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AN EXTENSION OF A THEOREM OF HARTOGS

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Abstract. Hartogs proved that every function which is holomorphic on the boundary of the unit ball in C^n, $n>1$, can be extended to a function holomorphic on the ball itself. It is conjectured that a real k-dimensional $\mathscr{C}^\infty$ compact submanifold of C^n, $k>n$, is extendible over a manifold of real dimension $(k+1)$. This is known for hypersurfaces (i.e., $k=2n-1$) and submanifolds of real codimension 2. It is the purpose of this paper to prove this conjecture and to show that we actually get C-R extendibility.

1. **Introduction.** Let M^k be a real k-dimensional compact $\mathscr{C}^\infty$ manifold embedded in C^n, $k, n \geqq 2$. Hartogs proved that every function holomorphic in an open neighborhood of M^{2n-1} can be extended to a function holomorphic in some open subset of C^n. Bochner proved a similar theorem for functions which satisfy the induced Cauchy-Riemann equations on M^{2n-1}. It has been conjectured that any real k-dimensional compact $\mathscr{C}^\infty$ submanifold of C^n is extendible to a manifold of real dimension $(k+1)$ if $k>n$. This has been proved for real-analytic submanifolds of C^n in [3] and generic C-R submanifolds in [2]. It is the purpose of this paper to prove the conjecture with extendibility being replaced by C-R extendibility.

The early work for the higher codimensional study was done by Bishop [1], Wells [6] and Greenfield [2]. A recent article due to Nirenberg [4] led to the results in this paper.

2. **Definitions.** Let M^k be a real k-dimensional $\mathscr{C}^\infty$ manifold embedded in C^n, $k, n \geqq 2$. Suppose $T(M^k)$ is the tangent bundle to M^k, and J denotes the almost complex tensor $J: T(C^n) \to T(C^n)$, with $J^2=-I$. Then we define

$$H_p(M^k) = T_p(M^k) \cap JT_p(M^k),$$

the vector space of *holomorphic tangent vectors* to M^k at p. Then $H_p(M^k)$ is the maximal complex subspace of $T_p(C^n)$ which is contained in $T_p(M^k)$. It is well known that

$$\max(k-n, 0) \leqq \dim_C H_p(M^k) \leqq [k/2].$$

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There is another way of examining almost complex structures which we shall use. Let f denote the embedding of M^k into $\boldsymbol{C}^n$, and let $J(f)$ be the complex Jacobian of f. If $q = \min(n, k)$, a point p in M^k is said to be an *exceptional point of order* l, $0 \leqq l \leqq [k/2] - \max(k-n, 0)$ if the complex rank of $J(f)|_p$ is equal to $q-l$.

A point p in M^k is *generic* if p is an exceptional point of order 0. The manifold M^k is *locally generic* at p if every point in some open neighborhood of p is generic, and is *locally C-R* at p if every point in some open neighborhood of p is an exceptional point of the same order.

Suppose M^k is locally C-R at p and $H_p(M^k)$ is nonempty. Then we define the *Levi form* at any x near p

$$L_x(M^k)\colon H_x(M^k) \to (Tx(M^k) \otimes \boldsymbol{C})/(H_x(M^k) \otimes \boldsymbol{C})$$

by $L_x(M^k)(t) = \pi_x\{[Y, \bar{Y}]_x\}$, where Y is a local section of the fiber bundle $H(M^k)$ (with fiber $H_x(M^k)$) such that $Y_x = t$, $[Y, \bar{Y}]_x$ is the Lie bracket evaluated at x, and

$$\pi_x\colon T_x(M^k) \otimes \boldsymbol{C} \to (T_x(M^k) \otimes \boldsymbol{C})/(H_x(M^k) \otimes \boldsymbol{C})$$

is the projection.

Denote by $\mathcal{O}_{\boldsymbol{C}^n} = \mathcal{O}$ the sheaf of germs of holomorphic functions on $\boldsymbol{C}^n$. Let K be a compact subset of $\boldsymbol{C}^n$ and V an open subset of $\boldsymbol{C}^n$ containing K. We set

$$\mathcal{O}(K) = \operatorname*{ind\,lim}_{V \supset K} \mathcal{O}(V),$$

where $\mathcal{O}(V)$ is the Fréchet algebra of holomorphic functions on V. We say that K is *extendible* to a connected set $K' \supsetneqq K$ if the map $r\colon \mathcal{O}(K') \to \mathcal{O}(K)$ is onto.

Suppose $f \in \mathscr{C}^\infty(M^k)$. We say f is a *C-R function at* $p \in M^k$ if $\bar{X}f(y) = 0$, for y near p and X any section of $H(M^k)$. If M^k is locally C-R at p it suffices to verify the equality just for X in a local basis for $H(M^k)$ at p. We note that our manifold need not be globally C-R. Thus we may have points which are not locally C-R. But obviously, the set of such points is nowhere dense in M^k.

DEFINITION 2.1. Let $f \in \mathscr{C}^\infty(M^k)$. Then f is a *C-R function on* M^k if f is a C-R function at each point of M^k. The C-R functions are denoted by CR (M^k).

We say that M^k is *C-R extendible* to a connected set $K = M^k \cup K'$, where $K' \neq \varnothing$, if for every $f \in \mathrm{CR}(M^k)$ there exists an $F\colon M^k \cup K' \to \boldsymbol{C}$ continuous so that $F|_{M^k} = f$ and $F|_{K'} \in \mathcal{O}(K')$. We observe that C-R extendibility implies extendibility.

Let K be a compact subset of $\boldsymbol{C}^n$. We shall call a point $x \in K$ a *holomorphic peak point* if there exists a function $f \in \mathcal{O}(K)$ such that, for any $y \in K - \{x\}$, we have $|f(y)| < |f(x)|$.

3. **Local equations and the Levi form.** Again let M^k be a real k-dimensional $\mathscr{C}^\infty$ manifold embedded in C^n, k, $n \geqq 2$. Suppose M^k is locally C-R at p, and p is an exceptional point of order l. If $k > n$ the local equations of M^k in a neighborhood of p are (after a suitable choice of coordinates)

$$
(1)\quad
\begin{aligned}
z_1 &= x_1 + ih_1(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}) \\
&\vdots \\
z_{2(n-l)-k} &= x_{2(n-l)-k} + ih_{2(n-l)-k}(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}) \\
z_{2(n-l)-k+1} &= u_1 + iv_1 = w_1 \\
&\vdots \\
z_{n-l} &= u_{k-n+l} + iv_{k-n+l} = w_{k-n+l} \\
z_{n-l+1} &= g_1(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}) \\
&\vdots \\
z_n &= g_l(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}),
\end{aligned}
$$

where $x_1, \ldots, x_{2(n-l)-k}, u_1, v_1, \ldots, u_{k-n+l}, v_{k-n+l}$ are local coordinates for M^k in a neighborhood of p vanishing at p, and $z_1, \ldots, z_n$ are coordinates for C^n vanishing at p. The real-valued functions $h_1, \ldots, h_{2(n-l)-k}$ as well as the complex-valued functions $g_1, \ldots, g_l$ vanish to order 2 at p. Because M^k is locally C-R at p, the functions $g_1, \ldots, g_l$ must be complex-analytic functions of $w_1, \ldots, w_{k-n+l}$ (see [3]).

Letting $g_j = g_j' + ig_j''$, $j = 1, \ldots, l$, we find from [5] that the Levi form vanishes at p if and only if the complex Hessians at p of each of the functions $h_1, \ldots, h_{2(n-l)-k}$, $g_1', g_1'', \ldots, g_l', g_l''$ with respect to the variables $w_1, \ldots, w_{k-n+l}$ all have zero eigenvalues.

Fix $x_1, \ldots, x_{2(n-l)-k}$ and expand each g_j in a Taylor series in $w_1, \ldots, w_{k-n+l}$,

$$g_j = \sum_\alpha a_{j,\alpha} w^\alpha,$$

where $w = (w_1, \ldots, w_{k-n+l})$ and $\alpha = (\alpha_1, \ldots, \alpha_{k-n+l})$. Replacing z_{n-l+j} by $z_{n-l+j} - \sum_\alpha a_{j,\alpha} w^\alpha$, we have that $z_{n-l+1} = 0, \ldots, z_n = 0$ in our new local equations. Thus the Levi form vanishes at p if and only if the complex Hessians at p of each of the functions $h_1, \ldots, h_{2(n-l)-k}$ are all zero matrices.

Suppose M^k is compact in C^n. It is shown in [5] that there exists an open set of holomorphic peak points on M^k which is nonempty. By the remarks before Definition 2.1, we can find a holomorphic peak point $p \in M^k$ such that p is an exceptional point of some order l, and M^k is locally C-R at p. Assume $p = 0$ and M^k near p is given by the equations in (1). Wells proves that through p we can put a hyperplane which intersects M^k at only the point p. If $z_j = x_j + iy_j$, $j = 1, \ldots, 2(n-l)-k, n-l+1, \ldots, n$, we can assume the hyperplane is defined by $y_1 = 0$ (the information about the g_j's in this section forces our arbitrary choice to $y_1, \ldots, y_{2(n-l)-k}$).

Let Q denote the 1-dimensional real subspace of $T_0(C^n)$ generated by $\partial/\partial y_1$. Set

$W = Q \oplus T_0(M^k)$ and let π be the projection from C^n to W. Under this projection the manifold M^k projects to a manifold with local equations

$$
\begin{aligned}
z_1 &= x_1 + ih_1(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}) \\
z_2 &= x_2 \\
&\vdots \\
z_{2(n-l)-k} &= x_{2(n-l)-k} \\
z_{2(n-l)-k+1} &= u_1 + iv_1 = w_1 \\
&\vdots \\
z_{n-l} &= u_{k-n+l} + iv_{k-n+l} = w_{k-n+l}.
\end{aligned} \tag{2}
$$

Wells shows that

$$\frac{\partial^2 h_1}{\partial x_1^2}, \ldots, \frac{\partial^2 h_1}{\partial x_{2(n-l)-k}^2}, \frac{\partial^2 h_1}{\partial u_1^2}, \ldots, \frac{\partial^2 h_1}{\partial u_{k-n+l}^2}, \frac{\partial^2 h_1}{\partial v_1^2}, \ldots, \frac{\partial^2 h_1}{\partial v_{k-n+l}^2}$$

are all >0 on some open neighborhood U of p in M^k. In particular

$$\frac{\partial^2 h_1}{\partial w_1\, \partial \bar{w}_1}, \ldots, \frac{\partial^2 h_1}{\partial w_{k-n+l}\, \partial \bar{w}_{k-n+l}}$$

are positive on the set U. By diagonalizing, we find that the Hessian of h_1 with respect to $w_1, \ldots, w_{k-n+l}$ is positive definite.

4. **The main result.** Assume M^k is a real k-dimensional $\mathscr{C}^\infty$ manifold embedded in C^n, and M^k is locally C-R at $p \in M^k$. Suppose at least one of the following conditions is satisfied.

(I) There is a real hypersurface containing M^k whose Levi form restricted to $H(M^k)$ has at p at least one positive and one negative eigenvalue.

(II) There is a real hypersurface containing M^k whose Levi form restricted to $H(M^k)$ has at p all its eigenvalues of the same sign different from zero.

Then we have the following theorem due to Nirenberg [4].

THEOREM 4.1. *Let M^k be locally C-R at $p \in M$ and assume either* (I) *or* (II) *holds. Then M^k is locally C-R extendible to a manifold $\tilde{M}$ of real dimension one higher than that of M^k.*

We are now able to prove the main result.

THEOREM 4.2. *Let M^k be a real k-dimensional compact $\mathscr{C}^\infty$ manifold embedded in C^n, $k > n \geqq 2$. Then M^k is C-R extendible to a real* $(k+1)$*-dimensional submanifold of C^n.*

Proof. We showed in the previous section that there exists a point $p \in M^k$ such that:

(i) M^k is locally C-R at p,

(ii) M^k is given by the local equations (1) near p, and

(iii) the complex Hessian of the function h_1 with respect to the variables $w_1, \ldots, w_{k-n+l}$ has all positive eigenvalues at p.

Consider the real hypersurface containing M^k defined by the function $\rho = y_1 - h_1$. The Levi form of this hypersurface restricted to $H(M^k)$ is the negative of the complex Hessian of h_1 with respect to the variables $w_1, \ldots, w_{k-n+l}$. Then this hypersurface satisfies condition (II) at the point p, and we apply Theorem 4.1. Q.E.D.

THEOREM 4.3. *Let M^k be a real k-dimensional compact $\mathscr{C}^\infty$ manifold embedded in C^n, $k > n \geqq 2$. Then M^k is extendible to a real $(k+1)$-dimensional submanifold of C^n.*

REMARK 1. The manifold $\tilde{M}$ of Theorem 1 can be taken to have $\mathscr{C}^q$ structure, $1 \leqq q < \infty$.

REMARK 2. If $k \leqq n$, then there are examples of totally real submanifolds which are always holomorphically convex. Thus, from the standpoint of dimension, Theorems 4.2 and 4.3 are the best possible.

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