A CLASS OF COMPLETE ORTHOGONAL SEQUENCES OF BROKEN LINE FUNCTIONS

BY

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Abstract. A class of orthonormal sets of continuous broken line functions is defined. Each member is shown to be complete in $L_2(0, 1)$ and pointwise convergence theorems are obtained for the Fourier expansions relative to these sets.

1. Introduction. It was shown in [2] that each sequence of points which is dense in $[0, 1]$ determines a complete orthonormal set of step functions in $L_2(0, 1)$. In this paper we prove that each such sequence of points also determines a complete orthonormal set of continuous broken line functions similar to that constructed by Franklin [1]. The Fourier expansion of a function $f \in L_2(0, 1)$ relative to a set of this class is found to converge at each point of continuity of $f$ and is shown to converge uniformly on $[0, 1]$ when $f$ is continuous on this interval.

2. Definitions. Suppose that $A = \{a_n\}_{n=1}^\infty$ is a sequence of distinct points in $(0, 1)$ which is dense in $[0, 1]$ and let $\{h_n\}_{n=0}^\infty$ be the set of linear functions defined by

\[
    h_0(x) = 1, \quad h_1(x) = x, \quad x \in [0, 1); \\
    h_{n+1}(x) = 0, \quad x \in [0, a_n), \\
    h_{n+1}(x) = x - a_n, \quad x \in [a_n, 1].
\]

Since it is evident that no $h_i$ is a linear combination of the other functions in the set, we see that the $h_i$ are linearly independent on $[0, 1]$. Thus, one can employ the Gram-Schmidt process to construct an orthonormal sequence $\{u_n(x)\}$ such that each $u_n$ is a linear combination of the $h_i$, $i \leq n$. Because of the triangular nature of this construction, each $h_n$ can also be expressed as a linear combination of the $u_i$, $i \leq n$.

3. Completeness of $\{u_n\}$. To prove that the sequence of functions $\{u_n\}$ is complete in $L_2(0, 1)$, one needs an obvious property of the sequence $A$ which is given in Lemma 1. In this lemma and throughout this paper the term “adjacent points” of a finite subset $A_N \subset A$ will be used to denote successive elements of the subset when its elements are arranged in order of magnitude; i.e. $a_m$ and $a_n$ are adjacent points of $A_N$ if and only if there is no $a_k \in A_N$ such that $a_m < a_k < a_n$ or $a_n < a_k < a_m$. 

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293
Lemma 1. Let $A = \{a_1, a_2, \ldots\}$ be a sequence of distinct points of $(0, 1)$ which is dense in $[0, 1]$. Then for each $\delta > 0$ there is an integer $N_\delta$ such that if $N > N_\delta$, (i) any pair of adjacent points $a_n$ and $a_{n+1}$ in the subset $A_N = \{a_1, a_2, \ldots, a_N\}$ satisfy $|a_n - a_{n+1}| < \delta$; (ii) $d(x, A_N) < \delta$ for $x \in [0, 1]$. ($d(x, A_N)$ is the distance from $x$ to the $A_N$ defined in the usual manner.)

Theorem 1. The orthonormal sequence of functions $\{u_n\}$ is complete in $L_2(0, 1)$.

Proof. Let $0 < a_1 < a_2 < \cdots < a_N < 1$ be the points of $\{0, a_1, a_2, \ldots, a_N, 1\}$ arranged in order of magnitude. If $P_N$ is any continuous polygonal function (broken line function) which is linear on each subinterval $[a_{i-1}, a_i]$ of the partition of $[0, 1]$ determined by these points, it is clear that $P_N$ can be expressed as a linear combination of the $h_i$, $i \leq N$. Thus since each $h_i$, $i \leq N$, is a linear combination of the $u_i$, $i \leq N$, any such $P_N$ is a linear combination of the $u_i$, $i \leq N$.

Now suppose that $F$ is any continuous function on $[0, 1]$ and let $\delta$ be a positive number such that $|F(x_1) - F(x_2)| < \varepsilon/2$ when $x_1, x_2 \in [0, 1]$ and $|x_1 - x_2| < \delta$. By Lemma 1 we can choose an integer $N_\delta$ such that if $N > N_\delta$, the norm of the partition of $[0, 1]$ determined by the points of $A_N$ is less than $\delta$. Therefore, the broken line function $P_N$ which equals $F$ at each point of this partition and is linear elsewhere in $[0, 1]$ satisfies $|P_N(x) - F(x)| < \varepsilon$ for $x \in [0, 1]$. It follows from the preceding remarks that there is a linear combination of $u_i$, $i \leq N$, say $T_N$, such that $|T_N(x) - F(x)| < \varepsilon$ if $x \in [0, 1]$ or such that

$$\|T_N - F\|_2 = \int_0^1 (T_N - F)^2 \, dx < \varepsilon^2.$$

Since the set of continuous functions on $[0, 1]$ is dense in $L_2(0, 1)$, we conclude from the last inequality that the set of linear combinations of the $u_i$ is also dense in this space. This statement, of course, implies that the sequence $\{u_n\}$ is complete in $L_2(0, 1)$.

4. Convergence of the Fourier $\{u_n\}$ expansion. Since $\{u_n\}$ is a complete orthonormal sequence in $L_2(0, 1)$, each $f \in L_2(0, 1)$ has the norm-convergent Fourier expansion

$$f(x) \sim \sum c_k u_k(x)$$

where

$$c_k = \int_0^1 f u_k \, dx.$$

We next investigate the pointwise convergence of this expansion.

Theorem 2. The Fourier $-u_n$ expansion of $f \in L_2(0, 1)$ converges to $f(x)$ at each point $x \in [0, 1]$ at which $f$ is continuous.

Proof. Let $S_N(x, f)$ denote the $N$th partial sum of (1). Since each $u_i$ is a linear
combination of the \( h_k, k \leq i \), \( S_N \) itself is a linear combination of the \( h_i, i \leq N \), and thus is a continuous broken line function which is linear on each subinterval of the partition of \([0, 1]\) determined by the points of \( A_N = \{a_1, a_2, \ldots, a_N\} \). Suppose \( 0 < a_{i1} < a_{i2} < \cdots < a_{in} < 1 \) are the points of \( A_N \) arranged in order of magnitude and let \( K_0, K_1, \ldots, K_N \) denote the characteristic functions of the intervals \([0, a_{i1}), [a_{i1}, a_{i2}), \ldots, [a_{in}, 1]\). Then

\[
S_N(x, f) = \sum_{i=0}^{N} c_i u_i = \sum_{i=0}^{N} (\alpha_i + \beta_i h_i) K_i
\]

where the \( \alpha \)'s and \( \beta \)'s are constants. To determine \( \alpha_i \) and \( \beta_i \) we use the well-known fact that if \( T_N \) is any linear combination of the \( u_i, i \leq N \), \( \int_0^1 (f-T_N)^2 \, dx \) assumes its minimum value when \( T_N = S_N \). Thus \( \alpha_i \) and \( \beta_i \) must have values which minimize

\[
\int_0^1 \left[ f - \sum_{i=0}^{N} (\alpha_i + \beta_i h_i) K_i \right]^2 \, dx
\]

and when the partial derivatives of this integral with respect to \( \alpha_m \) and \( \beta_m \) are equated to 0, one has for each \( m=0, 1, 2, \ldots, N \),

\[
\int_0^1 \left[ f - \sum_{i=0}^{N} (\alpha_i + \beta_i h_i) K_i \right] K_m \, dx = 0,
\]

\[
\int_0^1 \left[ f - \sum_{i=0}^{N} (\alpha_i + \beta_i h_i) K_i \right] K_m h_m \, dx = 0.
\]

Now if \( I = [a_{im}, a_{im+1}) \), we obtain from (2) and (3) respectively

\[
\int_I f \, dx = \alpha_m |I| + \beta_m \frac{|I|^2}{2}
\]

and

\[
\int_I f h_m \, dx = \alpha_m \frac{|I|^2}{2} + \beta_m \frac{|I|^3}{3}.
\]

Thus

\[
\alpha_m = \frac{2}{|I|^3} \int_I (2|I| - 3h_m) f \, dx
\]

and

\[
\beta_m = \frac{6}{|I|^3} \int_I (2h_m - |I|) f \, dx.
\]

Since

\[
\int_I (2|I| - 3h_m) \, dx = \frac{|I|^2}{2}
\]

and

\[
\int_I (2h_m - |I|) \, dx = 0,
\]

we have if \( x_0 \in I \),

\[
|S_m(x_0, f) - f(x_0)| = |\alpha_m + \beta_m h_m(x_0) - f(x_0)|
\]

\[
= \left| \frac{2}{|I|^3} \int_I [2|I| - 3h_m(x)] [f(x) - f(x_0)] \, dx + \frac{6h_m(x_0)}{|I|^3} \int_I [2h_m(x) - |I|] [f(x) - f(x_0)] \, dx \right|.
\]
If \( x_0 \) is a point of continuity of \( f \), there exists a positive number \( \delta \) such that
\[
|f(x) - f(x_0)| < \varepsilon \quad \text{when} \quad x \in I \quad \text{and} \quad |I| < \delta.
\]
By Lemma 1 there is an integer \( N_0 \) such that if \( N > N_0 \), \( |I| < \delta \) and since \( |h_m(x)| \leq |I| \) when \( x \in I \), we find from (6) if \( N > N_0 \),
\[
|S_N(x_0,f) - f(x_0)| < 10\varepsilon + 18\varepsilon = 28\varepsilon.
\]

**Theorem 3.** If \( f \) is continuous on \([0, 1]\), the Fourier \( -u_n \) expansion (1) converges uniformly to \( f(x) \) on \([0, 1]\).

**Proof.** If \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( |f(x_1) - f(x_2)| < \varepsilon/28 \) when \( x_1, x_2 \in [0, 1] \) and \( |x_1 - x_2| < \delta \). By Lemma 1 we can choose an integer \( N_0 \) such that if \( N > N_0 \), the norm of the partition of \([0, 1]\) determined by \( A_N \) is less than \( \delta \). Then from equation (6) of the preceding proof we see that \( |S_N(x_0,f) - f(x_0)| < \varepsilon \) for any \( x_0 \in [0, 1] \).

In closing it should be pointed out that if the set \( A \) involved in the definition of \( \{u_n\} \) is taken to be the particular set described in §7(B) of [2], the resulting \( \{u_n\} \) is the orthonormal sequence of functions defined by Franklin [1].

**References**


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