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UNIFORMLY BOUNDED REPRESENTATIONS FOR THE LORENTZ GROUPS

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Abstract. A family of uniformly bounded class 1 representations of the Lorentz groups is constructed. This family of representations includes, but is larger than, a similar family of representations constructed by Lipsman. The construction technique relies on a multiplicative analysis of various operators under a Mellin transform.

1. **Introduction.** Let $G=SO_e(1, n+1)$ $(n\geqq 2)$ be one of the Lorentz groups. The class 1 principal series of representations of G may be regarded as a family of unitary representations $T(\cdot, s)$ for $s\in i\boldsymbol{R}$. Lipsman constructs a family of representations $R(\cdot, s)$ for $-1<\operatorname{Re} s<1$ with the following properties:

(1) $R(\cdot, s)$ is unitarily equivalent to $T(\cdot, s)$ for $s\in i\boldsymbol{R}$.

(2) $\sup_{g\in G} \|R(g, s)\|<\infty$ for all s.

(3) $s\to R(g, s)$ is an analytic operator-valued function for all $g\in G$.

The main result of this paper is the construction of a family of representations $R(\cdot, s)$ satisfying (1)–(3) for $-n/2<\operatorname{Re} s<n/2$.

2. **Multiplicative analysis of the Fourier transform on $\boldsymbol{R}^n$.** Let $n\geqq 2$ be fixed. Denote the standard inner product of vectors w and z in $\boldsymbol{R}^n$ by $w\cdot z$ and the length of a vector w by $|w|=(w\cdot w)^{1/2}$. Let $S=\{w\in\boldsymbol{R}^n : |w|=1\}$. Identify $\boldsymbol{R}^n-\{0\}$ with the Cartesian product $\boldsymbol{R}^+\times S$ where $\boldsymbol{R}^+$ is the multiplicative group of positive real numbers. Denote the usual Lebesgue measures on $\boldsymbol{R}^n$ and S by dw and $d\xi$. Define a measure d^*w on $\boldsymbol{R}^n-\{0\}$ by $d^*w=dw/|w|^n$. If dx is Lebesgue measure on the real line, then dx/x is a Haar measure for $\boldsymbol{R}^+$. Let $L_2(\boldsymbol{R}^n)$, $L_2(\boldsymbol{R}^n-\{0\})$, $L_2(S)$ and $L_2(\boldsymbol{R}^+)$ be the spaces of square integrable functions defined by these measures. By a classical result,

$$L_2(S) = \sum_{k=0}^{\infty} \oplus H_k \tag{2.1}$$

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where H_k is the space of spherical harmonics of order k. We may identify $L_2(\boldsymbol{R}^n - \{0\})$ with the Hilbert space tensor product $L_2(\boldsymbol{R}^+) \otimes L_2(S)$ and hence with

$$\sum_{k=0}^{\infty} \bigoplus (L_2(\boldsymbol{R}^+) \otimes H_k).$$

For f a function on $\boldsymbol{R}^n$, define $\mathcal{U}f$ on $\boldsymbol{R}^n - \{0\}$ by

$$\mathcal{U}f(w) = |w|^{n/2} f(w). \tag{2.2}$$

$\mathcal{U}$ is then an isometry of $L_2(\boldsymbol{R}^n)$ onto $L_2(\boldsymbol{R}^n - \{0\})$. For

$$f \in L_1(\boldsymbol{R}^n - \{0\}) \cap L_2(\boldsymbol{R}^n - \{0\})$$

define $\tilde{f}$ on $i\boldsymbol{R} \times S$ by

$$\tilde{f}(t, \xi) = \int_0^\infty x^t f(x, \xi) \frac{dx}{x}. \tag{2.3}$$

The mapping $f \to \tilde{f}$ may be viewed as the tensor product of the identity mapping on $L_2(S)$ with the group theoretic Fourier transform mapping integrable functions on $\boldsymbol{R}^+$ to functions on the dual group $\hat{\boldsymbol{R}}^+ = i\boldsymbol{R}$. By the Plancherel theorem for locally compact abelian groups, this tensor product of maps has a unique extension to a mapping $\mathcal{M}$ from $L_2(\boldsymbol{R}^+) \otimes L_2(S)$ onto $L_2(i\boldsymbol{R}) \otimes L_2(S)$. Up to a scalar multiple, $\mathcal{M}$ is an isometry which will be referred to as the *Mellin transform* on $L_2(\boldsymbol{R}^n - \{0\})$.

Let $\mathcal{I}$ be the unitary operator on $L_2(\boldsymbol{R}^n - \{0\})$ defined by $(\mathcal{I}f)(x, \xi) = f(1/x, \xi)$. Let $\mathcal{F}$ denote the Fourier transform on $L_2(\boldsymbol{R}^n)$. For $f \in L_1(\boldsymbol{R}^n) \cap L_2(\boldsymbol{R}^n)$, $\mathcal{F}f$ is defined by

$$(\mathcal{F}f)(w) = \frac{1}{(2\pi)^{n/2}} \int_{\boldsymbol{R}^n} f(z) e^{-iw\cdot z}\, dz. \tag{2.4}$$

Set $\hat{\mathcal{F}} = \mathcal{I}\mathcal{U}\mathcal{F}\mathcal{U}^{-1}$ and $\tilde{\mathcal{F}} = \mathcal{M}\hat{\mathcal{F}}\mathcal{M}^{-1}$. Then $\hat{\mathcal{F}}$ and $\tilde{\mathcal{F}}$ are unitary operators on $L_2(\boldsymbol{R}^+) \otimes L_2(S)$ and $L_2(i\boldsymbol{R}) \otimes L_2(S)$, respectively. The following lemma provides the desired multiplicative analysis of $\mathcal{F}$.

LEMMA 1. *For* $\tilde{f} \in L_2(i\boldsymbol{R}) \otimes H_k$ $(k = 0, 1, 2, \ldots)$,

$$\tilde{\mathcal{F}}\tilde{f}(t, \xi) = \omega(t, k) \tilde{f}(t, \xi) \tag{2.5}$$

where

$$\omega(t, k) = i^k \frac{2^{-t}\Gamma((k + n/2 - t)/2)}{\Gamma((k + n/2 + t)/2)}. \tag{2.6}$$

Proof. Let φ be in H_k $(k = 0, 1, 2, \ldots)$ and g a C^∞ function on $\boldsymbol{R}^+$ with support in a compact subset of the interval $(0, \infty)$. Since linear combinations of functions of the form $f = g\varphi$ are dense in $L_2(\boldsymbol{R}^+) \otimes H_k$, it suffices to show that formula (2.5) holds for the function $\tilde{f} = \mathcal{M}f$. Let $\hat{f} = \hat{\mathcal{F}}f$. It follows from Theorem (2.6.1) in Bochner [1, p. 38] that

$$\hat{f}(x, \xi) = i^k g_1(x) \varphi(\xi) \tag{2.7}$$

where

$$g_1(x) = \int_0^\infty g(yx)J_{k+(n-2)/2}(y)\,dy. \tag{2.8}$$

Set $\nu = k+(n-2)/2$ for convenience in notation. J_ν is the Bessel function of order ν. Now define

$$\tilde{g}_1(z) = \int_0^\infty x^z g_1(x)\frac{dx}{x} = \int_0^\infty\int_0^\infty x^z g(yx)J_\nu(y)\,dy\,\frac{dx}{x}. \tag{2.9}$$

It can be readily verified that the integral in (2.9) converges absolutely for $\operatorname{Re} z < \nu+1$ and defines an analytic function in this domain. From Titchmarsh [5, p. 182],

$$\int_0^\infty x^\alpha J_\nu(x)\frac{dx}{x} = \frac{2^{\alpha-1}\Gamma((\nu+\alpha)/2)}{\Gamma((\nu-\alpha+2)/2)} \tag{2.10}$$

for $-\nu < \operatorname{Re}\alpha < \frac{1}{2}$. For $\frac{1}{2} < \operatorname{Re} z < \nu+1$ interchange of the order of integration in (2.9) is valid and yields the result

$$\tilde{g}_1(z) = \frac{2^{-z}\Gamma((\nu+1-z)/2)}{\Gamma((\nu+1+z)/2)}\tilde{g}(z) \tag{2.11}$$

where

$$\tilde{g}(z) = \int_0^\infty x^z g(x)\frac{dx}{x}.$$

Since both sides of (2.11) define analytic functions of z in the domain $\operatorname{Re} z < \nu+1$, the result remains valid for $z = t \in i\boldsymbol{R}$. Equation (2.5) now follows from (2.7) and (2.11). This completes the proof of the lemma.

3. **Mellin transforms of A and B operators.** For $s \in i\boldsymbol{R}$, let $B(s)$ be the unitary operator on $L_2(\boldsymbol{R}^n)$ defined by

$$\begin{aligned} B(s)f(w) &= |w|^{-s}f(w) && \text{for } w \neq 0,\\ &= 0 && \text{for } w = 0, \end{aligned}$$

and set $A(s) = \mathscr{F}^{-1}B(s)\mathscr{F}$.

This section is devoted to establishing a multiplicative analysis of the operators $A(s)$ and $B(s)$. Prior to this, estimates on the growth of the quotient of gamma functions in Lemma 1 must be obtained.

LEMMA 2. *For $c \in \boldsymbol{R}^+$ and z in the strip $-c < \operatorname{Re} z < c$, define*

$$F(c, z) = \frac{\Gamma((c+z)/2)}{\Gamma((c-z)/2)}.$$

There exists a constant C such that whenever $c > 0$, $\varepsilon > 0$ and $z = x+iy$ with $|x| < c-\varepsilon$, then

$$|F(c,z)| \leqq Ce^{3|x|}2^{-x}(1+2|x|/\varepsilon)|c+2+iy|^x. \tag{3.1}$$

Proof. The proof is modeled after the argument given by Kunze and Stein [3, p. 758] to prove a similar result. From Stirling's formula, for $\alpha>0$ and $w=u+iv\in \boldsymbol{C}$ with $|u|<\alpha$,

$$\log\frac{\Gamma(\alpha+w)}{\Gamma(\alpha-w)} = (\alpha+w-\tfrac{1}{2})\log\frac{\alpha+w}{\alpha+iv}-(\alpha-w-\tfrac{1}{2})\log\frac{\alpha-w}{\alpha-iv} -2w+i(2\alpha-1)\arg(\alpha+iv)+2w\log|\alpha+iv|+B(w) \tag{3.2}$$

where $B(w)$ is a bounded function. For $\alpha>\frac{1}{2}$ and $|u|<\alpha-\frac{1}{2}$, routine estimates show that

$$|(\alpha\pm w-\tfrac{1}{2})\log((\alpha\pm w)/(\alpha\pm iv))| \leqq 2|u|. \tag{3.3}$$

It follows from (3.2) and (3.3) that there exists a constant C independent of α such that

$$|\Gamma(\alpha+w)/\Gamma(\alpha-w)| \leqq Ce^{6|u|}|\alpha+iv|^{2u} \quad \text{for } |u| < \alpha-\tfrac{1}{2}. \tag{3.4}$$

Now let $c>0$, $\varepsilon>0$ and $z=x+iy$ with $|x|<c-\varepsilon$. Replacing α by $(c+2)/2$ and w by $z/2$ in (3.4), we obtain the estimate

$$\begin{aligned} |F(c,z)| &= \left|\frac{c-z}{c+z}\right|\left|\frac{\Gamma((c+2+z)/2)}{\Gamma((c+2-z)/2)}\right| \\ &\leqq \left|\frac{c-z}{c+z}\right| Ce^{3|x|}\left|\frac{c+2+iy}{2}\right|^{x} \leqq C\left(1+\frac{2|x|}{\varepsilon}\right)e^{3|x|}2^{-x}|c+2+iy|^{x}. \end{aligned} \tag{3.5}$$

LEMMA 3. *Let* $s=\sigma_0+i\tau_0$ *with* $|\sigma_0|<n/2-\varepsilon$. *Set* $C_n=C^2e^{3n}2^n$. *For* ω *as in Lemma* 1,

$$\sup_{k\in \boldsymbol{Z}^+;t\in i\boldsymbol{R}} |\omega(t+s,k)\omega(t-s,k)| \leqq C_n(1+n/\varepsilon)^2(1+4/n|\tau_0|)^{n/2}.$$

Proof. From Lemma 2,

$$\begin{aligned} \sup_{k\in \boldsymbol{Z}^+;t=i\tau\in i\boldsymbol{R}} |\omega(t+s,k)\omega(t-s,k)| &= \sup_{k,\tau}|F(k+n/2,-s-t)F(k+n/2,s-t)| \\ &\leqq C_n(1+n/\varepsilon)^2\sup_{k,\tau}\left|\frac{k+n/2+i(\tau_0-\tau)}{k+n/2-i(\tau_0+\tau)}\right|^{\sigma_0} \\ &\leqq C_n(1+n/\varepsilon)^2\sup_{k,\tau}\left|1+\frac{2|\tau_0|}{|k+n/2-i\tau|}\right|^{|\sigma_0|} \\ &\leqq C_n(1+n/\varepsilon)^2(1+4/n|\tau_0|)^{n/2}. \end{aligned}$$

THEOREM 1. (1) $s\to A(s)B(s)$ *is a commutative family of operators for* $s\in i\boldsymbol{R}$.

(2) *There exists an operator-valued function* $s\to C(s)$ *defined on the strip* $-n/2<\operatorname{Re} s<n/2$ *such that*

$$C(s) = A(s)B(s)A(-s)B(-s) \quad \textit{for } \operatorname{Re} s = 0.$$

(3) $\|C(s)\|\leqq C_n(1+n/\varepsilon)^2(1+4/n|\operatorname{Im} s|)^{n/2}$ *for* $\varepsilon>0$ *and* $|\operatorname{Re} s|<n/2-\varepsilon$.

Proof. Let $\tilde{B}(s)$ and $\tilde{A}(s)$ be the unitary operators on $L_2(i\boldsymbol{R})\otimes L_2(S)$ defined by

$$\tilde{B}(s) = (\mathscr{M}\mathscr{U})B(s)(\mathscr{M}\mathscr{U})^{-1}, \qquad \tilde{A}(s) = (\mathscr{M}\mathscr{U})A(s)(\mathscr{M}\mathscr{U})^{-1}.$$

It is easily checked that for $f \in L_2(i\boldsymbol{R}) \otimes L_2(S)$,

$$\tilde{B}(s)f(t, \xi) = f(t-s, \xi), \tag{3.6}$$

$$\tilde{A}(s) = \mathscr{F}^{-1}\tilde{B}(-s)\mathscr{F}. \tag{3.7}$$

From (3.6), (3.7) and Lemma 1, it follows that for $f \in L_2(i\boldsymbol{R}) \otimes H_k$ $(k=0, 1, 2, \ldots)$,

$$\tilde{A}(s)\tilde{B}(s)f(t, \xi) = (\omega(t+s, k)/\omega(t, k))f(t, \xi). \tag{3.8}$$

(1) is an immediate consequence of (3.8). Now for $|\mathrm{Re}\, s| < n/2$, define $\tilde{C}(s)$ on $L_2(i\boldsymbol{R}) \otimes H_k$ by

$$\tilde{C}(s)f(t, \xi) = \frac{\omega(t+s, k)}{\omega(t, k)} \frac{\omega(t-s, k)}{\omega(t, k)} f(t, \xi). \tag{3.9}$$

From Lemma 3, the multiplier in (3.9) is a bounded function of $t \in i\boldsymbol{R}$ and $k \in \boldsymbol{Z}^+$ for each s in the strip $|\mathrm{Re}\, s| < n/2$. Hence (3.9) defines a bounded operator on $L_2(i\boldsymbol{R}) \otimes L_2(S)$. Clearly $s \to \tilde{C}(s)$ is analytic. Defining $C(s) = (\mathscr{M}\mathscr{U})^{-1}\tilde{C}(s)\mathscr{M}\mathscr{U}$, we obtain an analytic operator-valued function on $L_2(\boldsymbol{R}^n)$ satisfying (2) and (3).

4. **Construction of uniformly bounded representations for the Lorentz groups.** For $n \geqq 1$, define $G = SO_e(1, n+1)$ to be the connected component of the identity in the group of $(n+2) \times (n+2)$ real matrices g for which ${}^t g p_0 g = p_0$ where ${}^t g$ is the transpose of g and

$$p_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -I_n & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The family of *Lorentz groups* is the collection $\{SO_e(1, n+1) : n \geqq 1\}$. For $n=1$, the group $SO_e(1, 2)$ is locally isomorphic to $SL(2, \boldsymbol{R})$. Uniformly bounded representations of this group are constructed by Kunze and Stein [2]. A modified version of the Mellin transform analysis used here applies to this group but yields no new representations. We shall therefore assume $n \geqq 2$ in the remainder of this paper.

Let M, A, N and V be the subgroups of G defined as follows:

$$A = \left\{ a = a(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 1/x \end{bmatrix} : x \in \boldsymbol{R}^+ \right\},$$

$$M = \left\{ m = m(h) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1 \end{bmatrix} : h \in SO(n, \boldsymbol{R}) \right\},$$

$$N = \left\{ u = u(w) = \begin{bmatrix} 1 & w & \frac{1}{2}|w|^2 \\ 0 & I_n & {}^t w \\ 0 & 0 & 1 \end{bmatrix} : w \in \boldsymbol{R}^{1 \times n} \right\},$$

$$V = \{ v = v(w) = {}^t u(w) \}.$$

Set $B = MAN$. For n odd, set $J_n = -I_n$ and for n even, set

$$J_n = \begin{bmatrix} -1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & -1 & \\ & & & & 1 \end{bmatrix}.$$

Now set

$$p_n = \begin{bmatrix} 0 & 0 & 1 \\ 0 & J_n & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Throughout the following discussion, n will be fixed and we write J for J_n and p for p_n. Then p is a representative of the nontrivial element of the Weyl group for the rank 1 semisimple Lie group G. B is a closed subgroup of G and by the Bruhat Lemma, $G = B \cup BpB = B \cup BVp$. Set $\tilde{B} = p^{-1}Bp$. Then $\tilde{B} = MAV$ and $G = \tilde{B} \cup \tilde{B}p\tilde{B}$. The set BV is a dense set of full measure in G.

For $s \in i\boldsymbol{R}$, let $T(\cdot, s)$ be the unitary representation of G induced from the character $ma(x)u \to x^s$ of B. The family of representations $T(\cdot, s)$ is the class 1 principal series of G. These representations may be realized as operators on $L_2(V)$ defined by the formulae

$$\text{(4.1)} \qquad \begin{aligned} T(g, s)f(v) &= x^{n/2+s}f(v\cdot g) \quad \text{if } vg = (ma(x)u)(v\cdot g) \in BV, \\ &= 0 \qquad\qquad\qquad\quad \text{if } vg \notin BV. \end{aligned}$$

Since $w \to v(w)$ is a measure preserving homeomorphism of $\boldsymbol{R}^n$ onto V, we may identify $L_2(V)$ with $L_2(\boldsymbol{R}^n)$ by writing $f(w)$ for $f(v(w))$. Note that

$$\text{(4.2)} \qquad v(w)a(x) = a(x)v(xw) \quad \text{for all } (x, w) \in \boldsymbol{R}^+ \times \boldsymbol{R}^n$$

and

$$\text{(4.3)} \qquad v(w)m(h) = m(h)v(wh) \quad \text{for all } (h, w) \in SO(n, \boldsymbol{R}) \times \boldsymbol{R}^n.$$

In (4.3) wh is to be interpreted as the product of the row matrix w and the orthogonal matrix h. From (4.1)–(4.3), it follows that the operators $T(g, s)$ for $g = m(h)a(x)v(w_0) \in \tilde{B}$ are defined on $L_2(\boldsymbol{R}^n)$ by

$$\text{(4.4)} \qquad T(m(h)a(x)v(w_0), s)f(w) = x^{n/2+s}f(xwh + w_0).$$

An easy matrix calculation shows that for $w \neq 0$,

$$v(w)p = m(J - 2^t wwJ/|w|^2)a(2/|w|^2)u(-wJ)v(2wJ/|w|^2).$$

For $f \in L_2(\boldsymbol{R}^n)$ it follows that

$$\text{(4.5)} \qquad \begin{aligned} T(p, s)f(w) &= (2/|w|^2)^{n/2+s}f(2wJ/|w|^2) \quad \text{for } w \neq 0, \\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{if } w = 0. \end{aligned}$$

Since $G=\tilde{B}\cup\tilde{B}p\tilde{B}$, the representation $T(\cdot, s)$ is completely determined by (4.4) and (4.5). Now define $R(\cdot, s)=A(s)T(\cdot, s)A(-s)$.

LEMMA 4. $R(\cdot, s)|_{\tilde{B}}=T(\cdot, 0)|_{\tilde{B}}$.

Proof. Since $A(s)=\mathscr{F}^{-1}B(s)\mathscr{F}$, it suffices to show that

$$B(s)\hat{T}(g, s) = \hat{T}(g, 0)B(s) \tag{4.6}$$

for all $g\in\tilde{B}$ where $\hat{T}(g, s)\equiv\mathscr{F}T(g, s)\mathscr{F}^{-1}$. Let $g=m(h)a(x)v(w_0)$. It follows easily from (4.4) that

$$\hat{T}(g, s)f(w) = x^{-n/2+s}e^{iw_0\cdot x^{-1}wh}f(x^{-1}wh). \tag{4.7}$$

Formula (4.6) is an immediate consequence of (4.7).

LEMMA 5. $A(2s)T(\cdot, s)=T(\cdot, -s)A(2s)$.

Proof. From (4.7) it follows easily that

$$B(2s)\hat{T}(g, s) = \hat{T}(g, -s)B(2s) \quad \text{for all } g\in\tilde{B}.$$

Hence it suffices to show that

$$A(2s)T(p, s) = T(p, -s)A(2s). \tag{4.8}$$

Let $\tilde{T}(p, s)=(\mathscr{M}\mathscr{U})T(p, s)(\mathscr{M}\mathscr{U})^{-1}$. For $f\in L_2(i\boldsymbol{R})\otimes L_2(S)$, it follows from (4.5) that

$$\tilde{T}(p, s)f(t, \xi) = 2^{t-s}f(2s-t, \xi J). \tag{4.9}$$

Note that if $\xi\to g(\xi)$ is a function in H_k, then $\xi\to g(\xi J)$ is also a function in H_k. Hence the subspaces $L_2(i\boldsymbol{R})\otimes H_k$ $(k=0, 1, 2, \dots)$ are invariant under $\tilde{T}(p, s)$. From (2.5), (3.7) and (4.8), it follows that for $f\in L_2(i\boldsymbol{R})\otimes H_k$,

$$\tilde{A}(2s)\tilde{T}(p, s)f = \tilde{T}(p, -s)\tilde{A}(2s)f. \tag{4.10}$$

Formula (4.8) now follows easily from (4.10).

LEMMA 6. (1) $R(\cdot, s)=R(\cdot, -s)$ *for all* $s\in i\boldsymbol{R}$.
(2) $R(p, s)=A(s)B(s)A(-s)B(-s)T(p, 0)$ *for all* $s\in i\boldsymbol{R}$.

Proof. (1)

$$\begin{aligned} R(\cdot, s) &= A(s)T(\cdot, s)A(-s) = A(-s)A(2s)T(\cdot, s)A(-2s)A(s) \\ &= A(-s)T(\cdot, -s)A(s) = R(\cdot, -s). \end{aligned}$$

(2) Formula (4.5) implies that $T(p, s)=2^sB(2s)T(p, 0)$. From Lemma 5 and the fact that $T(p, 0)^2$ is the identity, it follows that

$$T(p, 0)A(\cdot, -s)T(p, 0) = 2^{-s}B(-s)A(-s)B(-s). \tag{4.11}$$

Hence

$$\begin{aligned} R(p, s) &= 2^sA(s)B(2s)T(p, 0)A(-s)T(p, 0)^2 \\ &= A(s)B(s)A(-s)B(-s)T(p, 0). \end{aligned}$$

THEOREM 2. *For each $g \in G$, the function $s \to R(g, s)$ initially defined for $s \in i\boldsymbol{R}$ has an extension to an analytic function on the strip $D=\{s \in \boldsymbol{C} : |\mathrm{Re}\, s| < n/2\}$. The resulting operators $R(g, s)$ have the property that $R(g, s)=R(g, -s)$ for all $g \in G$ and $s \in D$. For all $s \in D$, $g \to R(g, s)$ is a uniformly bounded representation of G.*

Proof. Let $s \in D$. For $g \in \tilde{B}=MAV$, define $R(g, s)=T(g, 0)$. Define $R(p, s) =C(s)T(p, 0)$ where $C(s)$ is the operator-valued function defined in Theorem 1. If $g \in G$ is of the form $g=g_1pg_2$ for g_1 and g_2 in $\tilde{B}$, set $R(g, s)=R(g_1, s)R(p, s) R(g_2, s)$. By the preservation of functional equations under analytic continuation and the fact that $G=\tilde{B} \cup \tilde{B}p\tilde{B}$, it follows that the operators $R(g, s)$ are well defined for all $g \in G$ and $s \in D$ and satisfy the symmetry condition $R(g, s)=R(g, -s)$. Moreover, $g \to R(g, s)$ is a representation of G for all $s \in D$ and $s \to R(g, s)$ is analytic for all $g \in G$. Since $\sup_{g \in G} \|R(g, s)\| = \|R(p, s)\| = \|C(s)\|$, the representations $R(\cdot, s)$ are uniformly bounded.

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