TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 166, April 1972

G_0 OF A GRADED RING(1)

BY
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Abstract. We consider the Grothendieck group G_0 of various graded rings, including $G_0(A_n^r)$ where A is a commutative noetherian ring, and A_n^r is the A-subalgebra of the polynomial ring $A[X_0, \ldots, X_n]$ generated by monomials of degree r. If A is regular, then $G_0(A_n^r)$ has a ring structure. The ideal class groups of these rings are also considered.

Introduction. Let A be a commutative noetherian ring with 1, and let $G_0(A)$ denote the Grothendieck group of A-modules of finite type, as in [1]. Let A_n^r be the A-subalgebra of $A[X_0, \ldots, X_n]$ generated by the monomials of degree r (the X_i are indeterminants). Then we consider $G_0(A_n^r)$. My original attack on this problem made use of the exact sequence in Theorem 6.2, p. 492 of [1]. This method is sufficient to handle the case where A is regular, and an exposition for the case where A is a field can be found in [11]. The method used in this paper, that of blowing up the cone Spec (A_n^r) at the vertex, was suggested to me by Grothendieck, and is sufficient to handle the case where A is not regular, as well as other graded rings besides A_n^r. The case where A is not commutative, but is regular, can be handled by my original approach—this is done in §8.

The motivation for this paper comes from several sources. The groups $G_0(A_n^r)$ are closely related to the groups $K_0(A_n)$ ($K_0(A_n')$) considered in [5], and I had hoped originally to obtain information about the latter groups by calculating $G_0(A_n^r)$. This relation is discussed in §6. Furthermore, the ideal class groups of the A_n^r are calculated by Samuel for A a field in [9] (with various restrictions on the field). We have a surjection $G_0(A_n^r) \to Z \oplus c(A_n^r)$ (by [4, §4, Proposition 16]), and it will be shown that this map is not necessarily an isomorphism.

Unless otherwise stated, the notation will be that of E.G.A. [6], [7]. Throughout, Z denotes the ring of integers.

2. **G_0 of a graded ring.** Let $B = \bigoplus_{n \geq 0} B_n$ be a commutative graded ring, such that $B_0 = A$ is noetherian, and B is generated as an A-algebra by a finite number of elements of B_1. Let $X = \text{Proj}(B)$, and let $f\colon X \to \text{Spec } A$ be the structure morphism.

Received by the editors April 27, 1971.

AMS 1970 *subject classifications*. Primary 13D15, 14F15, 18F25; Secondary 16A54.

Key words and phrases. Grothendieck group, G_0, graded ring, ideal class group.

(1) The preparation of this paper was partially supported by the National Research Council of Canada, grant A7209.

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Let $\mathscr{L}=\mathcal{O}_X(1)$ and let S be the A-algebra $S=A\oplus(\bigoplus_{n\geqq 1}\Gamma(\mathscr{L}^{\otimes n}))$. Then by Proposition 8.8.2, p. 178 of [6], we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{j} & V(\mathscr{L}) & \xrightarrow{\pi} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} A & \xrightarrow[\varepsilon]{} & \operatorname{Spec}(S) & \xrightarrow[\psi]{} & \operatorname{Spec} A \end{array}$$

where ψ and π are the structure morphisms and ε is the closed immersion corresponding to the augmentation homomorphism $S\to A$ (that is, the homomorphism sending elements of positive degree to zero). The morphism j is the zero section of $V(\mathscr{L})$. Furthermore the restriction of g to $V(\mathscr{L})-j(X)$ gives an isomorphism $g\colon V(\mathscr{L})-j(X)\to \operatorname{Spec}(S)-\varepsilon(\operatorname{Spec} A)$.

The canonical homomorphism of graded rings $\alpha\colon B\to S$ is a TN-isomorphism by Proposition 2.3.1, p. 102 of [7]. That is, there exists N so that, for $n\geqq N$, $\alpha_n\colon B_n\to S_n=\Gamma(\mathscr{L}^{\otimes n})$ is an isomorphism. Let $x_1,\ldots,x_r$ be elements in B_1 which generate B as an A-algebra. Then the $D(x_i)$ give an open covering of $\operatorname{Spec} B-V(B_+)$, and $D(\alpha(x_i))$ give an open covering of $\operatorname{Spec} S-V(S_+)=\operatorname{Spec}(S)-\varepsilon(\operatorname{Spec} A)$. (Here, as usual, $+$ denotes the elements of positive degree, and $D(f)=\operatorname{Spec}-V(f)$.) Since α is a TN-isomorphism, it follows readily that α induces isomorphisms $\alpha_i\colon B_{x_i}\to S_{\alpha(x_i)}$, hence isomorphisms ${}^a\alpha\colon D(\alpha(x_i))\to D(x_i)$. Thus α gives an isomorphism ${}^a\alpha\colon \operatorname{Spec} S-\varepsilon(\operatorname{Spec} A)\to \operatorname{Spec} B-V(B_+)$. If we denote the structure morphism and the augmentation morphism of B by ψ and ε also, then we get, by substituting $\operatorname{Spec} B$ in place of $\operatorname{Spec}(S)$, a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{j} & V(\mathscr{L}) & \xrightarrow{\pi} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} A & \xrightarrow[\varepsilon]{} & \operatorname{Spec} B & \xrightarrow[\psi]{} & \operatorname{Spec} A \end{array}$$

where $h={}^a\alpha\cdot g$. Also restriction of h to $V(\mathscr{L})-j(X)$ gives an isomorphism $h\colon V(\mathscr{L})-j(X)\to \operatorname{Spec} B-\varepsilon(\operatorname{Spec} A)$. This substitution is unnecessary if α is already an isomorphism, as is the case if $B=A[X_0,\ldots,X_n]$ (X_i indeterminants) or if B is integrally closed.

The scheme $V(\mathscr{L})$ is obtained by blowing up S at S_+. Since Proj is unchanged if the graded ring is changed in a finite number of degrees, $V(\mathscr{L})$ is also obtained by blowing up B at the ideal B_+. (Blowing up is defined on p. 153 of [6].)

For a noetherian scheme, let $K^{\bullet}$ denote the Grothendieck group of locally free sheaves of finite type, and $K_{\bullet}$ denote the Grothendieck group of coherent sheaves,

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as in [10]. Thus $K_.(\operatorname{Spec} B)=G_0(B)$. From the above commutative diagram of schemes, we get a commutative diagram

$$\begin{array}{ccccccc} K_.(X) & \xrightarrow{j_*} & K_.(V(\mathscr{L})) & \longrightarrow & K_.(V(\mathscr{L})-j(X)) & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle h_*} & & \cong\downarrow{\scriptstyle f_*} & & \\ G_0(A) & \xrightarrow[\varepsilon_*]{} & G_0(B) & \longrightarrow & K_.(\operatorname{Spec} B-\varepsilon(\operatorname{Spec} A)) & \longrightarrow & 0 \end{array}$$

The horizontal rows are exact sequences by Proposition 1.1, Exposé IX of [10]. The morphisms f and h are proper, so the homomorphisms f_* and h_* can be defined as in [3, p. 110] (there called $f_!$).

By Proposition 1.6, Exposé IX of [10], $\pi^*\colon K_.(X)\to K_.(V(\mathscr{L}))$ is an isomorphism. (π^* is defined since π is a flat morphism.) Since $\pi\circ j$ is the identity on X, we have that j^* is an inverse for π^*. By Proposition 1.8 Exposé IX of [10], $j^*\circ j_*$ is multiplication by $1-\mathscr{O}(1)\in K^{\cdot}(X)$. Identify $K_.(X)$ and $K_.(V(\mathscr{L}))$ by isomorphism j^*. Then the above diagram yields:

THEOREM 1. *Let B be a graded ring as at the beginning of this section, and let $X=\operatorname{Proj}(B)$. Then we have an exact sequence*

$$G_0(A)\xrightarrow[\varepsilon_*]{} G_0(B)\longrightarrow K_.(X)/(1-\mathscr{O}(1))K_.(X)\longrightarrow 0,$$

where ε_* is induced by the augmentation homomorphism $B\to B_0=A$.

COROLLARY. *If ε_* is zero, then $G_0(B)\cong K_.(X)/(1-\mathscr{O}(1))K_.(X)$.*

3. **Applications.** If the augmentation homomorphism $B\xrightarrow{\varepsilon} A$ factors through a pure transcendental extension, $B\xrightarrow{\varepsilon_1} A[X]\xrightarrow{\varepsilon_2} A$, with ε_1 surjective, then ε_* is zero. For ε_* is the composition

$$G_0(A)\xrightarrow{G_0(\varepsilon_2)} G_0(A[X])\xrightarrow{G_0(\varepsilon_1)} G_0(B),$$

and I claim that $G_0(\varepsilon_2)$ is zero. For let M be any A-module. Then we have an exact sequence of A-modules $0\longrightarrow M[X]\xrightarrow{X} M[X]\longrightarrow M\longrightarrow 0$, where $M[X]=M\otimes_A A[X]$. This shows that the class of M in $G_0(A[X])$ is zero. Some examples are the following:

(1) $B=A^r_n$, as defined in the introduction. (If the degrees are divided by r, then the above hypotheses on B are satisfied.) Then $A^r_n\to A$ factors through $A[X]$ by sending X^r_0 to X and any term involving X_i $(i>0)$ to zero. Also $\operatorname{Proj}(A^r_n)=P^n_A$ (projective n-space over A), and

$$K_.(P^n_A) = G_0(A)[X]/(X^{n+1}) \qquad (= G_0(A)\otimes_Z Z[X]/(X^{n+1}))$$

(Proposition 3.1, Exposé IX of [10]). We may choose X so that the class of $\mathscr{O}(1)$ in $K_.(P^n_A)$ is $(1+X)^r$. Thus we have

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THEOREM 2. *Let A be a commutative noetherian ring with unit, and let A_n^r be as defined in the introduction. Then* $G_0(A_n^r)=G_0(A)\otimes_Z Z[X]/(X^{n+1}, (1+X)^r-1)$.

COROLLARY. *If $G_0(A)=Z$, then* $G_0(A_n^r)=Z[X]/(X^{n+1}, (1+X)^r-1)$.

(2) Let B be the homogeneous coordinate ring of an algebraic variety V over an algebraically closed field $K=B_0$. Let P be any closed point of Spec B (other than the origin). Then the straight line through the origin and P is an affine line which is a closed subscheme of Spec B. Thus we have a surjection $B\to K[X]$ and the corollary to Theorem 1 again applies, yielding $G_0(B)=K_.(V)/(1-\mathcal{O}(1))K_.(V)$. For example, if V is a complete nonsingular curve, then $K_.(V)=Z\oplus \operatorname{Pic}(V) =Z\oplus Z\oplus \operatorname{Pic}_0(V)$, and $\mathcal{O}(1)$ is the class of a hyperplane section. If $\mathcal{O}(1)$ is of degree r, the direct sum decomposition $\operatorname{Pic}(V)=Z\oplus\operatorname{Pic}_0(V)$ can be chosen so that the class of $\mathcal{O}(1)$ is $r\oplus 0$. Thus we get $G_0(B)=Z\oplus Z/rZ\oplus\operatorname{Pic}_0(V)$.

More generally, let $B=K[x_1,\dots,x_n]$ be the homogeneous coordinate ring of an algebraic variety over field K that is not necessarily algebraically closed, with the x_i homogeneous of degree one. Let P be a closed point (other than the vertex) with residue field K_P. Then K_P is a finite extension of K, of degree d_P, and we have a surjection $e_P: K[x_1,\dots,x_n]\to K_P$. Suppose $e_P(x_i)=\bar{x}_i$. Then we have a homomorphism $K[x_1,\dots,x_n]\to K_P[X]$ (X an indeterminant) defined by $x_i\to\bar{x}_iX$. This map is well defined since $K[x_1,\dots,x_n]=K[X_1,\dots,X_n]/I$, where the X_i are indeterminants, and I is a homogeneous ideal. It is easily seen that $K_P[X]$ is a $K[x_1,\dots,x_n]$-module of finite type. Thus we have a commutative diagram

$$\begin{array}{ccc} K[x_1,\dots,x_n] & \longrightarrow & K_P[X] \\ \varepsilon\big\downarrow & & \big\downarrow \\ K & \longrightarrow & K_P \end{array}$$

and the induced maps on G_0 show that the image of ε_* is killed by d_P. Thus the image of ε_* is killed by the greatest common divisor of the d_P, as P ranges over all the closed points of Spec B (other than the vertex). In particular, if there is a K-rational point, then $\varepsilon_*=0$ and the corollary to Theorem 1 applies.

4. **The ring structure on G_0.** Suppose A is a commutative noetherian regular ring of finite Krull dimension d. Then $G_0(A)=K_0(A)$ is a ring, so

$$G_0(A_n^r) = K_0(A)\otimes_Z Z[X]/(X^{n+1}, (1+X)^r-1) = K_0(A)[X]/(X^{n+1}, (1+X)^r-1)$$

is a ring. If $r>1$, then A_n^r is not a regular ring, so one might not expect $G_0(A_n^r)$ to have a ring structure. However, Spec A_n^r is regular outside of $V((A_n^r)_+)$, and we have shown that $G_0(A_n^r)=K_.(\operatorname{Spec} A_n^r-V((A_n^r)_+))$ and the latter equals $K^{\cdot}(\operatorname{Spec} A_n^r-V((A_n^r)_+))$ which has a ring structure. The ring structure can be described explicitly in terms of A_n^r-modules of finite type as follows. Let M and N be A_n^r-modules of finite type, and let $[M]$ and $[N]$ denote their classes in $G_0(A_n^r)$.

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Then $[M][N]=\sum_{i=0}^{n+d+1}(-1)^i[\mathrm{Tor}_i(M,N)]$. This is a well-defined multiplication, since if $0\to N_1\to N\to N_2\to 0$ is a short exact sequence, then we have an exact sequence $0\to X\to \mathrm{Tor}_{n+d+1}(M,N_1)\to\cdots\to M\otimes N\to M\otimes N_2\to 0$. The module X has support contained in $V((A_n^r)_+)$ since A_n^r is of Krull dimension $n+d+1$ and is regular outside of $V((A_n^r)_+)$. Thus $[X]=0$ in $G_0(A_n^r)$. Note that if M and N are locally free outside of $\varepsilon(\mathrm{Spec}\,A)$, then $[M][N]=[M\otimes N]$.

If A is any noetherian ring of finite Krull dimension, and any module whose support is contained in the singular locus has zero image in $G_0(A)$, then one can put a ring structure on $G_0(A)$ in the same manner as above. The above examples indicate that such rings are fairly common.

5. **The abelian group structure.** We consider the abelian group structure of the ring $Z[X]/(X^{n+1},(1+X)^r-1)=B_{n,r}$. Multiplication by X gives a homomorphism $B_{n-1,r}\xrightarrow{X}B_{n,r}$ with cokernel Z. Let $F\in Z[X]$ represent an element in the kernel. Then $XF=AX^{n+1}+B((1+X)^r-1)$ with $A,B\in Z[X]$, and $F=AX^n+BG$, where $G=((1+X)^r-1)/X$. But X^n is already zero in $B_{n-1,r}$. Thus the kernel is principal, generated by the class of $G=r+(\text{terms involving } X)$. This element of $B_{n-1,r}$ is of infinite order so we have an exact sequence

$$0\longrightarrow Z\longrightarrow B_{n-1,r}\xrightarrow{\;X\;}B_{n,r}\longrightarrow Z\longrightarrow 0.$$

Clearly $B_{n,r}$ is of rank one, so write $B_{n,r}=Z\oplus T_n^r$. The last homomorphism in the exact sequence is projecting onto the first direct summand. Thus we have an exact sequence $0\longrightarrow Z\xrightarrow{G}Z\oplus T_{n-1}^r\longrightarrow T_n^r\longrightarrow 0$. Form the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & T_{n-1}^r & \xrightarrow{\;1\;} & T_{n-1}^r & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle i_2} & & & & \\
0 & \longrightarrow & Z & \xrightarrow{\;G\;} & Z\oplus T_{n-1}^r & \longrightarrow & T_n^r & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \pi_1} & & & & \\
0 & \longrightarrow & Z & \longrightarrow & Z & \longrightarrow & Z/rZ & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

All rows and columns are exact. Therefore there exist maps in the last column

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making the diagram commute and the last column exact. Therefore we have an exact sequence

$$0 \to T^r_{n-1} \to T^r_n \to Z/rZ \to 0.$$

Since $T^r_0 = 0$, we have $T^r_1 = Z/rZ$, and more generally that T^r_n is of order r^n.

Now suppose that $r = r_1 r_2$ where r_1 and r_2 are relatively prime. Then we have a surjection of rings $B_{n,r_1r_2} \to B_{n,r_1} \to 0$ and hence a surjection $T^{r_1r_2}_n \to T^{r_1}_n \to 0$. This splits since $T^{r_1}_n$ has order r_1^n, the kernel has order r_2^n, and $(r_1, r_2) = 1$. Thus $T^{r_1}_n$ is the subgroup of $T^{r_1r_2}_n$ consisting of all elements killed by r_1^n. Similarly for r_2, so $T^{r_1r_2}_n = T^{r_1}_n \oplus T^{r_2}_n$. This reduces the problem to r a prime power.

The abelian group T^r_n has generators $X, X^2, \ldots, X^n$, and relations $\sum_{i=1}^r C^r_i X^{i+s}$, $0 \leqq s \leqq n-1$, where the C^r_i are the binomial coefficients, and the summations stop with the term X^n.

Suppose first that r is a prime. Then $C^r_1 = r$, and C^r_i has a factor r, $2 \leqq i < r-1$. By subtracting the above relations from each other in a suitable way, we get a new equivalent set of relations of the form

$$rX^{1+s} + X^{r+s} + \sum_{i>r} a_i X^{i+s}, \qquad 0 \leqq s < n-1,$$

(again stopping each summation with the term X^n). By induction define a new set of generators $e_1, \ldots, e_n$ by $e_i = X^i$, $1 \leqq i \leqq r-1$, and $re_i = -e_{i+r-1}$. By induction we prove that $e_i = X^i + \sum_{j>i} b_j X^j$. For suppose $e_i = X^i + \sum_{j>i} b_j X^j$. Then $re_i = rX^i + \sum_{j>i} b_j r X^j = -X^{i+r-1} - \sum_{j>i+r-1} b'_j X^j$, so $e_{i+r-1} = X^{i+r-1} + \sum_{j>i+r-1} b'_j X^j$ as required. Thus $e_1, \ldots, e_n$ generate T^r_n. Assuming that these relations among the e_i are all there are, then T^r_n has the following abelian group structure. If $1 \leqq n \leqq r-1$, then T^r_n is the direct sum of n copies of Z/rZ (with generators $X, X^2, \ldots, X^n$ respectively). As n varies from r to $2r-1$, the copies of Z/rZ are changed one after the other into Z/r^2Z. Then they are changed one after the other into Z/r^3Z, and so on. We get a group of the required order, hence we have enough relations among the e_i. In general, the generators of the cyclic groups will be $X, X^2, \ldots, X^t$ where $t = \inf(n, r-1)$.

Now assume $r = p^a$, $a > 1$. Each of the C^r_i, $2 \leqq i < p$, has a factor r. Thus, as above, we can change our relations to $rX^{1+s} + \sum_{i \geqq p} a_i X^{i+s}$, $0 \leqq s < n-1$. The integer a_p is divisible by p^{a-1}, but not by p^a. Thus, for $1 \leqq n \leqq p-1$, T^r_n is the direct sum of n copies of Z/rZ, and in T^r_p one of the copies of Z/rZ is changed into $Z/p^{a+1}Z \oplus Z/p^{a-1}Z$. After this, things seem more complicated.

Added in proof. The abelian group structure of $B_{n,r}$ has been completely determined. See A. Chabour, C. R. Acad. Sci. Paris Sér. A **272** (1971), p. 462.

6. **Relation with open subsets of** proj. Let $f \in A^r_n$ be homogeneous of degree r. Then $A^r_n[f^{-1}]$ is graded (in positive and negative degrees). Let G^r_n be the degree 0 part. Then $A^r_n[f^{-1}] = G^r_n[f, f^{-1}]$ where f is an indeterminant over G^r_n. Also $G^r_n \simeq A^r_n/(f-1)A^r_n$ (Proposition 2.2.5, p. 24 of [6]). Finally $D_+(f) = \operatorname{Spec}(G^r_n)$. We now

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have a surjection $G_0(A_n^r) \to G_0(A_n^r[f^{-1}]) = G_0(G_n^r[f, f^{-1}]) \cong G_0(G_n^r)$. This yields the theorem on p. 299 of [5], but with A no longer required to be regular, and the Grothendieck group G_0 replacing K_0.

Now suppose that $r=2$, A is regular, and $G_0(A)=K_0(A)=Z$. This puts us in the situation discussed at the end of §6 in [5]. The above surjection reads

$$Z[X]/(X^{n+1}, (1+X)^2-1) \to Z[X]/(X^{l+1}, (1+x)^2-1) = K_0(G_n^r), \quad l \leqq n.$$

It was originally hoped that calculating $G_0(A_n^r)$ would lower the upper bound on l, but this has turned out not to be the case. At least, the possibility remains that l could be as large as n, for suitable choice of A and f.

7. **The ideal class groups.** The results in this section are included for comparison with G_0. Let B be a graded Krull ring (in positive degrees.) Let $DH(B)$ be the free abelian group on homogeneous prime ideals of height one, and let $FH(B)$ be the subgroup of $DH(B)$ generated by the divisors of homogeneous elements of B. Then $c(B)=DH(B)/FH(B)$ by Proposition 7.1, p. 24 of [9]. Let $X=\operatorname{Proj}(B)$, and $K=\{f/g \mid f, g$ are homogeneous elements in B of same degree$\}$ = stalk of X at the generic point. The irreducible closed subsets of codimension one in X correspond to homogeneous primes of height one in B (assuming B_+ is not of height $\leqq 1$). Let X_1 be the set of generic points of irreducible subsets of codimension one in X. Then for all $P \in X_1$, the stalk of X at P is a discrete valuation ring, so we can define the divisor of $h \in K$ by $(h)=\sum v_P(h)\cdot P$ as P ranges over X_1. If, as above, we identify the P's with the homogeneous primes of height one in B it is readily seen that this divisor of h is the same as the divisor of h regarded as an element in the quotient field of B. (The last observation shows that $v_P(h)=0$ for all but a finite number of $P \in X_1$.) Thus if we let $c(X)$ = free abelian group on irreducible subsets of codimension one of X, factored out by the subgroup generated by the divisors of functions (i.e., elements of K), then $c(X)=DH(B)/FH_0(B)$, where $FH_0(B)$ is the subgroup of $DH(B)$ generated by the divisors of elements of K.

We now have an exact sequence

$$0 \to FH(B)/FH_0(B) \to c(X) \to c(B) \to 0.$$

The homomorphism degree: $FH(B) \to Z$ has kernel $FH_0(B)$, and the image will be isomorphic to Z. Thus $FH(B)/FH_0(B) \cong Z$ (noncanonically) and our exact sequence reads $0 \to Z \to c(X) \to c(B) \to 0$.

Now take $X=P_A^n=\operatorname{Proj} A[X_0, \ldots, X_n]$, where A is a Krull ring. Then $c(B)=c(A)$, so the exact sequence reads

$$0 \to Z \to c(P_A^n) \to c(A) \to 0.$$

Let L be the quotient field of A. Then we have a splitting homomorphism $c(P_A^n) \to c(P_L^n)=Z$. Thus $c(P_A^n)=Z \oplus c(A)$. We also have $P_A^n=\operatorname{Proj}(A_n^r)$. The ring A_n^r is a Krull ring since $A_n^r=K_n^r \cap A[X_0, \ldots, X_n]$, where K_n^r is the quotient field of A_n^r.

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The exact sequence reads $0 \to Z \to Z \oplus c(A) \to c(A_n^r) \to 0$. The image of $1 \in Z$ is r, so we get $c(A_n^r) = Z/rZ \oplus c(A)$. This shows that the homomorphism $G_0(A_n^r) \to Z \oplus c(A_n^r)$ (where A is noetherian integrally closed) is not an isomorphism if $r > 1$.

Divisorial ideals (of A_n^r) corresponding to the elements of Z/rZ can be taken as $I_i = X_0^{r-i} J_i$, where J_i is the A_n^r-submodule of $A[X_0, \ldots, X_n]$ generated by the monomials of degree i, $0 \leqq i \leqq r-1$. The ideal I_i is divisorial since $I_i = A_n \cap (X_0^{r-i}/X_1^{r-i})$, but is not invertible if $i > 0$.

Now we remark that if $B = \bigoplus_{n \geqq 0} B_i$ is a graded Krull ring, then Pic B = Pic B_0. The inclusion $B_0 \to B$ induces, by $\otimes_{B_0} B$, a monomorphism Pic $(B_0) \to$ Pic B which we need only show is onto. Proposition 7.1, p. 24 of [9], shows that it is sufficient to consider graded projective modules of rank one, and by Proposition 3.3, p. 637 of [1], these are all of the form $P_0 \otimes_{B_0} B$. This is a slight generalization of Lemma 5.1 of [8].

The results of this section can be summarized as follows:

THEOREM 3. *Let B be a graded Krull ring (in positive degrees) such that the elements of positive degree do not form a prime ideal of height* $\leqq 1$. *Let* $X = \text{Proj}\,(B)$. *Then there is an exact sequence* $0 \to Z \to c(X) \to c(B) \to 0$. *If A is any Krull ring, then* $c(A_n^r) = Z/rZ \oplus c(A)$. *Finally, if B is any graded Krull ring in positive degrees, then* Pic B = Pic B_0.

8. **The noncommutative case.** Let A be a right noetherian ring (as defined in [1, p. 122]). Let A_n^r be the A-subalgebra of $B_n = A[X_0, \ldots, X_n]$ generated by monomials of degree r. Let S be the (central) multiplicative set generated by X_n^r. Then $S^{-1}A_n^r = A[X_0/X_n, X_1/X_n, \ldots, X_{n-1}/X_n, X_n^r, X_n^{-r}]$, which is right regular. Thus we may apply Theorem 6.2, p. 492 of [1], to get an exact sequence

$$K_1(S^{-1}A_n^r) \xrightarrow{\partial} G_0(A_n^r/X_n^r A_n^r) \longrightarrow G_0(A_n^r) \longrightarrow K_0(S^{-1}A_n^r) \longrightarrow 0.$$

But $K_1(S^{-1}A_n^r) = K_0(A) \oplus K_1(A)$ and $K_0(S^{-1}A_n^r) = K_0(A)$ by Theorem 2 and Theorem 1 of [2], respectively. We have a surjection $A_n^r/X_n^r A_n^r \to A_{n-1}^r$ with nilpotent kernel. Thus by Proposition 2.3, p. 454 of [1], we have an isomorphism $G_0(A_{n-1}^r) \to G(A_n^r/X_n^r)$. Furthermore ∂ kills $K_1(A)$, since in the statement of Proposition 6.1, p. 492 of [1], α can already be taken as an automorphism. Thus we get an exact sequence

$$K_0(A) \to G_0(A_{n-1}^r) \to G_0(A_n^r) \to K_0(A) \to 0.$$

If M is a Z_n^r module, then $M \to M \otimes_Z A - \text{Tor}_1^Z(M, A)$ defines a homomorphism $G_0(Z_n^r) \to G_0(A_n^r)$. We now have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \longrightarrow & G_0(Z_{n-1}^r) & \longrightarrow & G_0(Z_n^r) & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K_0(A) & \longrightarrow & G_0(A_{n-1}^r) & \longrightarrow & G_0(A_n^r) & \longrightarrow & K_0(A) & \longrightarrow & 0
\end{array}$$

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In §2, the rings $G_0(Z_n^r)$ were found to be $Z[X]/(X^{n+1}, (1+X)^r)=B_{n,r}$. If we tensor the top row with $K_0(A)$ the sequence remains exact (except at the left) since the top row is the direct sum of two short exact sequences $0\to Z\to B_{n-1,r}\to T_n^r\to 0$ and $0\to Z\to Z\to 0$. The groups in the bottom row are $K_0(A)$-modules. Thus we get a commutative diagram

$$\begin{array}{ccccccccc} K_0(A) & \longrightarrow & B_{n-1,r}\otimes_Z K_0(A) & \longrightarrow & B_{n,r}\otimes_Z K_0(A) & \longrightarrow & K_0(A) & \longrightarrow & 0 \\ \approx\big\downarrow & & \big\downarrow f_{n-1} & & \big\downarrow f_n & & \big\downarrow\approx & & \\ K_0(A) & \longrightarrow & G_0(A^r_{n-1}) & \longrightarrow & G_0(A^r_n) & \longrightarrow & K_0(A) & \longrightarrow & 0 \end{array}$$

The diagram is commutative by the universal mapping property of $\otimes$. Now f_0 is an isomorphism. Therefore we may use the five-lemma to conclude by induction that f_n is an isomorphism for all n. This proves

THEOREM 4. *Let A be a right noetherian regular ring. Then* $G_0(A_n)\cong K_0(A)\otimes_Z Z[X]/(X^{n+1}, (1+X)^r-1)$.

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