

G_0 OF A GRADED RING⁽¹⁾

BY
LESLIE G. ROBERTS

Abstract. We consider the Grothendieck group G_0 of various graded rings, including $G_0(A'_n)$ where A is a commutative noetherian ring, and A'_n is the A -subalgebra of the polynomial ring $A[X_0, \dots, X_n]$ generated by monomials of degree r . If A is regular, then $G_0(A'_n)$ has a ring structure. The ideal class groups of these rings are also considered.

Introduction. Let A be a commutative noetherian ring with 1, and let $G_0(A)$ denote the Grothendieck group of A -modules of finite type, as in [1]. Let A'_n be the A -subalgebra of $A[X_0, \dots, X_n]$ generated by the monomials of degree r (the X_i are indeterminants). Then we consider $G_0(A'_n)$. My original attack on this problem made use of the exact sequence in Theorem 6.2, p. 492 of [1]. This method is sufficient to handle the case where A is regular, and an exposition for the case where A is a field can be found in [11]. The method used in this paper, that of blowing up the cone $\text{Spec}(A'_n)$ at the vertex, was suggested to me by Grothendieck, and is sufficient to handle the case where A is not regular, as well as other graded rings besides A'_n . The case where A is not commutative, but is regular, can be handled by my original approach—this is done in §8.

The motivation for this paper comes from several sources. The groups $G_0(A'_n)$ are closely related to the groups $K_0(A_n)$ ($K_0(A'_n)$) considered in [5], and I had hoped originally to obtain information about the latter groups by calculating $G_0(A'_n)$. This relation is discussed in §6. Furthermore, the ideal class groups of the A'_n are calculated by Samuel for A a field in [9] (with various restrictions on the field). We have a surjection $G_0(A'_n) \rightarrow Z \oplus c(A'_n)$ (by [4, §4, Proposition 16]), and it will be shown that this map is not necessarily an isomorphism.

Unless otherwise stated, the notation will be that of E.G.A. [6], [7]. Throughout, Z denotes the ring of integers.

2. G_0 of a graded ring. Let $B = \bigoplus_{n \geq 0} B_n$ be a commutative graded ring, such that $B_0 = A$ is noetherian, and B is generated as an A -algebra by a finite number of elements of B_1 . Let $X = \text{Proj}(B)$, and let $f: X \rightarrow \text{Spec } A$ be the structure morphism.

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Let $\mathcal{L} = \mathcal{O}_X(1)$ and let S be the A -algebra $S = A \oplus (\bigoplus_{n \geq 1} \Gamma(\mathcal{L}^{\otimes n}))$. Then by Proposition 8.8.2, p. 178 of [6], we have a commutative diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{j} & V(\mathcal{L}) & \xrightarrow{\pi} & X \\
 f \downarrow & & g \downarrow & & \downarrow f \\
 \text{Spec } A & \xrightarrow{\varepsilon} & \text{Spec } (S) & \xrightarrow{\psi} & \text{Spec } A
 \end{array}$$

where ψ and π are the structure morphisms and ε is the closed immersion corresponding to the augmentation homomorphism $S \rightarrow A$ (that is, the homomorphism sending elements of positive degree to zero). The morphism j is the zero section of $V(\mathcal{L})$. Furthermore the restriction of g to $V(\mathcal{L}) - j(X)$ gives an isomorphism $g: V(\mathcal{L}) - j(X) \rightarrow \text{Spec } (S) - \varepsilon(\text{Spec } A)$.

The canonical homomorphism of graded rings $\alpha: B \rightarrow S$ is a TN -isomorphism by Proposition 2.3.1, p. 102 of [7]. That is, there exists N so that, for $n \geq N$, $\alpha_n: B_n \rightarrow S_n = \Gamma(\mathcal{L}^{\otimes n})$ is an isomorphism. Let x_1, \dots, x_r be elements in B_1 which generate B as an A -algebra. Then the $D(x_i)$ give an open covering of $\text{Spec } B - V(B_+)$, and $D(\alpha(x_i))$ give an open covering of $\text{Spec } S - V(S_+) = \text{Spec } (S) - \varepsilon(\text{Spec } A)$. (Here, as usual, $+$ denotes the elements of positive degree, and $D(f) = \text{Spec} - V(f)$.) Since α is a TN -isomorphism, it follows readily that α induces isomorphisms $\alpha_i: B_{x_i} \rightarrow S_{\alpha(x_i)}$, hence isomorphisms ${}^a\alpha: D(\alpha(x_i)) \rightarrow D(x_i)$. Thus α gives an isomorphism ${}^a\alpha: \text{Spec } S - \varepsilon(\text{Spec } A) \rightarrow \text{Spec } B - V(B_+)$. If we denote the structure morphism and the augmentation morphism of B by ψ and ε also, then we get, by substituting $\text{Spec } B$ in place of $\text{Spec } (S)$, a commutative diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{j} & V(\mathcal{L}) & \xrightarrow{\pi} & X \\
 f \downarrow & & h \downarrow & & \downarrow f \\
 \text{Spec } A & \xrightarrow{\varepsilon} & \text{Spec } B & \xrightarrow{\psi} & \text{Spec } A
 \end{array}$$

where $h = {}^a\alpha \cdot g$. Also restriction of h to $V(\mathcal{L}) - j(X)$ gives an isomorphism $h: V(\mathcal{L}) - j(X) \rightarrow \text{Spec } B - \varepsilon(\text{Spec } A)$. This substitution is unnecessary if α is already an isomorphism, as is the case if $B = A[X_0, \dots, X_n]$ (X_i indeterminants) or if B is integrally closed.

The scheme $V(\mathcal{L})$ is obtained by blowing up S at S_+ . Since Proj is unchanged if the graded ring is changed in a finite number of degrees, $V(\mathcal{L})$ is also obtained by blowing up B at the ideal B_+ . (Blowing up is defined on p. 153 of [6].)

For a noetherian scheme, let K' denote the Grothendieck group of locally free sheaves of finite type, and K , denote the Grothendieck group of coherent sheaves,

as in [10]. Thus $K_*(\text{Spec } B) = G_0(B)$. From the above commutative diagram of schemes, we get a commutative diagram

$$\begin{array}{ccccccc}
 K_*(X) & \xrightarrow{j_*} & K_*(V(\mathcal{L})) & \longrightarrow & K_*(V(\mathcal{L}) - j(X)) & \longrightarrow & 0 \\
 f_* \downarrow & & \downarrow h_* & & \cong \downarrow f_* & & \\
 G_0(A) & \xrightarrow{\varepsilon_*} & G_0(B) & \longrightarrow & K_*(\text{Spec } B - \varepsilon(\text{Spec } A)) & \longrightarrow & 0
 \end{array}$$

The horizontal rows are exact sequences by Proposition 1.1, Exposé IX of [10]. The morphisms f and h are proper, so the homomorphisms f_* and h_* can be defined as in [3, p. 110] (there called f_i).

By Proposition 1.6, Exposé IX of [10], $\pi^*: K_*(X) \rightarrow K_*(V(\mathcal{L}))$ is an isomorphism. (π^* is defined since π is a flat morphism.) Since $\pi \circ j$ is the identity on X , we have that j^* is an inverse for π^* . By Proposition 1.8 Exposé IX of [10], $j^* \circ j_*$ is multiplication by $1 - \mathcal{O}(1) \in K^*(X)$. Identify $K_*(X)$ and $K_*(V(\mathcal{L}))$ by isomorphism j^* . Then the above diagram yields:

THEOREM 1. *Let B be a graded ring as at the beginning of this section, and let $X = \text{Proj}(B)$. Then we have an exact sequence*

$$G_0(A) \xrightarrow{\varepsilon_*} G_0(B) \longrightarrow K_*(X)/(1 - \mathcal{O}(1))K_*(X) \longrightarrow 0,$$

where ε_* is induced by the augmentation homomorphism $B \rightarrow B_0 = A$.

COROLLARY. *If ε_* is zero, then $G_0(B) \cong K_*(X)/(1 - \mathcal{O}(1))K_*(X)$.*

3. Applications. If the augmentation homomorphism $B \xrightarrow{\varepsilon} A$ factors through a pure transcendental extension, $B \xrightarrow{\varepsilon_1} A[X] \xrightarrow{\varepsilon_2} A$, with ε_1 surjective, then ε_* is zero. For ε_* is the composition

$$G_0(A) \xrightarrow{G_0(\varepsilon_2)} G_0(A[X]) \xrightarrow{G_0(\varepsilon_1)} G_0(B),$$

and I claim that $G_0(\varepsilon_2)$ is zero. For let M be any A -module. Then we have an exact sequence of A -modules $0 \rightarrow M[X] \xrightarrow{x} M[X] \rightarrow M \rightarrow 0$, where $M[X] = M \otimes_A A[X]$. This shows that the class of M in $G_0(A[X])$ is zero. Some examples are the following:

(1) $B = A_n^r$, as defined in the introduction. (If the degrees are divided by r , then the above hypotheses on B are satisfied.) Then $A_n^r \rightarrow A$ factors through $A[X]$ by sending X_0^r to X and any term involving X_i ($i > 0$) to zero. Also $\text{Proj}(A_n^r) = P_A^n$ (projective n -space over A), and

$$K_*(P_A^n) = G_0(A)[X]/(X^{n+1}) \quad (= G_0(A) \otimes_Z Z[X]/(X^{n+1}))$$

(Proposition 3.1, Exposé IX of [10]). We may choose X so that the class of $\mathcal{O}(1)$ in $K_*(P_A^n)$ is $(1 + X)^r$. Thus we have

THEOREM 2. *Let A be a commutative noetherian ring with unit, and let A_n^r be as defined in the introduction. Then $G_0(A_n^r) = G_0(A) \otimes_Z Z[X]/(X^{n+1}, (1+X)^r - 1)$.*

COROLLARY. *If $G_0(A) = Z$, then $G_0(A_n^r) = Z[X]/(X^{n+1}, (1+X)^r - 1)$.*

(2) Let B be the homogeneous coordinate ring of an algebraic variety V over an algebraically closed field $K = B_0$. Let P be any closed point of $\text{Spec } B$ (other than the origin). Then the straight line through the origin and P is an affine line which is a closed subscheme of $\text{Spec } B$. Thus we have a surjection $B \rightarrow K[X]$ and the corollary to Theorem 1 again applies, yielding $G_0(B) = K(V)/(1 - \mathcal{O}(1))K(V)$. For example, if V is a complete nonsingular curve, then $K(V) = Z \oplus \text{Pic}(V) = Z \oplus Z \oplus \text{Pic}_0(V)$, and $\mathcal{O}(1)$ is the class of a hyperplane section. If $\mathcal{O}(1)$ is of degree r , the direct sum decomposition $\text{Pic}(V) = Z \oplus \text{Pic}_0(V)$ can be chosen so that the class of $\mathcal{O}(1)$ is $r \oplus 0$. Thus we get $G_0(B) = Z \oplus Z/rZ \oplus \text{Pic}_0(V)$.

More generally, let $B = K[x_1, \dots, x_n]$ be the homogeneous coordinate ring of an algebraic variety over field K that is not necessarily algebraically closed, with the x_i homogeneous of degree one. Let P be a closed point (other than the vertex) with residue field K_P . Then K_P is a finite extension of K , of degree d_P , and we have a surjection $e_P: K[x_1, \dots, x_n] \rightarrow K_P$. Suppose $e_P(x_i) = \bar{x}_i$. Then we have a homomorphism $K[x_1, \dots, x_n] \rightarrow K_P[X]$ (X an indeterminate) defined by $x_i \rightarrow \bar{x}_i X$. This map is well defined since $K[x_1, \dots, x_n] = K[X_1, \dots, X_n]/I$, where the X_i are indeterminates, and I is a homogeneous ideal. It is easily seen that $K_P[X]$ is a $K[x_1, \dots, x_n]$ -module of finite type. Thus we have a commutative diagram

$$\begin{array}{ccc} K[x_1, \dots, x_n] & \longrightarrow & K_P[X] \\ \varepsilon \downarrow & & \downarrow \\ K & \longrightarrow & K_P \end{array}$$

and the induced maps on G_0 show that the image of ε_* is killed by d_P . Thus the image of ε_* is killed by the greatest common divisor of the d_P , as P ranges over all the closed points of $\text{Spec } B$ (other than the vertex). In particular, if there is a K -rational point, then $\varepsilon_* = 0$ and the corollary to Theorem 1 applies.

4. The ring structure on G_0 . Suppose A is a commutative noetherian regular ring of finite Krull dimension d . Then $G_0(A) = K_0(A)$ is a ring, so

$$G_0(A_n^r) = K_0(A) \otimes_Z Z[X]/(X^{n+1}, (1+X)^r - 1) = K_0(A)[X]/(X^{n+1}, (1+X)^r - 1)$$

is a ring. If $r > 1$, then A_n^r is not a regular ring, so one might not expect $G_0(A_n^r)$ to have a ring structure. However, $\text{Spec } A_n^r$ is regular outside of $V((A_n^r)_+)$, and we have shown that $G_0(A_n^r) = K(\text{Spec } A_n^r - V((A_n^r)_+))$ and the latter equals $K(\text{Spec } A_n^r - V((A_n^r)_+))$ which has a ring structure. The ring structure can be described explicitly in terms of A_n^r -modules of finite type as follows. Let M and N be A_n^r -modules of finite type, and let $[M]$ and $[N]$ denote their classes in $G_0(A_n^r)$.

Then $[M][N] = \sum_{i=0}^{n+d+1} (-1)^i [\text{Tor}_i(M, N)]$. This is a well-defined multiplication, since if $0 \rightarrow N_1 \rightarrow N \rightarrow N_2 \rightarrow 0$ is a short exact sequence, then we have an exact sequence $0 \rightarrow X \rightarrow \text{Tor}_{n+d+1}(M, N_1) \rightarrow \dots \rightarrow M \otimes N \rightarrow M \otimes N_2 \rightarrow 0$. The module X has support contained in $V((A'_n)_+)$ since A'_n is of Krull dimension $n+d+1$ and is regular outside of $V((A'_n)_+)$. Thus $[X]=0$ in $G_0(A'_n)$. Note that if M and N are locally free outside of $\varepsilon(\text{Spec } A)$, then $[M][N]=[M \otimes N]$.

If A is any noetherian ring of finite Krull dimension, and any module whose support is contained in the singular locus has zero image in $G_0(A)$, then one can put a ring structure on $G_0(A)$ in the same manner as above. The above examples indicate that such rings are fairly common.

5. The abelian group structure. We consider the abelian group structure of the ring $Z[X]/(X^{n+1}, (1+X)^r - 1) = B_{n,r}$. Multiplication by X gives a homomorphism $B_{n-1,r} \xrightarrow{X} B_{n,r}$ with cokernel Z . Let $F \in Z[X]$ represent an element in the kernel. Then $XF = AX^{n+1} + B((1+X)^r - 1)$ with $A, B \in Z[X]$, and $F = AX^n + BG$, where $G = ((1+X)^r - 1)/X$. But X^n is already zero in $B_{n-1,r}$. Thus the kernel is principal, generated by the class of $G = r + (\text{terms involving } X)$. This element of $B_{n-1,r}$ is of infinite order so we have an exact sequence

$$0 \longrightarrow Z \longrightarrow B_{n-1,r} \xrightarrow{X} B_{n,r} \longrightarrow Z \longrightarrow 0.$$

Clearly $B_{n,r}$ is of rank one, so write $B_{n,r} = Z \oplus T_n^r$. The last homomorphism in the exact sequence is projecting onto the first direct summand. Thus we have an exact sequence $0 \rightarrow Z \xrightarrow{G} Z \oplus T_{n-1}^r \rightarrow T_n^r \rightarrow 0$. Form the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & 0 & \longrightarrow & T_{n-1}^r & \xrightarrow{1} & T_{n-1}^r \longrightarrow 0 \\
 & & \downarrow & & \downarrow i_2 & & \\
 0 & \longrightarrow & Z & \xrightarrow{G} & Z \oplus T_{n-1}^r & \longrightarrow & T_n^r \longrightarrow 0 \\
 & & \downarrow 1 & & \downarrow \pi_1 & & \\
 0 & \longrightarrow & Z & \longrightarrow & Z & \longrightarrow & Z/rZ \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

All rows and columns are exact. Therefore there exist maps in the last column

making the diagram commute and the last column exact. Therefore we have an exact sequence

$$0 \rightarrow T_{n-1}^r \rightarrow T_n^r \rightarrow Z/rZ \rightarrow 0.$$

Since $T_0^r=0$, we have $T_1^r=Z/rZ$, and more generally that T_n^r is of order r^n .

Now suppose that $r=r_1r_2$ where r_1 and r_2 are relatively prime. Then we have a surjection of rings $B_{n,r_1r_2} \rightarrow B_{n,r_1} \rightarrow 0$ and hence a surjection $T_{n,r_1r_2}^r \rightarrow T_{n,r_1}^r \rightarrow 0$. This splits since T_{n,r_1}^r has order r_1^n , the kernel has order r_2^n , and $(r_1, r_2)=1$. Thus T_{n,r_1}^r is the subgroup of $T_{n,r_1r_2}^r$ consisting of all elements killed by r_1^n . Similarly for r_2 , so $T_{n,r_1r_2}^r=T_{n,r_1}^r \oplus T_{n,r_2}^r$. This reduces the problem to r a prime power.

The abelian group T_n^r has generators X, X^2, \dots, X^n , and relations $\sum_{i=1}^r C_i^r X^{i+s}$, $0 \leq s \leq n-1$, where the C_i^r are the binomial coefficients, and the summations stop with the term X^n .

Suppose first that r is a prime. Then $C_1^r=r$, and C_i^r has a factor r , $2 \leq i < r-1$. By subtracting the above relations from each other in a suitable way, we get a new equivalent set of relations of the form

$$rX^{1+s} + X^{r+s} + \sum_{i>r} a_i X^{i+s}, \quad 0 \leq s < n-1,$$

(again stopping each summation with the term X^n). By induction define a new set of generators e_1, \dots, e_n by $e_i = X^i$, $1 \leq i \leq r-1$, and $re_i = -e_{i+r-1}$. By induction we prove that $e_i = X^i + \sum_{j>i} b_j X^j$. For suppose $e_i = X^i + \sum_{j>i} b_j X^j$. Then $re_i = rX^i + \sum_{j>i} b_j r X^j = -X^{i+r-1} - \sum_{j>i+r-1} b'_j X^j$, so $e_{i+r-1} = X^{i+r-1} + \sum_{j>i+r-1} b'_j X^j$ as required. Thus e_1, \dots, e_n generate T_n^r . Assuming that these relations among the e_i are all there are, then T_n^r has the following abelian group structure. If $1 \leq n \leq r-1$, then T_n^r is the direct sum of n copies of Z/rZ (with generators X, X^2, \dots, X^n respectively). As n varies from r to $2r-1$, the copies of Z/rZ are changed one after the other into Z/r^2Z . Then they are changed one after the other into Z/r^3Z , and so on. We get a group of the required order, hence we have enough relations among the e_i . In general, the generators of the cyclic groups will be X, X^2, \dots, X^t where $t = \inf(n, r-1)$.

Now assume $r=p^a$, $a > 1$. Each of the C_i^r , $2 \leq i < p$, has a factor r . Thus, as above, we can change our relations to $rX^{1+s} + \sum_{i \geq p} a_i X^{i+s}$, $0 \leq s < n-1$. The integer a_p is divisible by p^{a-1} , but not by p^a . Thus, for $1 \leq n \leq p-1$, T_n^r is the direct sum of n copies of Z/rZ , and in T_p^r one of the copies of Z/rZ is changed into $Z/p^{a+1}Z \oplus Z/p^{a-1}Z$. After this, things seem more complicated.

Added in proof. The abelian group structure of $B_{n,r}$ has been completely determined. See A. Chabour, C. R. Acad. Sci. Paris Sér. A **272** (1971), p. 462.

6. Relation with open subsets of proj. Let $f \in A_n^r$ be homogeneous of degree r . Then $A_n^r[f^{-1}]$ is graded (in positive and negative degrees). Let G_n^r be the degree 0 part. Then $A_n^r[f^{-1}] = G_n^r[f, f^{-1}]$ where f is an indeterminate over G_n^r . Also $G_n^r \cong A_n^r/(f-1)A_n^r$ (Proposition 2.2.5, p. 24 of [6]). Finally $D_+(f) = \text{Spec}(G_n^r)$. We now

have a surjection $G_0(A_n^r) \rightarrow G_0(A_n^r[f^{-1}]) = G_0(G_n^r[f, f^{-1}]) \cong G_0(G_n^r)$. This yields the theorem on p. 299 of [5], but with A no longer required to be regular, and the Grothendieck group G_0 replacing K_0 .

Now suppose that $r=2$, A is regular, and $G_0(A) = K_0(A) = Z$. This puts us in the situation discussed at the end of §6 in [5]. The above surjection reads

$$Z[X]/(X^{n+1}, (1+X)^2-1) \rightarrow Z[X]/(X^{l+1}, (1+x)^2-1) = K_0(G_n^r), \quad l \leq n.$$

It was originally hoped that calculating $G_0(A_n^r)$ would lower the upper bound on l , but this has turned out not to be the case. At least, the possibility remains that l could be as large as n , for suitable choice of A and f .

7. The ideal class groups. The results in this section are included for comparison with G_0 . Let B be a graded Krull ring (in positive degrees.) Let $DH(B)$ be the free abelian group on homogeneous prime ideals of height one, and let $FH(B)$ be the subgroup of $DH(B)$ generated by the divisors of homogeneous elements of B . Then $c(B) = DH(B)/FH(B)$ by Proposition 7.1, p. 24 of [9]. Let $X = \text{Proj}(B)$, and $K = \{f/g \mid f, g \text{ are homogeneous elements in } B \text{ of same degree}\} = \text{stalk of } X \text{ at the generic point}$. The irreducible closed subsets of codimension one in X correspond to homogeneous primes of height one in B (assuming B_+ is not of height ≤ 1). Let X_1 be the set of generic points of irreducible subsets of codimension one in X . Then for all $P \in X_1$, the stalk of X at P is a discrete valuation ring, so we can define the divisor of $h \in K$ by $(h) = \sum v_P(h) \cdot P$ as P ranges over X_1 . If, as above, we identify the P 's with the homogeneous primes of height one in B it is readily seen that this divisor of h is the same as the divisor of h regarded as an element in the quotient field of B . (The last observation shows that $v_P(h) = 0$ for all but a finite number of $P \in X_1$.) Thus if we let $c(X) = \text{free abelian group on irreducible subsets of codimension one of } X$, factored out by the subgroup generated by the divisors of functions (i.e., elements of K), then $c(X) = DH(B)/FH_0(B)$, where $FH_0(B)$ is the subgroup of $DH(B)$ generated by the divisors of elements of K .

We now have an exact sequence

$$0 \rightarrow FH(B)/FH_0(B) \rightarrow c(X) \rightarrow c(B) \rightarrow 0.$$

The homomorphism $\text{degree: } FH(B) \rightarrow Z$ has kernel $FH_0(B)$, and the image will be isomorphic to Z . Thus $FH(B)/FH_0(B) \cong Z$ (noncanonically) and our exact sequence reads $0 \rightarrow Z \rightarrow c(X) \rightarrow c(B) \rightarrow 0$.

Now take $X = P_A^n = \text{Proj } A[X_0, \dots, X_n]$, where A is a Krull ring. Then $c(B) = c(A)$, so the exact sequence reads

$$0 \rightarrow Z \rightarrow c(P_A^n) \rightarrow c(A) \rightarrow 0.$$

Let L be the quotient field of A . Then we have a splitting homomorphism $c(P_A^n) \rightarrow c(P_L^n) = Z$. Thus $c(P_A^n) = Z \oplus c(A)$. We also have $P_A^n = \text{Proj}(A_n^r)$. The ring A_n^r is a Krull ring since $A_n^r = K_n^r \cap A[X_0, \dots, X_n]$, where K_n^r is the quotient field of A_n^r .

The exact sequence reads $0 \rightarrow Z \rightarrow Z \oplus c(A) \rightarrow c(A_n^r) \rightarrow 0$. The image of $1 \in Z$ is r , so we get $c(A_n^r) = Z/rZ \oplus c(A)$. This shows that the homomorphism $G_0(A_n^r) \rightarrow Z \oplus c(A_n^r)$ (where A is noetherian integrally closed) is not an isomorphism if $r > 1$.

Divisorial ideals (of A_n^r) corresponding to the elements of Z/rZ can be taken as $I_i = X_0^{r-i} J_i$, where J_i is the A_n^r -submodule of $A[X_0, \dots, X_n]$ generated by the monomials of degree i , $0 \leq i \leq r-1$. The ideal I_i is divisorial since $I_i = A_n^r \cap (X_0^{r-i}/X_1^{r-i})$, but is not invertible if $i > 0$.

Now we remark that if $B = \bigoplus_{n \geq 0} B_n$ is a graded Krull ring, then $\text{Pic } B = \text{Pic } B_0$. The inclusion $B_0 \rightarrow B$ induces, by $\otimes_{B_0} B$, a monomorphism $\text{Pic}(B_0) \rightarrow \text{Pic } B$ which we need only show is onto. Proposition 7.1, p. 24 of [9], shows that it is sufficient to consider graded projective modules of rank one, and by Proposition 3.3, p. 637 of [1], these are all of the form $P_0 \otimes_{B_0} B$. This is a slight generalization of Lemma 5.1 of [8].

The results of this section can be summarized as follows:

THEOREM 3. *Let B be a graded Krull ring (in positive degrees) such that the elements of positive degree do not form a prime ideal of height ≤ 1 . Let $X = \text{Proj}(B)$. Then there is an exact sequence $0 \rightarrow Z \rightarrow c(X) \rightarrow c(B) \rightarrow 0$. If A is any Krull ring, then $c(A_n^r) = Z/rZ \oplus c(A)$. Finally, if B is any graded Krull ring in positive degrees, then $\text{Pic } B = \text{Pic } B_0$.*

8. The noncommutative case. Let A be a right noetherian ring (as defined in [1, p. 122]). Let A_n^r be the A -subalgebra of $B_n = A[X_0, \dots, X_n]$ generated by monomials of degree r . Let S be the (central) multiplicative set generated by X_n^r . Then $S^{-1}A_n^r = A[X_0/X_n, X_1/X_n, \dots, X_{n-1}/X_n, X_n^r, X_n^{-r}]$, which is right regular. Thus we may apply Theorem 6.2, p. 492 of [1], to get an exact sequence

$$K_1(S^{-1}A_n^r) \xrightarrow{\partial} G_0(A_n^r/X_n^r A_n^r) \longrightarrow G_0(A_n^r) \longrightarrow K_0(S^{-1}A_n^r) \longrightarrow 0.$$

But $K_1(S^{-1}A_n^r) = K_0(A) \oplus K_1(A)$ and $K_0(S^{-1}A_n^r) = K_0(A)$ by Theorem 2 and Theorem 1 of [2], respectively. We have a surjection $A_n^r/X_n^r A_n^r \rightarrow A_{n-1}^r$ with nilpotent kernel. Thus by Proposition 2.3, p. 454 of [1], we have an isomorphism $G_0(A_{n-1}^r) \rightarrow G_0(A_n^r/X_n^r A_n^r)$. Furthermore ∂ kills $K_1(A)$, since in the statement of Proposition 6.1, p. 492 of [1], α can already be taken as an automorphism. Thus we get an exact sequence

$$K_0(A) \rightarrow G_0(A_{n-1}^r) \rightarrow G_0(A_n^r) \rightarrow K_0(A) \rightarrow 0.$$

If M is a Z_n^r module, then $M \rightarrow M \otimes_Z A - \text{Tor}_1^Z(M, A)$ defines a homomorphism $G_0(Z_n^r) \rightarrow G_0(A_n^r)$. We now have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & G_0(Z_{n-1}^r) & \longrightarrow & G_0(Z_n^r) & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & K_0(A) & \longrightarrow & G_0(A_{n-1}^r) & \longrightarrow & G_0(A_n^r) & \longrightarrow & K_0(A) & \longrightarrow & 0 \end{array}$$

In §2, the rings $G_0(Z_n^r)$ were found to be $Z[X]/(X^{n+1}, (1+X)^r) = B_{n,r}$. If we tensor the top row with $K_0(A)$ the sequence remains exact (except at the left) since the top row is the direct sum of two short exact sequences $0 \rightarrow Z \rightarrow B_{n-1,r} \rightarrow T_n^r \rightarrow 0$ and $0 \rightarrow Z \rightarrow Z \rightarrow 0$. The groups in the bottom row are $K_0(A)$ -modules. Thus we get a commutative diagram

$$\begin{array}{ccccccc}
 K_0(A) & \longrightarrow & B_{n-1,r} \otimes_Z K_0(A) & \longrightarrow & B_{n,r} \otimes_Z K_0(A) & \longrightarrow & K_0(A) \longrightarrow 0 \\
 \approx \downarrow & & \downarrow f_{n-1} & & \downarrow f_n & & \downarrow \approx \\
 K_0(A) & \longrightarrow & G_0(A_{n-1}^r) & \longrightarrow & G_0(A_n^r) & \longrightarrow & K_0(A) \longrightarrow 0
 \end{array}$$

The diagram is commutative by the universal mapping property of \otimes . Now f_0 is an isomorphism. Therefore we may use the five-lemma to conclude by induction that f_n is an isomorphism for all n . This proves

THEOREM 4. *Let A be a right noetherian regular ring. Then $G_0(A_n) \cong K_0(A) \otimes_Z Z[X]/(X^{n+1}, (1+X)^r - 1)$.*

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