is a function in class C^∞ on the plane and periodic of period 2π in each variable.

It follows immediately from (3.25) that for each k, $|\hat{\lambda}_k(M)| = O(|M|^{-15})$ as $|M| \to \infty$. Consequently, the sequence $\{\lambda_k(X)\}_{k=1}^{\infty}$ meets condition (3.2) in the hypothesis of Lemma 5.

To show that the sequence $\{\lambda_k(X)\}_{k=1}^{\infty}$ meets condition (3.1) in the hypothesis of Lemma 5, we fix k and observe from (3.20) that for (x_0, y_0) in E, either one (or both) of the following situations prevail:

(3.26) there exists a j_0 such that $x_0 v_k^{j_0}$ is not in $Q_{2j_0-1}^*$, or

(3.27) there exists a j_0 such that $y_0 w_k^{j_0}$ is not in $Q_{2j_0}^*$.

Suppose (3.26) prevails. Then it follows from (3.18), (3.19), and (3.21) that there exists an $h_0 > 0$ such that if $|x - x_0| < h_0$, then $xv_k^{j_0}$ is not in $[\alpha_{j_0}', \beta_{j_0}']$ mod 2π. Therefore we have from the 2π periodicity of η_{2j_0-1} that $\eta_{2j_0-1}(xv_k^{j_0}) = 0$ for $|x - x_0| < h_0$.

Consequently, it follows from (3.24) that for fixed k,

(3.28) there exists an $h_0>0$ such that $\lambda_k(X)=0$ for X in $B(X_0, h_0)$.

In case (3.27) prevails, similar reasoning will once again lead us to (3.28). Since X_0 was an arbitrary point in E, it follows from (3.28) and from the compactness of E in the torus topology and from (3.25) that for each k there exists an open set in the plane D_k^* such that $E^*\subset D_k^*$ and $\lambda_k(X)=0$ for X in D_k^*. We conclude that the sequence $\{\lambda_k(X)\}_{k=1}^{\infty}$ meets condition (3.1) in the hypothesis of Lemma 5.

To establish conditions (3.3), (3.4), and (3.5) in the hypothesis of Lemma 5, we set

(3.29) $\lambda_k'(x)=\eta_1(v_k^1x)\cdots\eta_{2J-1}(v_k^Jx)$ and $\lambda_k''(y)=\eta_2(w_k^1y)\cdots\eta_{2J}(w_k^Jy)$

and observe from (3.24) that

(3.30) $\lambda_k(x, y)=\lambda_k'(x)\lambda_k''(y)$.

Next we observe that

$$(3.31)\qquad \begin{aligned} \hat{\lambda}_k'(m) &= \sum_{v_k^1m_1+\cdots+v_k^Jm_J=m} \hat{\eta}_1(m_1)\cdots\hat{\eta}_{2J-1}(m_J), \\ &\qquad m_j = 0, \pm 1, \pm 2, \ldots, j = 1, \ldots, J, \quad \text{and} \\ \hat{\lambda}_k''(n) &= \sum_{w_k^1n_1+\cdots+w_k^Jn_J=n} \hat{\eta}_2(n_1)\cdots\hat{\eta}_{2J}(n_J), \\ &\qquad n_j = 0, \pm 1, \pm 2, \ldots, j = 1, \ldots, J. \end{aligned}$$

It follows from (3.30) that

$$(3.32)\qquad \hat{\lambda}_k(M) = \hat{\lambda}_k'(m)\hat{\lambda}_k''(n)$$

and therefore from (3.31) that

$$(3.33)\qquad \sum_M |\hat{\lambda}_k(M)| \leqq \sum_{m=-\infty}^{\infty} |\hat{\lambda}_k'(m)| \sum_{n=-\infty}^{\infty} |\lambda_k''(n)|.$$

On the other hand from (3.31) it follows that

$$(3.34)\qquad \begin{aligned} \sum_{m=-\infty}^{\infty} |\hat{\lambda}_k'(m)| &\leqq \sum_{m_1=-\infty}^{\infty} |\hat{\eta}_1(m_1)|\cdots\sum_{m_J=-\infty}^{\infty} |\hat{\eta}_{2J-1}(m_J)|, \quad \text{and} \\ \sum_{n=-\infty}^{\infty} |\hat{\lambda}_k''(n)| &\leqq \sum_{n_1=-\infty}^{\infty} |\hat{\eta}_2(n_1)|\cdots\sum_{n_J=-\infty}^{\infty} |\hat{\eta}_{2J}(n_J)|. \end{aligned}$$

Setting $C=c_1\cdots c_{2J}$, it follows from (3.23), (3.33), (3.34) that

$$\sum_M |\hat{\lambda}_k(M)| \leqq C < \infty \quad \text{for } k = 1, 2, \ldots.$$

Consequently condition (3.3) in the hypothesis of Lemma 5 is met by the sequence $\{\lambda_k(X)\}_{k=1}^{\infty}$.

To show that the sequence $\{\lambda_k(X)\}_{k=1}^{\infty}$ meets condition (3.4) in the hypothesis of the lemma, let $M=(m, n)$ be a fixed integral lattice point different from the origin. Then either one or both of the following situations prevail: $m\neq 0$ or $n\neq 0$. We shall assume $m\neq 0$ with similar reasoning prevailing in case $n\neq 0$.

Given an $\varepsilon>0$, choose a positive integer γ such that

$$(3.35)\qquad \sum_{\gamma<|m_j|} |\hat{\eta}_{2j-1}(m_j)| < \varepsilon J^{-1}(c_1+1)^{-1}\cdots(c_{2J}+1)^{-1} \quad \text{for } j=1,\ldots,J,$$

where $c_1, c_2, c_3, \ldots, c_{2J}$ are defined in (3.23).

Next, using the fact that $\{(v_k^1,\ldots,v_k^J)\}_{k=1}^{\infty}$ is a normal sequence, choose k_0 so that the following holds:

(3.36) if $0<|m_1|+\cdots+|m_J|\leqq J\gamma$ and $k>k_0$, then $|v_k^1m_1+\cdots+v_k^Jm_J|>|m|+1$.

From (3.36), it follows that if $v_k^1m_1+\cdots+v_k^Jm_J=m$ and $k>k_0$, then at least one entry in the J-tuple $(m_1,\ldots,m_J)$ is in modulus greater than γ. We consequently conclude from (3.31) and (3.23) that

$$|\hat{\lambda}_k'(m)| \leqq \Bigg\{\sum_{|m_1|>\gamma} |\hat{\eta}_1(m_1)|c_3\cdots c_{2J-1} + \cdots + \sum_{|m_J|>\gamma} |\hat{\eta}_{2J-1}(m_J)|c_1\cdots c_{2J-3}\Bigg\} \quad \text{for } k>k_0,$$

and therefore from (3.35) that

(3.37) $|\hat{\lambda}_k'(m)|\leqq\varepsilon(c_2+1)^{-1}(c_4+1)^{-1}\cdots(c_{2J}+1)^{-1}$ for $k>k_0$.

Also, it follows from (3.31) and (3.23) that

(3.38) $|\hat{\lambda}_k''(n)|\leqq c_2c_4\cdots c_{2J}$ for all k.

We therefore obtain from (3.32), (3.37), and (3.38) that

(3.39) $|\hat{\lambda}_k(M)|\leqq\varepsilon$ for $k>k_0$.

We conclude from (3.39) that $\lim_{k\to\infty}|\hat{\lambda}_k(M)|=0$ for $M\neq0$. Consequently, the sequence $\{\lambda_k(X)\}_{k=1}^{\infty}$ meets condition (3.4) in the hypothesis of Lemma 5.

In order to complete the proof of Theorem 2 all that remains to be shown is that the sequence $\{\lambda_k(X)\}_{k=1}^{\infty}$ meets condition (3.5) in the hypothesis of Lemma 5. To accomplish this fact, we observe from (3.22′) and (3.31) that

$$(3.40)\qquad \hat{\lambda}_k'(0) = 1+\sideset{}{^*}\sum_{v_k^1m_1+\cdots+v_k^Jm_J=0} \hat{\eta}_1(m_1)\cdots\hat{\eta}_{2J-1}(m_J)$$

where $\sum^*_{v_k^1m_1+\cdots+v_k^Jm_J=0}$ represents a summation over those J-tuples $(m_1,\ldots,m_J)$ such that $v_k^1m_1+\cdots+v_k^Jm_J=0$ with the J-tuple $(0,\ldots,0)$ omitted. It follows from (3.23) and the normality of $\{V_k^J\}_{k=1}^{\infty}$ exactly as in the last case considered that

$$(3.41)\qquad \lim_{k\to\infty}\sideset{}{^*}\sum_{v_k^1m_1+\cdots+v_k^Jm_J=0} \hat{\eta}_1(m_1)\cdots\hat{\eta}_{2J-1}(m_J) = 0.$$

We conclude from (3.40) and (3.41) that

(3.42) $\lim_{k\to\infty}\hat{\lambda}_k'(0)=1$.

In a similar manner, we obtain that

(3.43) $\lim_{k\to\infty}\hat{\lambda}_k''(0)=1$.

But from (3.32), we have that $\hat{\lambda}_k(0)=\hat{\lambda}_k'(0)\hat{\lambda}_k''(0)$, and consequently from (3.42) and (3.43) that $\lim_{k\to\infty}\hat{\lambda}_k(0)=1$. Therefore, the sequence $\{\lambda_k(X)\}$ meets condition (3.5) in the hypothesis of Lemma 5, and the proof of Theorem 2 is complete.

We now prove Theorem 1. Let E be an $H^{(J)}$-set on the 2-torus which with no loss in generality we take to be closed in the torus sense. We let $\chi_E(X)$ designate the characteristic function of E in the torus sense, i.e. the function which is one on E, zero on T_2-E, and then is extended by periodicity of period 2π in each variable to the rest of the plane. If X_0 is in T_2-E, then there exists $h_0>0$ such that $\chi_E(X)=0$ for X in $B(X_0, h_0)$. Consequently [6, Theorem 2, pp. 55–56] $\sum_M \hat{\chi}_M(M)e^{i(M,X)-|M|t} \to 0$ for X in $B(X_0, h_0)$. We conclude therefore that the Fourier series of χ_E is Abel summable to zero in T_2-E. Also $\lim_{|M|\to\infty} \hat{\chi}_E(M)=0$. Therefore if the two-dimensional Lebesgue measure of E, designated by $|E|$, were positive we would have a contradiction to Theorem 2. We conclude that $|E|=0$.

Now, suppose

(3.44) $\lim_{R\to\infty} \sum_{|M|\leq R} a_M e^{i(M,X)}=0$ for X in T_2-E.

Cooke has shown, [2] and [3], that if $\lim_{k\to\infty} \sum_{|M|^2=k} a_M e^{i(M,X)}=0$ almost everywhere on T_2 then $\sum_{|M|^2=k} |a_M|^2=o(1)$ as $k\to\infty$. We conclude from this result, from (3.44), and from the fact that $|E|=0$ that

(3.45) $a_M=o(1)$ as $|M|\to\infty$.

From (3.44), we obtain easily that

(3.46) $\lim_{t\to 0} \sum a_M e^{i(M,X)-|M|t}=0$ for X in T_2-E.

Since E is an $H^{(J)}$-set on the 2-torus we conclude from (3.45), (3.46), and Theorem 2 that $a_M=0$ for all integral lattice points. Consequently, E is a set of uniqueness of type (C), and the proof of Theorem 1 is complete.

To prove Theorem 3, we suppose

(3.47) $a_M=o(1)$ as $|M|\to\infty$

and set for $t>0$

(3.48) $f(X, t)=\sum_M a_M e^{i(M,X)-|M|t}$.

Also, we set

(3.49) $E^*=\bigcup_{k=1}^{\infty} E_k^*$

and suppose that

(3.50) $\lim_{t\to 0} f(X, t)=0$ for X in $\mathscr{P}-E^*$, where $\mathscr{P}$ designates the plane.

The proof will be complete when we establish the fact that

(3.51) $a_M=0$ for all M.

To establish (3.51), we see from the start, using an argument similar to that in [6, p. 65], that with no loss in generality we can also assume

(3.52) $a_M=\bar{a}_{-M}$ for all M.

Next, using the fact that $f(X, t)$ defined in (3.48) is continuous for $t>0$ and periodic of period 2π in the x and y variables, we select a sequence $1=t_1>t_2>\cdots>t_k\cdots\to 0$ such that

(3.53) $\sup_{X \text{ in } \mathscr{P}, t_k\leq t\leq t_{k+1}} |f(X, t)-f(X, t_k)|\leq 1$.

Set

$$Z_{j,k} = \{X : |f(X, t_k)| > j\}, \qquad Z_j = \bigcup_{k=1}^{\infty} Z_{j,k}, \qquad Z = \bigcap_{j=1}^{\infty} Z_j.$$

Then

(3.54) Z is a G_δ-set and

(3.55) $Z=\{X : X \text{ in } \mathscr{P}, \limsup_{t\to 0} |f(X, t)| = +\infty\}$.

If we show

(3.56) Z is empty,

then the proof to Theorem 3 will be complete. For the fact that E_k^* is of two-dimensional Lebesgue measure zero for $k=1, 2, \ldots$ along with (3.47), (3.49), (3.50), (3.52), (3.55), (3.56) and Lemma 3 implies that $\lim_{t\to 0} f(X, t)=0$ uniformly for X in $\mathscr{P}$. This fact in conjunction with (3.48) implies that $a_M=0$ for all M.

We now establish (3.56). Suppose that Z is not empty. Then it follows from (3.50) and (3.55) that $Z\subset E^*$. But then it follows from (3.49) that

(3.57) $Z=\bigcup_{k=1}^{\infty} Z \cap E_k^*$.

From (3.54), (3.57), and the Baire Category Theorem [5, p. 54], we consequently obtain that there is a $B(X_0, h_0)$, $0<h_0<1$, such that

(3.58) $B(X_0, h_0) \cap Z \neq 0$

and there is an $E_{k_0}^*$ such that $E_{k_0}^*$ is dense in $B(X_0, h_0) \cap Z$. However, $E_{k_0}^*$ is closed. We have, therefore, that

(3.59) $B(X_0, h_0) \cap Z \subset E_{k_0}^*$.

From (3.50), (3.58), (3.59), and the definition of Z, we conclude therefore that

(3.60) $\lim_{t\to 0} f(X, t)=0$ almost everywhere in $B(X_0, h_0)$, and

(3.61) $\limsup_{t\to 0} |f(X, t)| < \infty$ in $B(X_0, h_0)-E_{k_0}^*$.

From (3.60), (3.61), Lemma 3, and the fact that E_k^* is closed we obtain

(3.62) $\lim_{t\to 0} f(X, t)=0$ in $B(X_0, h_0)-E_{k_0}^*$.

If (3.58) is indeed true then there is an h_3 such that

(3.63) $0<h_3<h_2<h_1<h_0<1$, and

(3.64) $B(X_0, h_3) \cap Z \neq 0$.

Let $\eta(X)$ be a function in $C^\infty(\mathscr{P})$ which is defined as follows for X in T_2.

$$\begin{aligned} \eta(X) &= 1 \quad \text{in } B(0, h_2), \\ &= 0 \quad \text{in } T_2-B(0, h_1) \end{aligned} \tag{3.65}$$

and is extended by periodicity of period 2π in each variable to the rest of the plane.

We set

(3.66) $\lambda(X)=\eta(X+X_0)$, and

(3.67) $A_M=\sum_P a_P\hat{\lambda}(M-P)$.

If X_1 is in $(T_2+X_0)-\bar{B}(X_0, h_1)$, then it follows from (3.65) and (3.66) that λ is identically zero in a small neighborhood of X_1. Consequently, we obtain from [1, Theorem 2, p. 330] that $\sum A_M e^{i(M,X)}$ is (B-R, 1) summable to zero for $X=X_1$. We conclude therefore in particular that

(3.68) $\lim_{t\to 0} \sum_M A_M e^{i(M,X)-|M|t}=0$ for X in $(T_2+X_0)-\bar{B}(X_0, h_1)$.

Likewise, if X_1 is $B(X_0, h_3)$, $\lambda(X)$ is identically one in a small neighborhood of X_1, and we conclude once again from [1, Theorem 2, p. 330] that

(3.69) $\lim_{t\to 0} \sum_M [A_M - \lambda(X)a_M]e^{i(M,X)-|M|t} = 0$ for X in $B(X_0, h_3)$.

On the other hand, if X_1 is in $\bar{B}(X_0, h_1) - E^*_{k_0}$, there exists $\rho_1 > 0$ such that $B(X_1, \rho_1) \subset B(X_0, h_0)$ and $B(X_1, \rho_1) \cap E^*_{k_0}$ is empty, we obtain therefore from (3.62) and Lemma 4 that

$$\lim_{R\to\infty} \sum_{|M|\leq R} [A_M - \lambda(X_1)a_M] \exp(i(M, X_1))(1 - |M|^2R^{-2})^3 = 0.$$

We consequently conclude from this last fact that

(3.70) $\lim_{R\to\infty} \{\sum_M A_M e^{i(M,X)-|M|t} - \lambda(X)f(X, t)\} = 0$ for X in $\bar{B}(X_0, h_1) - E^*_{k_0}$.

We conclude from (3.62), (3.63), (3.68), and (3.70) that $\lim_{t\to 0} \sum_M A_M e^{i(M,X)-|M|t} = 0$ in $(T_2 + X_0) - E^*_{k_0}$ and therefore by periodicity that

$$\lim_{t\to 0+} \sum_M A_M e^{i(M,X)-|M|t} = 0 \quad \text{for } X \text{ in } \mathscr{P} - E^*_{k_0}. \tag{3.71}$$

But from (3.47), (3.67), and [1, Lemma 1, p. 325] we have that $A_M = o(1)$. Also, E_{k_0} is a set of uniqueness of type (A). We conclude therefore from (3.71) that

(3.72) $A_M = 0$ for all M.

Recalling from (3.63), (3.65), and (3.66) that $\lambda(X) = 1$ for X in $B(X_0, h_3)$, we conclude from (3.69) and (3.72) that

(3.73) $\lim_{t\to 0} f(X, t) = 0$ for X in $B(X_0, h_3)$.

But this last fact, namely (3.73), is a contradiction to (3.64), and we conclude that Z must be empty. The proof of Theorem 3 is therefore complete.

4. **Appendix.** Rajchman's formal product theorem [11, 4.9, p. 331] is used both implicitly and explicitly throughout the one-dimensional approach to sets of uniqueness (see [11, p. 345 and p. 349]). We show here that the direct analogue in two dimensions of Rajchman's one-dimensional formal product theorem is false. In particular we show here that the following statement is *false*.

STATEMENT. *Let* $\sum a_M e^{i(M,X)}$ *be a double trigonometric series with*

(4.1) $a_M = o(1)$ *as* $|M| \to \infty$.

Let $\sum \lambda_M e^{i(M,X)}$ *be a double trigonometric series with the property that*

(4.2) $\sum_M |M|^j |\lambda_M| < \infty$ *for* $j = 1, 2, \ldots$.

Set $A_M = \sum_P a_P \lambda_{M-P}$ *and* $\lambda(X) = \sum_M \lambda_M e^{i(M,X)}$. *Then*

$$\lim_{R\to\infty} \sum_{|M|\leq R} [A_M - \lambda(0)a_M] = 0. \tag{4.3}$$

(It will be apparent from the proof to be given that the above Statement remains false when circular convergence in (4.3) is replaced by Bochner-Riesz summability of any order or even Abel summability.)

To show that the above Statement is false, we choose a function $\eta(X)$ which is in class C^∞ in the plane, periodic of period 2π in each variable, and which furthermore satisfies the following conditions:

$$\begin{aligned} \eta(X) &= 1 \quad \text{in } B(0, \tfrac{1}{2}) \\ &= 0 \quad \text{in } T_2 - B(0, 1). \end{aligned} \tag{4.4}$$

We define

(4.5) $\lambda(X)=x\eta(X)$ for X in T_2

and then define $\lambda(X)$ throughout the plane by periodicity of period 2π in each variable.

We define

(4.6) $f(X)=x|X|^{-5/2}\eta(X)$ for X in T_2-0

and then define $f(X)$ throughout the plane punctured at the integral lattice points by periodicity of period 2π in each variable.

Now both $\lambda(X)$ and $f(X)$ are in $L^1(T_2)$. We define

(4.7) $a_M=\hat{f}(M)$, and

(4.8) $\lambda_M=\hat{\lambda}(M)$.

It follows from the Riemann-Lebesgue lemma and (4.7) that a_M meets (4.1). Also since $\lambda(X)$ is of class C^∞ in the plane and periodic of period 2π in each variable, it follows from (4.8) that λ_M meets (4.2).

Next, we set

(4.9) $g(X)=f(X)\lambda(X)$ for X in the plane and X not equal to an integral lattice point.

Also, we observe from (4.2), (4.7), and (4.9) that there is a finite constant $C>0$ such that

$$\left|\sum_{|M|\leqq R} \lambda_M f(X)e^{i(M,X)}\right| \leqq C|f(X)| \quad \text{for } X \text{ in } T_2-0 \tag{4.10}$$

and that

$$\lim_{R\to\infty} \sum_{|M|\leqq R} \lambda_M f(X)e^{i(M,X)} = g(X) \quad \text{for } X \text{ in } T_2-0. \tag{4.11}$$

We obtain from (4.10), (4.11), and the Lebesgue dominated convergence theorem that

$$\begin{aligned}\int_{T_2} g(X)e^{-i(M,X)} &= \lim_{R\to\infty} \sum_{|P|\leqq R} \lambda_P \int_{T_2} f(X)e^{-i(M-P,X)}\,dX \\ &= \lim_{R\to\infty} \sum_{|P|\leqq R} \lambda_P a_{M-P}4\pi^2.\end{aligned}$$

We therefore conclude that

(4.12) $\hat{g}(M)=\sum_P \lambda_P a_{M-P}=A_M$.

Next we observe from (4.4) and (4.5) that

(4.13) $\lambda(0)=0$.

Also, we see from (4.9) that for $0<|X|<\frac{1}{2}$, $g(X)=x^2|X|^{-5/2}$. Consequently

$$h^{-2}\int_{B(0,h)} g(X)\,dX \to +\infty \quad \text{as } h\to 0$$

and we conclude from [6, Theorem 2, pp. 55–56] that

(4.14) $\lim_{t\to 0} \sum_M \hat{g}(M)e^{-|M|t}=+\infty$.

But then from (4.12), (4.13), and (4.14), we have that

(4.15) $\lim_{t\to 0} \sum_M [A_M - \lambda(0)a_M]e^{-|M|t} = +\infty$.

From (4.15), it follows immediately that (4.3) does not hold, and therefore that the above Statement is indeed false.

In closing, we point out that the above Statement becomes true if we replace (4.1) by the condition $\sum_{R \leq |M| \leq R+1} |a_M| = o(1)$ as $R \to \infty$. However, this condition has no relevance for this paper.

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