TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 166, April 1972

AUTOMORPHISM GROUPS OF BOUNDED DOMAINS IN BANACH SPACES(1)

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Abstract. We prove a weak Schwarz lemma in Banach space and use it to show that in Hilbert space a Siegel domain of type II is not necessarily biholomorphic to a bounded domain. We use a strong Schwarz lemma of L. Harris to find the full group of automorphisms of the infinite dimensional versions of the Cartan domains of type I. We then show that all domains of type I are holomorphically inequivalent, and are different from k-fold products of unit balls ($k \geqq 2$). Other generalizations and comments are given.

0. **Introduction.** In this paper we use a generalized Schwarz lemma (a result of Harris [6]) to analyze the groups of holomorphic automorphisms of certain bounded domains in Hilbert space. We find the full group of automorphisms of the infinite dimensional versions of the Cartan domains of type I (in the notation of Hua [7]). Since the unit ball in Hilbert space is one of these domains, we find the full group of holomorphic automorphisms of the unit ball. This result seems to be known already to R. S. Phillips [11] and certainly to Hayden and Suffridge [8].

We also derive a necessary condition for an open domain in a Banach space to be holomorphically equivalent to a bounded domain. We use this condition to prove that the infinite dimensional version of the polydisc is not equivalent to a bounded domain. Thus in Hilbert space a Siegel domain of type II is not necessarily holomorphically equivalent to a bounded domain.

We find the full group of holomorphic automorphisms of the k-fold product of the Hilbert ball with itself. We prove that all domains of type I are distinct and are inequivalent to k-fold products of balls of $k \geqq 2$. In particular, we show that the unit ball in Hilbert space, B, and $B \times B$ are not holomorphically equivalent (answering a question of D. Burghelea).

A quantity of comments and questions are appended.

1. **Schwarz's lemma.** Let V and W be complex Banach spaces. We give a generalization of Schwarz's lemma for maps from V to W:

Received by the editors October 30, 1970 and, in revised form, June 6, 1971.

AMS 1969 *subject classifications*. Primary 5755, 3260, 2270.

Key words and phrases. Hilbert space, Banach space, unit balls, inequivalence of polyballs, Cartan domains, Siegel domains, Schwarz lemma.

(1) This research was partially supported by the NSF under Grants GP-27959 and GP-20647.

PROPOSITION 1.1. *Let K be a starlike circular domain in V and K' a convex circular domain in W. Suppose that $f: K \to K'$ is a holomorphic mapping with $f(0)=0$. Then $(1/k!)\, d^k f(0)(z^k) \in \bar{K}'$ for $z \in K$. (Here holomorphic means that the Fréchet derivative of f is complex linear at each point in K, $d^k f$ is the kth derivative of f, and z^k is the k-tuple $(z, \ldots, z)$. See Dieudonné* [3].)

Proof. Let

$$G(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-kt\theta} f(e^{i\theta}z)\, d\theta, \quad \text{for } z \in K.$$

By definition of the Riemann integral $G(z)$ is a limit of convex combinations of elements of K'. Hence $G(z) \in \bar{K}'$. If $z \in K$, then the map of the closed unit disc to K' given by $t \to f(tz)$ is holomorphic. It therefore has a power series expansion: $f(tz) = \sum_{k=1}^{\infty} (t^k/k!)\, d^k f(0)(z^k)$ which converges uniformly and absolutely on the unit circle. This immediately implies that $G(z) = (1/k!)\, d^k f(0)(z^k)$.

COROLLARY 1.1 (SCHWARZ LEMMA). *Let $B = \{z \in V \mid \|z\| < 1\}$ and $B' = \{z \in W \mid \|z\| < 1\}$. If $f: B \to B'$ is holomorphic and $f(0)=0$, then $df(0)$ has operator norm less than or equal to* 1.

Proof. $df(0)B \subset \bar{B}$. Thus if $\|z\|=1$, $t \in R$, $|t|<1$ then $\|df(0)tz\| \leqq 1$. Hence $\|df(0)z\| \leqq 1$, as was to be proved.

Proposition 1.1 also gives a necessary condition that a domain in V be holomorphically equivalent to a bounded domain.

COROLLARY 1.2. *Let K be an open subset in V. If K is holomorphically equivalent to a bounded domain in V and if, for each $z \in K$,*

$$\mathscr{F}(z) = \{f: K \to K \mid f \text{ holomorphic and } f(z) = z\}$$

then there are positive numbers $C(z)$ and $M(z)$ so that $M_k(z) \leqq C(z)(k!)^2 M(z)^k$, where $M_k(z) = \sup\{\|d^k f(z)\| \mid f \in \mathscr{F}(z)\}$. (Here, $\|d^k f(z)\| = \sup\{\|d^k f(z)(w^k)\| \mid \|w\|=1\}$.)

Note. If K is the unit disc in $\boldsymbol{C}$ then $M_k(0) = k!$.

Proof. We suppose that for each $z \in K$ there is a bounded open subset K_z in V and $G_z: K_z \to K$ a holomorphic isomorphism so that $G_z(0) = z$. Let

$$\mathscr{F}_z = \{f: K_z \to K_z \mid f \text{ holomorphic and } f(0) = 0\}.$$

Let $\tilde{M}_k(z) = \sup\{\|d^k f(0)\| \mid f \in \mathscr{F}_z\}$. Then the technique of the proof of Proposition 1.1 implies that $\tilde{M}_k(z) \leqq Ak!B^k$ (here A is the radius of a ball around 0 containing K_z and B^{-1} is the radius of a ball around 0 contained in K_z). The Cauchy estimates on $d^k G_z(0)$ imply that $\|d^k G_z(0)\| \leqq A'k!(B')^k$ for some constants A' and B'. A simple computation using the composite function formula [1, p. 7] yields the desired estimates on the elements of $\mathscr{F}(z)$ of the form $G_z \circ f \circ G_z^{-1}$, $f \in \mathscr{F}(z)$. Since every element of $\mathscr{F}(z)$ has this form the result is proven.

Let $l^2 = \{\{z_n\} \mid z_n \in \boldsymbol{C} \text{ and } \sum |z_n|^2 < \infty\}$. Let D^∞ be the subset defined by $D^\infty = \{\{z_n\} \in l^2 \mid \sup |z_n| < 1\}$. As an application of Corollary 1.2 we prove

PROPOSITION 1.2. *D^∞ is not equivalent to a bounded domain in l^2.*

Proof. Let $F_n(\{z_j\})=\{w_j\}$ be defined by $w_j=z_1$ for $j=1,\ldots,n$, $w_{n+j}=z_2$ for $j=1,\ldots,n$, and in general $w_{(k-1)n+j}=z_k$ for $j=1,\ldots,n$. Then $F_n\in\mathscr{F}(0)$ and $\|dF_n(0)\|=n^{1/2}$. Thus $M_1(0)\geqq n^{1/2}$ for all n. Corollary 1.2 now implies D^∞ is not equivalent to a bounded domain.

2. **Zero preserving automorphisms of the unit ball.** Let V and W be complex Banach spaces.

THEOREM 2.1 (HARRIS [6]). *Let B (resp. B') be the unit ball of V (resp. W). If $F\colon B\to B'$ is a holomorphic isomorphism of B so that $F(0)=0$, then $F=dF(0)|_B$. Furthermore $dF(0)$ is an isometry of V to W, i.e. $\|dF(0)z\|=\|z\|$ for $z\in V$.*

Proof. Let $F\colon B\to B'$ be a holomorphic isomorphism of B to B' with $F(0)=0$. Put $T=dF(0)$. Corollary 1.1 says that $\|T\|\leqq 1$. Since $dF^{-1}(0)=T^{-1}$ we see also that $\|T^{-1}\|\leqq 1$. Thus if $Ty=w$ with $\|y\|=1$ then $\|w\|\leqq 1$. Now $T^{-1}w=y$, so $1\geqq\|T^{-1}\|\geqq\|w\|^{-1}\geqq 1$, hence $\|w\|=1$. This proves that T is an isometry. The result now follows from a theorem of Harris [6] which says that if $f\colon B\to B$ is holomorphic and $df(0)$ is an isometry then $f=df(0)|_B$.

From Theorem 2.1 we derive

COROLLARY 2.1. *Let C be an open bounded convex circled domain in V. If $F\colon C\to C$ is a holomorphic automorphism so that $F(0)=0$ then $F=dF(0)|_C$. The group of all holomorphic automorphisms of C preserving 0 is a bounded subgroup of $GL(V)$.*

Proof. Let V_C be the Banach space whose underlying space is V and whose norm is defined by $\|x\|_C=\inf\{t\in R,\ t>0\mid x\in tC\}$. Then V_C is a Banach space (equivalent to V) having C as its unit ball. The result now follows from Theorem 2.1.

3. **Cartan domains of type I in Hilbert space.** Let H be a separable Hilbert space and let $L(\boldsymbol{C}^n, H)$ be the set of all linear maps from $\boldsymbol{C}^n$ to H. (We take the standard Hilbert space structure on $\boldsymbol{C}^n$.) We make $L(\boldsymbol{C}^n, H)$ into a Hilbert space by defining $\langle Z, W\rangle=\operatorname{tr} W^*Z$ for $Z, W\in L(\boldsymbol{C}^n, H)$. Let $\|Z\|=\langle Z,Z\rangle^{1/2}$ for $Z\in L(\boldsymbol{C}^n, H)$. Define $\|Z\|_1=\sup\{\|Zv\|\mid\|v\|=1\}$.

Let $D_n(H)$ be the unit ball in $L(\boldsymbol{C}^n, H)$ relative to the norm $\|\ \|_1$. We note that if $n=1$ then $D_n(H)$ is just the unit ball in H. If $\dim H<\infty$ then the $D_n(H)$ exhaust the standard Cartan domains of type I (see Hua [7]).

In this section we find the full group of holomorphic automorphisms of $D_n(H)$ fixing 0.

THEOREM 3.1. *Let F be a holomorphic automorphism of $D_n(H)$ with $F(0)=0$. Then there is $A\in U(H)$ and $B\in U(n)$ ($U(H)$, the unitary group of H, and $U(n)$, the unitary group of $\boldsymbol{C}^n$) so that $F(Z)=A\circ Z\circ B^{-1}$.*

The proof of this result will depend on a collection of lemmas and definitions. An element Z_0 of cl $(D_n(H))$ will be called an extreme point if whenever $Z_0 = aZ_1 + bZ_2$, $a>0$, $b>0$, $a+b=1$, and $Z_i \in \mathrm{cl}\,(D_n(H))$, then $Z_1 = Z_2$.

LEMMA 3.1. *$Z \in \mathrm{cl}\,(D_n(H))$ is extreme iff $Z^*Z = I_n$ (I_n is the identity map on $\boldsymbol{C}^n$).*

Proof. Suppose $Z^*Z = I$ and $Z = aZ_1 + bZ_2$, $a>0$, $b>0$, $a+b=1$, with $Z_i \in \mathrm{cl}\,(D_n(H))$. Then if $v \in \boldsymbol{C}^n$ is a unit vector Zv is a unit vector. Furthermore $\|Z_i v\| \leqq 1$ by the definition of $D_n(H)$. Thus for each $v \in \boldsymbol{C}^n$, $\|v\| = 1$, $aZ_1 v + bZ_2 v = Zv$. But if x, y, z are unit vectors in H so that $ax + by = z$ then $x = y = z$. Hence $Z = Z_1 = Z_2$.

Suppose now that $Z \in \mathrm{cl}\,(D_n(H))$, $Z^*Z \neq I$, and Z is extreme. Then

$$Z = \sum_{i=1}^{k} \phi_i \otimes v_i^*$$

where $\phi_1, \ldots, \phi_k$ is an orthonormal basis of Im Z, and $v_i^*(v) = \langle Zv, \phi_i \rangle$.

If $k \neq n$, let $v \in \boldsymbol{C}^n$ be a unit vector orthogonal to $v_1, \ldots, v_k$ and let ϕ be orthogonal to the ϕ_i. Then $Z + \phi \otimes v^*, Z - \phi \otimes v^* \in \mathrm{cl}\,(D_n(H))$ and $Z = \frac{1}{2}(Z + \phi \otimes v^*) + \frac{1}{2}(Z - \phi \otimes v^*)$. Thus $k = n$.

Since $Z^*Z \neq I$, $\|Z\|_1 \leqq 1$, we see there is i so that $0 < \|v_i\| < 1$. We assume $i = 1$. Let u_1, w_1 be elements of the unit ball in $\boldsymbol{C}^n$ (with $u_1 \neq v_1$, $w_1 \neq v_1$) so that $au_1 + bw_1 = z_1$, $a, b > 0$, $a + b = 1$. Then

$$a\left(\phi_1 \otimes u_1^* + \sum_{j=2}^{n} \phi_j \otimes v_j^*\right) + b\left(\phi_1 \otimes w_1^* + \sum_{j=2}^{n} \phi_j \otimes v_j^*\right) = Z.$$

So Z is not extreme.

LEMMA 3.2. *Let F be as in Theorem 3.1. Then $F = dF(0)|_{D_n(H)}$ and, if $Z \in D_n(H)$, there are elements $A_Z \in U(H)$, $B_Z \in U(n)$ so that $F(Z) = A_Z \circ Z \circ B_Z^{-1}$.*

Proof. By Theorem 2.1, $F = dF(0)|_{D_n(H)}$ and $dF(0)$ is an isometry of $\| \ \|_1$. Set $T = dF(0)$. Clearly T preserves extreme points of cl $(D_n(H))$. If Z is an extreme point, we can write (as a consequence of Lemma 3.1) $Z = \sum_{i=1}^{n} \phi_i \otimes v_i^*$ with $\{\phi_i\}$ and $\{v_i\}$ orthonormal sets. Let $A_i = T(\phi_i \otimes v_i^*)$. Then for each $(\theta_1, \ldots, \theta_n) \in \boldsymbol{R}^n$, $\sum_{j=1}^{n} \exp(i\theta_j)\phi_j \otimes v_j^*$ is also an extreme point of cl $(D_n(H))$. If $A_i = T(\phi_i \otimes v_i^*)$, we have

$$T\left(\sum_{j=1}^{n} \exp(i\theta_j)(\phi_j \otimes v_j^*)\right)^* T\left(\sum_{j=1}^{n} \exp(i\theta_j)(\phi_j \otimes v_j^*)\right) = I.$$

Thus $\sum_{k,l} \exp(-i\theta_k) \exp(i\theta_l) A_k^* A_l = I$ for all $(\theta_1, \ldots, \theta_n) \in \boldsymbol{R}^n$. This implies

$$\sum_{k=1}^{n} A_k^* A_k = I, \tag{1}$$

and (2) $A_k^* A_l = 0$, $k \neq l$. Now $\|\phi_i \otimes v_i^*\|_1 = 1$. Thus $\|A_k\| = 1$ (since T is an isometry of $\| \ \|_1$). Hence for each k, there is w_k so that $\|w_k\| = 1$ and $A_k^* A_k w_k = w_k$. (1)

implies that $A_k^* A_k w_l = \delta_{kl} w_k$, so that $w_1, \ldots, w_n$ are linearly independent. Let $\psi_k = A_k w_k$. (2) implies $\langle \psi_k, \psi_l \rangle = \delta_{kl}$. So $A_i = \psi_i \otimes w_i^*$, and $\sum_{i=1}^n A_i = \sum_{i=1}^n \psi_i \otimes w_i^*$. Therefore $w_1, \ldots, w_n$ are orthonormal. In particular, if $\phi \in H$, $v \in \boldsymbol{C}^n$, $T(\phi \otimes v^*) = \psi \otimes w^*$ for some $\psi \in H$, $w \in \boldsymbol{C}^n$. Furthermore, if $\phi_1, \phi_2 \in H$ and $v_1, v_2 \in \boldsymbol{C}^n$ are such that $\langle \phi_1, \phi_2 \rangle = 0$, and $\langle v_1, v_2 \rangle = 0$, then $T(\phi_i \otimes v_i) = \psi_i \otimes w_i$ $(i = 1, 2)$ and $\langle \psi_1, \psi_2 \rangle = 0$, and $\langle w_1, w_2 \rangle = 0$, and $\|\psi_i\| \, \|w_i\| = \|\phi_i\| \, \|v_i\|$ $(i = 1, 2)$.

If $Z \in L(\boldsymbol{C}^n, H)$, then there is a set $\{v_1, \ldots, v_k\}$ of orthonormal vectors so that $Z^*Z = \sum_{i=1}^k \lambda_i v_i \otimes v_i^*$ where $\lambda_i \geqq 0$. (Z^*Z is Hermitian positive semidefinite.) Let $(\lambda_i)^{1/2} \phi_i = Z v_i$, $i = 1, \ldots, k$, $\{\phi_i\}$ unit vectors. Then $Z^* = \sum_{i=1}^k (\lambda_i)^{1/2} \phi_i \otimes v_i^*$, and $Z^*Z = \sum_{i,j=1}^k (\lambda_i)^{1/2} (\lambda_j)^{1/2} \langle \phi_j, \phi_i \rangle v_i \otimes v_j^*$. Thus $\{\phi_1, \ldots, \phi_k\}$ are orthonormal. Hence $T(Z) = \sum_{i=1}^k (\lambda_i)^{1/2} \psi_i \otimes w_i^*$, with $\{\psi_j\}$ and $\{w_j\}$ orthonormal. Let $B_Z \in U(n)$ be defined so that $B_Z v_i = w_i$, and $A_Z \in U(H)$ defined so that $A_Z \phi_i = \psi_i$. Then $T(Z) = A_Z \circ Z \circ B_Z^{-1}$.

Note. We keep for further reference a part of the proof of Lemma 3.2. If T is an isometry of $\| \ \|_1$ on $L(\boldsymbol{C}^n, H)$, and if $\phi, \psi \in H$, $v, w \in \boldsymbol{C}^n$, $\langle v, w \rangle = \langle \phi, \psi \rangle = 0$, then $T(\phi \otimes v^*) = \xi \otimes u^*$, $T(\psi \otimes w^*) = \rho \otimes z^*$, and $\langle \xi, \rho \rangle = \langle w, z \rangle = 0$, furthermore

$$\|\xi\| \, \|u\| = \|\phi\| \, \|v\|.$$

We now proceed to prove Theorem 3.1. We note that Theorem 3.1 reduces by Lemma 3.2 to showing that if T is a linear isometry of $\| \ \|_1$, then $T(Z) = A \circ Z \circ B^{-1}$, $A \in U(H)$, $B \in U(n)$. Also we have shown that for each Z there are $A_Z \in U(H)$ and $B_Z \in U(n)$ so that $T(Z) = A_Z \circ Z \circ B_Z^{-1}$. Furthermore, we may assume that $A_{tZ} = A_Z$, and $B_{tZ} = B_Z$ for $t \in \boldsymbol{C}$.

(i) $A_{\phi \otimes v^*} \phi = \lambda A_{\phi \otimes w^*} \phi$, for $v, w \in \boldsymbol{C}^n$.

Proof. (a) Suppose $\langle v, w \rangle = 0$. $T(\phi \otimes v^*) = \psi \otimes u^*$, $T(\phi \otimes w^*) = \xi \otimes z^*$. We may assume $\|\phi\| = \|v\| = \|w\| = 1$. T preserves inner products by its expression. Thus we may assume $\|\psi\| = \|\xi\| = \|u\| = \|z\| = 1$. Furthermore $T(\phi \otimes (v^* + w^*)) = \delta \otimes t^*$, $\|\delta\| = 1$. Since T preserves inner products we see $\langle \delta, \psi \rangle \langle t, u \rangle = 1$ and $\langle \delta, \xi \rangle \langle t, z \rangle = 1$. Since $\langle \phi \otimes v^*, \phi \otimes w^* \rangle = 0$ we see that

$$\begin{aligned} \langle \delta \otimes t^*, \delta \otimes t^* \rangle = 2 &= |\langle \delta, \psi \rangle \langle t, u \rangle + \langle \delta, \xi \rangle \langle t, z \rangle| \\ &\leqq |\langle \delta, \psi \rangle| \, |\langle t, u \rangle| + |\langle \delta, \xi \rangle| \, |\langle t, z \rangle| \leqq |\langle t, u \rangle| + |\langle t, z \rangle| \\ &\leqq 2^{1/2} (|\langle t, u \rangle|^2 + |\langle t, z \rangle|^2)^{1/2} \leqq 2. \end{aligned}$$

Thus all inequalities are equalities. Hence $|\langle \delta, \psi \rangle| = 1$, $|\langle \delta, \xi \rangle| = 1$. And $\xi = \lambda \psi$.

(b) If $\langle v, w \rangle \neq 0$, we may assume $\|v\| = 1$ and set $u = w - \langle w, v \rangle v$. Then $\phi \otimes u^* = \phi \otimes w^* - \langle w, v \rangle^- \phi \otimes v^*$. We have by the above $T(\phi \otimes v^*) = \psi \otimes t^*$, $T(\phi \otimes u^*) = \psi \otimes z^*$. Thus $T(\phi \otimes v^*) = \psi \otimes x^*$ as was to be proven.

(ii) $B_{\phi \otimes v^*} v = \lambda B_{\psi \otimes v^*} v$.

Proof. Use the same argument with adjoints.

(iii) If $\langle \phi, \psi \rangle = 0$ then $\langle A_{\phi \otimes v^*} \phi, A_{\psi \otimes w^*} \psi \rangle = 0$.

Proof. By the above we may assume that $v = w$. The result now follows since T preserves inner products.

(iv) If $\langle v, w\rangle = 0$, then $\langle B_{\phi\otimes v^*}v, B_{\psi\otimes w^*}w\rangle = 0$.

Proof. Same argument as (iii).

Now let $\{\phi_j\}$ be an orthonormal basis of H, and $\{e_j\}$ an orthonormal basis of C^n. Then $\{\phi_j \otimes e_i^*\}$ is an orthonormal basis of $L(C^n, H)$. Let $A_1 \in U(H)$, $B_1 \in U(n)$ be defined by $A_1\phi_i = A_{\phi_i\otimes v_1^*}\phi_i$, and $B_1 v_i = B_{\phi_1\otimes v_i^*}v_i$. Then (iii), (iv) imply that A_1 and B_1 do indeed define unitary transformations of H and C^n respectively. Furthermore (i) and (ii) imply that if $\tilde{T}(Z) = A_1^{-1}T(Z)B_1$, then $\tilde{T}(\phi_i \otimes v_j^*) = \lambda_{ij}\phi_i \otimes v_j^*$ with $|\lambda_{ij}| = 1$. And $\lambda_{11} = 1$.

To complete the proof we need the following:

LEMMA 3.3. *Let T be an isometry of $L(C^n, C^k)$ relative to the operator norm. Suppose that these are orthonormal bases $\phi_1, \ldots, \phi_k$ of C^k and $v_1, \ldots, v_n$ of C^n so that $T(\phi_i \otimes v_j^*) = \lambda_{ij}\phi_i \otimes v_j^*$. Then there exist $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_n$ of norm* 1 *so that* (*) $\lambda_{ij} = \alpha_i\beta_j$. *Furthermore, every solution to* (*) *is of the form $\lambda\alpha_1, \ldots, \lambda\alpha_k$, $\bar{\lambda}\beta_1, \ldots, \bar{\lambda}\beta_n$, where λ is of norm* 1.

Proof. That $|\lambda_{ij}| = 1$ follows directly from Lemma 3.2. If $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_n$, and $\alpha_1', \ldots, \alpha_k', \beta_1', \ldots, \beta_k'$ are two solutions of (*) then $\alpha_i\beta_j = \alpha_i'\beta_j'$ for all i, j. Thus $\alpha_i(\alpha_i')^{-1} = \beta_j'\beta_j^{-1}$ for all i, j. Hence $\alpha_i = \lambda\alpha_i'$, $\beta_j = \bar{\lambda}\beta_j'$ for all i, j. We therefore only have to prove existence.

Consider first $k \geqq n$. If $Z^*Z = I$, then $T(Z)^*T(Z) = I$. If $Z = \sum a_{ij}\phi_i \otimes v_j^*$, then $Z^*Z = I$ implies $\sum_{i=1}^k \bar{a}_{ij}a_{is} = \delta_{js}$, $1 \leqq j, s \leqq n$. Thus $\sum_{i=1}^k (T(Z))_{ij}^- T(Z)_{is} = \delta_{js}$. If we define the maps $T_l\colon C^k \to C^k$ by $T_l(a_1, \ldots, a_k) = (a_1\lambda_{1l}, \ldots, a_k\lambda_{kl})$, the assumed form of $T(Z)$ implies: if $(a, b) = 0$, $a, b \in C^k$, then $(T_l a, T_p b) = 0$ $(l \neq p, 1 \leqq l, p \leqq n)$. Put $a = (0, \ldots, 1, \ldots, 1, \ldots, 0)$ ($+1$ in i and j coordinates) and $b = (0, \ldots, 1, \ldots, -1, \ldots, 0)$ ($+1$ in i coordinate and -1 in j coordinate). Then $(a, b) = 0$, and $(T_l a, T_p b) = \bar{\lambda}_{il}\lambda_{ip} - \bar{\lambda}_{jl}\lambda_{jp} = 0$. So $\lambda_{ip}\bar{\lambda}_{il} = \lambda_{jp}\bar{\lambda}_{jl}$.

Select α_i, β_j so that $\lambda_{1p} = \beta_p\alpha_1$, $\lambda_{p1} = \alpha_p$, $1 < p \leqq n$, and $\lambda_{11} = \alpha_1\beta_1 = 1$. Then the above equation shows that $\lambda_{ij} = \alpha_i\beta_j$.

The case $k < n$ follows by taking adjoints.

We now complete the proof of Theorem 3.1. Returning to the notation preceding Lemma 3.3, we note that $\tilde{T}(V_k) \subset V_k$, and $\tilde{T}\colon V_k \to V_k$ is an isometry where $V_k = \{Z \in L(C^n, H) \mid Z = \sum_{i=1,j=1}^{k,n} a_{ij}\phi_i \otimes v_j^*\}$. Clearly $V_k = L(C^n, C^k)$ and $\tilde{T}(\phi_i \otimes v_j^*) = \lambda_{ij}\phi_i \otimes v_j^*$. Thus Lemma 3.3 applies and there are $\alpha_1, \ldots \alpha_k, \beta_1, \ldots, \beta_k$ so that (*) $\lambda_{ij} = \alpha_i\beta_j$. Furthermore $\lambda_{11} = 1$, and we may take the solution such that $\alpha_1 = \beta_1 = 1$. By Lemma 3.3, this selection makes $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_n$ unique. Hence we have α_i, $i \in N$, and $\beta_1, \ldots, \beta_n$, so that $\alpha_i\beta_j = \lambda_{ij}$. Let $A_2\phi_i = \alpha_i\phi_i$ and $B_2 v_i = \beta_i v_i$. Then $A_2^{-1} \circ \tilde{T}(\phi_i \otimes v_j^*) \circ B_2 = \phi_i \otimes v_j^*$ for all i, j. Hence $T(Z) = AZB^{-1}$ with $A = A_1A_2$, $B = B_1B_2$, proving Theorem 3.1.

Note. It seems likely that Theorem 3.1 is true without any restriction on H. Possibly a proof using the lemmas above based on the hereditary collection of finite dimensional subspaces $L(C^n, W)$, $W \subset H$, $\dim W < \infty$, would work.

4. **The automorphism group of a Cartan domain of type I.** Let H be a Hilbert space and let Q be the quadratic form on $H \times \boldsymbol{C}^n$ given by $Q(z, v) = \|z\|^2 - \|v\|^2$. Let $U(H, n)$ be the set of all continuous endomorphisms A of $H \times \boldsymbol{C}^n$ so that $Q(A(z, w)) = Q(z, w)$, and $Q(A^*(z, w)) = Q(z, w)$. (It can be shown that the latter condition is implied by the former if we assume that A is invertible.)

Let $A = \left[\begin{smallmatrix} B & C \\ D & E \end{smallmatrix}\right] \in U(H, n)$ where $B: H \to H$, $C: \boldsymbol{C}^n \to H$, $D: \boldsymbol{C}^n \to H$, and $E: \boldsymbol{C}^n \to \boldsymbol{C}^n$. We list some relations between B, C, D, and E gotten directly from the definitions.

(1) $B^*B - DD^* = I_H = I$.

(2) $E^*E - C^*C = I_{\boldsymbol{C}^n} = I$.

(3) $C^*B - E^*D^* = 0$.

But $A^* = \left[\begin{smallmatrix} B^* & D \\ C^* & E \end{smallmatrix}\right]$, so (1), (2), (3) give immediately:

(1′) $BB^* - CC^* = I$.

(2′) $EE^* - D^*D = I_n$.

(3′) $D^*B^* - EC^* = 0$.

Setting $\tilde{A} = \left[\begin{smallmatrix} B^* & -D \\ -C^* & E \end{smallmatrix}\right]$ we see that $A\tilde{A} = \tilde{A}A = I$. Thus $U(H, n)$ is a subgroup of $GL(H \times \boldsymbol{C}^n)$. Furthermore if A is invertible and satisfies (1), (2), (3) or (1′), (2′), (3′) then $A \in U(H, n)$.

LEMMA 4.1. *Let $A = \left[\begin{smallmatrix} B & C \\ D & E \end{smallmatrix}\right] \in U(H, n)$. If $Z \in D_n(H)$, then $D^*Z + E \in GL(n, \boldsymbol{C})$.*

Proof. If $v \in \boldsymbol{C}^n$ is a unit vector, then by the definition of $U(H, n)$ we have (*) $\|BZv + Cv\|^2 - \|D^*Zv + Ev\|^2 = \|Zv\|^2 - \|v\|^2$. Since $Z \in D_n(H)$, $\|Zv\|^2 < 1$. Thus $\|BZv + Cv\|^2 - \|D^*Zv + Ev\|^2 < 0$. Hence $\|D^*Zv + Ev\|^2 \neq 0$. This proves the result.

If $A = \left[\begin{smallmatrix} B & C \\ D & E \end{smallmatrix}\right] \in U(H, n)$, and $Z \in D_n(H)$, put $A \cdot Z = (BZ + C) \circ (D^*Z + E)^{-1}$. We assert that $A \cdot Z \in D_n(H)$. Indeed, $\|A \cdot Zv\|^2 = \|(BZ + C) \circ (D^*Z + E)^{-1}v\|^2 < \|(D^*Z + E) \circ (D^*Z + E)^{-1}v\|^2$ (by (*) in the proof of Lemma 4.1) < 1. Thus we have $U(H, n)$ acting as a group of holomorphic automorphisms of $D_n(H)$.

THEOREM 4.1. *Every holomorphic automorphism of $D_n(H)$ is given by*

$$Z \to (BZ + C) \circ (D^*Z + E)^{-1},$$

where $\left[\begin{smallmatrix} B & C \\ D & E \end{smallmatrix}\right] \in U(H, n)$.

Proof. Clearly if $B \in U(H)$, $E \in U(n)$, then $\left[\begin{smallmatrix} B & 0 \\ 0 & E \end{smallmatrix}\right] \in U(H, n)$. Thus the automorphisms $Z \to B \circ Z \circ E^{-1}$ are gotten from $U(H, n)$. By Theorem 3.1 we need only prove that $U(H, n)$ acts transitively.

Let C be an arbitrary element of $L(\boldsymbol{C}^n, H)$. We assert that

$$A_C = \begin{bmatrix} (I + CC^*)^{1/2} & C \\ C^* & (I + C^*C)^{1/2} \end{bmatrix} \in U(H, n).$$

Indeed, it is obvious that A_C satisfies (1) and (2).

We show A_C satisfies (3). Since C^*C is a Hermitian positive semidefinite operator on $\boldsymbol{C}^n$ we see that there is an orthonormal basis $v_1, \ldots, v_n$ of $\boldsymbol{C}^n$ so that C^*C

$=\sum \lambda_i v_i \otimes v_i^*$, $\lambda_i \geqq 0$. Setting $(\lambda_i)^{1/2}\phi_i = Cv_i$, we see that $\{\phi_1, \ldots, \phi_n\}$ is an orthonormal set and $C=\sum (\lambda_i)^{1/2}\phi_i \otimes v_i^*$. Hence $CC^* = \sum \lambda_i \phi_i \otimes \phi_i^*$. Let $P: H \to \{\phi_1, \ldots, \phi_n\}^\perp$ be the orthogonal projection. Then

$$(I+CC^*)^{1/2} = \sum_{i=1}^{n} (1+\lambda_i)^{1/2}\phi_i \otimes \phi_i^* + P.$$

$(1+C^*C)^{1/2} = \sum_{i=1}^n (1+\lambda_i)^{1/2} v_i \otimes v_i^*$. $C^* = \sum (\lambda_i)^{1/2} v_i \otimes \phi_i^*$. Hence $C^*(1+CC^*)^{1/2} = \sum_{i=1}^n (1+\lambda_i)^{1/2} v_i \otimes \phi_i^*$ and $(1+CC^*)^{1/2}C^* = \sum_{i=1}^n (1+\lambda_i)^{1/2} v_i \otimes \phi_i^*$. This proves that A_C satisfies (3).

Now $A_C(0) = C \circ (I+C^*C)^{-1/2}$. Thus if $Z \in D_n(H)$ we must find $C \in L(\boldsymbol{C}^n, H)$ so that $Z = C \circ (I+C^*C)^{-1/2}$. In §3 we saw that Z could be written in the form $\sum_{i=1}^n \mu_i \phi_i \otimes v_i^*$ with $0 \leqq \mu_i < 1$ and $\{\phi_1, \ldots, \phi_n\}$, $\{v_1, \ldots, v_n\}$ orthonormal sets. Let $C = \sum_{i=1}^n (\lambda_i)^{1/2}\phi_i \otimes v_i^*$, $0 \leqq \lambda_i < \infty$. Then $(I+C^*C)^{-1/2} = \sum_{i=1}^n (1+\lambda_i)^{-1/2} v_i \otimes v_i^*$. Hence $C \circ (I+C^*C)^{-1/2} = \sum_{i=1}^n (\lambda_i)^{1/2}(1+\lambda_i)^{-1/2}\phi_i \otimes v_i^*$. Thus if the λ_i are defined by $(\lambda_i)^{1/2}(1+\lambda_i)^{-1/2} = \mu_i$ we see that $A_C(0) = Z$.

Note. Theorem 4.1 implies that $D_n(H)$ has the structure of a homogeneous complex Hilbert manifold with Hermitian metric $\langle\ ,\ \rangle$. And $\langle\ ,\ \rangle_0 = \langle\ ,\ \rangle$. With this normalization, $\langle\ ,\ \rangle_z$ is a direct generalization of the Bergman metric on $D_n(\boldsymbol{C}^k)$. In the case $n=1$, $\langle\ ,\ \rangle_z = \langle\ ,\ \rangle/(1-\|z\|^2)^2$. (Compare the invariant Hermitian metrics here with the Finsler metrics of Earle and Hamilton [4].) We leave it to the reader to show that this implies a Schwartz-Pick lemma for the unit ball of a separable Hilbert space.

5. **Polyballs.** Let H be a separable Hilbert space. Let $B = B_1(0) = D_1(H)$ be the unit ball of H. Let B^k be the k-fold product of B with itself. Then B^k is an open bounded domain in H^k (the k-fold product of H with itself). Let $\|(z_1, \ldots, z_k)\|_1 = \max\{\|z_i\| \mid i=1, \ldots, k\}$. Then B^k is clearly the unit ball of the norm $\|\cdots\|_1$ in H^k.

THEOREM 5.1. *Let S_k be the symmetric group in k letters. If $\sigma \in S_k$ let the corresponding permutation be given by $i \to \sigma i$. Let $f: B^k \to B^k$ be a holomorphic automorphism fixing* 0. *Then there are elements $A_i \in U(H)$, $i=1, \ldots, k$, and $\sigma \in S_k$ so that $f(z_1, \ldots, z_k) = (A_1 z_{\sigma_1}, \ldots, A_k z_{\sigma_k})$.*

Proof. By Theorem 2.1, f is given by a linear isometry T of $(H^n, \|\cdots\|_1)$. It is easy to see that the extreme points of $\bar{B}^k$ are the elements $(z_1, \ldots, z_k)$ so that $\|z_i\| = 1$, $i=1, \ldots, k$. Let T be given by the matrix (A_{ij}). That is $T(z_1, \ldots, z_k) = (w_1, \ldots, w_k)$ with $w_j = \sum_{i=1}^k A_{ij} z_i$. By the above, we see that if $\|z_i\| = 1$, $i=1, \ldots, k$, then $\|w_j\| = 1$, $j=1, \ldots, k$. Thus $\|\sum_j A_{ji} \exp(i\theta_j) z_j\|^2 = 1$ for all i, $\theta_1, \ldots, \theta_k \in \boldsymbol{R}$. Hence $\sum_{j,k} \exp(i(\theta_j - \theta_k))\langle A_{ji} z_j, A_{ki} z_k\rangle = 1$. This implies that

(a) $\sum_j \langle A_{ji} z_j, A_{ji} z_j\rangle = 1$.

(b) $\langle A_{ji} z_j, A_{ki} z_k\rangle = 0$, $k \neq j$.

Suppose $A_{ji} z_j \neq 0$, then $A_{ki} z_k = 0$ for $k \neq j$. Since z_k is an arbitrary unit vector this implies that, for each i, there is exactly one j so that $A_{ji} \neq 0$. Set $\sigma_i = j$ and the result follows.

THEOREM 5.2. *Let H_1, H_2 be separable Hilbert spaces.*

(i) *If* $\dim H_i=\infty$ $(i=1,2)$ *then* $D_n(H_1)$ *and* $D_m(H_2)$ *are holomorphically isomorphic if and only if* $n=m$.

(ii) *If* $H_1=\boldsymbol{C}^m$, $H_2=\boldsymbol{C}^l$ *then* $D_n(H_1)$ *is holomorphically isomorphic with* $D_k(H_2)$ *if and only if* $\{n, m\}=\{k, l\}$.

(iii) $D_n(H_1)$ *is inequivalent with* B^k *for* $n\geqq 1$, $k>1$.

Proof. Let D_1 and D_2 be any two of the above domains. Let T: $D_1\to D_2$ be a complex analytic isomorphism. We will show that D_1 must equal D_2. Since D_i has a transitive group of analytic isomorphisms, we may assume $T(0)=0$. By Theorem 2.1, T is a linear isomorphism of the underlying Banach spaces. If $I(D_i)$ is the group of 0-preserving automorphisms of D_i, T induces a map $\tilde{T}$: $I(D_1)\to I(D_2)$ by $\tilde{T}(f)=T\circ f\circ T^{-1}$. $\tilde{T}$ is a linear group homeomorphism, when $I(D_i)$ is given the usual sup norm topology.

If $D=B^k$, then $I(D)\approx S_k\times U(H)\times U(H)\times\cdots\times U(H)$ (k copies of $U(H)$). Note that the number of components of $I(D)$ is $k!$.

If $D=D_n(H)$, then consider $U(H)\times U(n)=G$. If (A, B) and (C, D) are in G, we say $(A, B)\sim(C, E)$ iff there is a real θ so that $A=e^{i\theta}C$ and $B=e^{i\theta}E$. Then $\sim$ is an equivalence relation, and $I(D)\approx G/\sim$. (See Theorem 3.1, and the way Lemma 3.3 was used in proving it.) If $J=U(H)\times SU(n)$, and $(A, B)\in H$, define $\pi((A, B))$ to be the $\sim$ equivalence class of (A, B) in $I(D)$. So $\pi: J\to I(D)$ is a group homomorphism.

If $\dim H=\infty$, we claim that (π, J) is a connected, simply-connected covering space of $I(D)$. Since $U(H)$ is contractible (Kuiper [10]), J is connected and simply-connected. If (A, B), $(C, E)\in J$, then $\pi((A, B))=\pi((C, E))$ if and only if $\lambda A=C$, $\lambda B=E$. This implies $\lambda^n=1$. Thus π is an n-to-1 map, and local cross sections are easy to construct. Hence, if $\dim H=\infty$, the universal covering group of $I(D)$ is $U(H)\times SU(n)$. The universal covering group has homotopy type that of $SU(n)$, and the first homotopy group of $I(D)$ is Z_n.

If $\dim H<\infty$, the universal covering group of $I(D)$ has homotopy type $SU(l)\times SU(n)$, where $l=\dim H$.

Returning to the proof of Theorem 5.1, $\tilde{T}$: $I(D_1)\to I(D_2)$. If $D_1=B^k$, then $(k>1)$ consideration of the number of components of $I(D_1)$ shows that $D_2=B^k$.

$\tilde{T}$ lifts to a homeomorphism of the universal covering groups, $(I(D_1))^\sim$ and $(I(D_2))^\sim$. The homotopy types of these covering groups must agree. Thus if $D_1=D_n(H)$, D_2 must also be $D_n(H)$.

The methods used in the above proof could also be used to prove

COROLLARY 5.1. *Let* $D\{k; m_i, q_i\}$ *be* $D_{m_1}(H_1)\times\cdots\times D_{m_k}(H_k)$, *with* $\dim H_i=q_i$ $(1\leqq q_i\leqq\infty)$. *Then* $D\{j; m_i', q_i'\}$ *and* $D\{k; m_i, q_i\}$ *are isomorphic iff* $j=k$ *and there is some* $\sigma\in S_k$ *so that* $m_{\sigma i}=m_i'$, $q_{\sigma i}=q_i'$, *or* $m_{\sigma i}=q_i'$, $q_{\sigma i}=m_i'$, $1\leqq i\leqq k$. B^l *and* $D\{k; m_i, q_i\}$ *are isomorphic if and only if* $l=k$ *and* $m_i\equiv 1$, $q_i\equiv\infty$.

6. **Siegel domains of type II.** Let M and N be Hilbert spaces. Let M_R be a real form of M. Let C be an open convex cone in M_R not containing a straight line. A

Hermitian form on N with values in M is a continuous sesquilinear form $F: N\times N\to M$. F is said to be C-positive if $F(n, n)\in \bar{C}$ and $F(n, n)=0$ if and only if $n=0$.

The domain $\mathscr{S}(M, N, C, F)=\{(m, n) \mid m\in M, n\in N,$ and $\operatorname{Im}(m)-F(n, n)\in C\}$ for F a Hermitian C-positive form is called a Siegel domain of type II.

The classical linear fractional transformations can be used to exhibit $D_n(H)$ as a Siegel domain of type II.

THEOREM 6.1. *Let $M=L(\boldsymbol{C}^n, \boldsymbol{C}^n)$. A real form of M is the Hermitian operators $\mathscr{H}$. Let $C=\{h\in\mathscr{H} \mid h$ positive definite$\}$. Then C is an open convex cone in $\mathscr{H}$ not containing any lines. Put $N=L(\boldsymbol{C}^n, H)$. If $F: N\times N\to M$ is defined by $F(n_1, n_2)=n_2^*n_1$, then F is C-positive. The domain $\mathscr{S}=\mathscr{S}(M, N, C, F)$ is isomorphic to $D_n(H)$, and the isomorphism is given by $G: \mathscr{S}\to D_n(H)$ where $G(m, n)=(m-iI)(m+iI)^{-1}+2n(m+iI)^{-1}$. ($G(m, n)\in L(\boldsymbol{C}^n, \boldsymbol{C}^n)\oplus L(\boldsymbol{C}^n, H)\approx L(\boldsymbol{C}^n, \boldsymbol{C}^n\oplus H)\approx L(\boldsymbol{C}^n, H)$.)*

Proof. If $m\in M$, then $h_1=\frac{1}{2}(m+m^*)\in\mathscr{H}$ and $h_2=-i\frac{1}{2}(m-m^*)\in\mathscr{H}$. And $m=h_1+ih_2$. So $\mathscr{H}$ is a real form of M. That C and F are as required is equally easy to see.

If $(m, n)\in\mathscr{S}$, then $m+iI$ is invertible. For if $(m+iI)v=0$ then $(m=h_1+ih_2, h_i\in H)$ we see $\langle h_1v, v\rangle+i\langle h_2v, v\rangle=-i\langle v, v\rangle$. But $\langle h_iv, v\rangle\in\boldsymbol{R}$, $i=1, 2$, $\langle h_1v, v\rangle=0$ and $\langle h_2v, v\rangle=-\langle v, v\rangle$.

Since $(m, n)\in\mathscr{S}$, $h_2-F(n, n)\in C$, therefore $h_2\in C$ and v must be 0.

Hence $G: \mathscr{S}\to L(\boldsymbol{C}^n, H)$ is well defined. If $(m, n)\in\mathscr{S}$, we will compute the norm of $G(m, n)\in L(\boldsymbol{C}^n, H)$. If $\|v\|=1$, $v\in\boldsymbol{C}^n$, then (being careful to use the correct norm)

$$\begin{aligned}\|G(m, n)v\|^2 &= \|(m-iI)(m+iI)^{-1}v\|^2+\|2n(m+iI)^{-1}v\|^2\\ &= \langle(m+iI)^{-1}v, (m^*m-2\operatorname{Im} m+I)(m+iI)^{-1}v\rangle\\ &\quad+\langle(m+iI)^{-1}v, 4n^*n(m+iI)^{-1}v\rangle\\ &= \langle T^{-1}v, CT^{-1}v\rangle,\end{aligned}$$

where $T=m+iI$, and $C=(m^*m+4(n^*n-\operatorname{Im}(m))+2\operatorname{Im} m+I)$.

We claim that $\|T^{*-1}CT^{-1}\|<1$ if and only if $(n, m)\in\mathscr{S}$. For, $T^*T=m^*m+2\operatorname{Im}(m)+I$. Thus $(n, m)\in\mathscr{S}$ if and only if $\langle Cv, v\rangle<\langle T^*Tv, v\rangle=\langle Tv, Tv\rangle$ which is the same as $\langle CT^{-1}v, T^{-1}v\rangle<\langle v, v\rangle$ or $\langle T^{*-1}CT^{-1}v, v\rangle<\langle v, v\rangle$. We conclude that the image of G is in $D_n(H)$ using the formula $\|Q\|=\sup\{|\langle Qv, v\rangle| \mid \|v\|=1\}$ (Q Hermitian) and the definiteness of C.

We leave to the reader the task of showing that G is onto and constructing G^{-1}.

In particular, Theorem 6.1 ($n=1$) realizes $B=D_1(H)$ as the set of all $(z, w)\in H\times\boldsymbol{C}$ so that $\operatorname{Im} w-\langle z, z\rangle>0$. If $\dim H=1$, the map G of Theorem 6.1 is the Cayley transform $K_i(z)=(z-i)/(z+i)$ taking the upper half plane to the unit disc, with $K_i(i)=0$. We can find similar maps K_a ($\operatorname{Im} a>0$) transforming the upper half plane to the unit disc, with $K_a(a)=0$.

Suppose now that $p \in l^2 = l^2_R \otimes C$, and $\inf(\operatorname{Im} p_k) > 0$. If we put $\tilde{K}_p(z) = (K_{p_1}(z_1), K_{p_2}(z_2), \ldots)$ for $z = (z_1, \ldots) \in l^2$ $(p = (p_1, \ldots))$ then $\tilde{K}_p: H^\infty \to D^\infty$ is a complex analytic isomorphism, with $\tilde{K}_p(p) = 0$. $(H^\infty = \{z \in l^2 \mid \inf(\operatorname{Im} z) > 0\}.)$ (In particular we can use the maps $\tilde{K}_p$ to show that the automorphisms of D^∞ act transitively.)

The relevance of Proposition 1.2 is that the classical result (Pjatetskii-Shapiro et al. [12]) that a Siegel domain of type II is holomorphically equivalent to a bounded domain is false in Hilbert space. Since H^∞ is clearly a Siegel domain of type II this gives the (unfortunate) counterexample.

7. **Some examples and generalizations.** (i) Let H be a separable Hilbert space. Let $\mathscr{H}$ denote the space of all Hilbert-Schmidt operators $A: H \to H$ (that is, A is continuous and given $\{\phi_n\}$ an orthonormal basis of H, $\operatorname{tr} A = \sum \langle A\phi_n, \phi_n \rangle < \infty$). Let $U(H, \infty)$ be the set of all $A = \left[\begin{smallmatrix} B & C \\ D & E \end{smallmatrix}\right]$ with $B, C, D, E: H \to H$ and $C, D \in \mathscr{H}$, subject to (1), (2), (3) and (1′), (2′), (3′) of §4. Then as in §4, $U(H, \infty)$ is a subgroup of $GL(H \times H)$. Let $D_\infty(H)$ be the unit ball in $\mathscr{H}$ relative to the norm

$$\|A\|_1 = \sup\{\|Av\| \mid \|v\| = 1\}.$$

Let $\langle\ ,\ \rangle$ be the Hilbert-Schmidt inner product on $\mathscr{H}$. Then if $A \in \mathscr{H}$, $\|A\|_1 \leqq \langle A, A \rangle^{1/2}$. Thus $D_\infty(H)$ is open in $(\mathscr{H}, \langle\ ,\ \rangle)$. The arguments of §4 imply that $U(H, \infty)$ act as a transitive group of holomorphic automorphisms of $D_\infty(H)$ by linear fractional transformations.

We note that an argument analogous to the proof of Proposition 1.2 shows that $D_\infty(H)$ is not holomorphically equivalent to a bounded domain in $\mathscr{H}$.

(ii) *An infinite dimensional analogue of a Siegel ball.* Let $H, \mathscr{H}$ be as above. Let H_R be a fixed real form of H and let $(\ ,\ )$ be the symmetric, complex bilinear extension of the inner product on H_R. Let $v \to \bar{v}$ be the conjugation of H, corresponding to H_R. Then $\langle v, w \rangle = (v, \bar{w})$. Let ${}^tA$ (for $A \in \mathscr{H}$) be defined by $(Av, w) = (v, {}^tAw)$. Let $\mathscr{S}$ be the set of all $A \in \mathscr{H}$ so that ${}^tA = A$. Let $B_{\mathscr{S}}$ be the set of all $A \in \mathscr{S}$ so that $\|A\|_1 < 1$ ($\|\cdots\|_1$ as in (i)), then $B_{\mathscr{S}}$ is an open subset of $(\mathscr{S}, \langle\ ,\ \rangle)$ ($\langle\ ,\ \rangle$ the Hilbert-Schmidt norm). Let $Sp(\infty)$ be defined to be the group of all elements g of $GL(H_R \times H_R)$ so that ${}^tg\left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]g = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$, and $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ with B, C Hilbert-Schmidt. Then following C. L. Siegel [13] (or R. S. Phillips [11]) we find that $Sp(\infty)$ acts as a transitive group of linear fractional transformations of $B_{\mathscr{S}}$.

8. **Questions, comments, and a result.** (i) It is known (S. Bergman [2], Fuks [5]), that Corollary 2.1 is true for finite dimensional circular domains that are not necessarily convex. Is the corresponding result in a Banach space true? Also, when $\dim V < \infty$, a bounded subgroup of the general linear group is conjugate to a subgroup of the unitary group. If $\dim V = \infty$, this is not necessarily true. But we can ask: if a bounded subgroup is the isotropy group of 0 arising as in Corollary 2.1 for V a Hilbert space, is it necessarily conjugate to a subgroup of the unitary group?

(ii) In [8] it was proved that every holomorphic automorphism of $D_1(H)$ has a fixed point in cl $(D_1(H))$. We prove

PROPOSITION 8.1. *Every holomorphic automorphism of B^k (see §5 for notation) has a fixed point in $\bar{B}^k$.*

Proof. By Theorem 4.1 and Theorem 5.1 every holomorphic automorphism of B^k is given by $F(z_1, \ldots, z_k) = (f_1(z_{\sigma_1}), \ldots, f_k(z_{\sigma_k}))$, where f_i is a holomorphic automorphism of $B = D_1(H)$ $(1 \leqq i \leqq k)$, and $\sigma \in S_k$. We prove the result by induction on k.

If $k=1$ the result is assumed true. Suppose the result is true for all $j<k$. If $1=\sigma^j 1$, for $0<j<k$, a fixed point for F reduces to a fixed point for maps on B^j and B^{k-j}. So we may assume that $\sigma^j 1 \neq 1$ if $0<j<k$, and, by renumbering, the map F is of the form $F(z_1, \ldots, z_k) = (f_1(z_2), f_2(z_3), \ldots, f_{k-1}(z_k), f_k(z_1))$. We can by the induction hypothesis find a fixed point $(w_1, \ldots, w_{k-1})$ for the map G of B^{k-1} defined by $G(z_1, \ldots, z_{k-1}) = (f_1(z_2), f_2(z_3), \ldots, f_{k-1}(f_k(z_1)))$. Then $(w_1, \ldots, w_{k-1}, f_k(w_1))$ is a fixed point for F.

(iii) Is the fixed point property above true for $D_n(H)$?

(iv) The fixed point result of Proposition 8.1 is related to a negative result of Kakutani [9]: if dim $H=\infty$, there is a homeomorphism f of $\bar{B} = \text{cl}\,(D_1(H))$ that is a diffeomorphism of B and has no fixed point in $\bar{B}$. Is there an ε-fixed point for f? That is, given $\varepsilon > 0$, is there $z \in \bar{B}$ so that $\|f(z) - z\| < \varepsilon$? It is simple to verify (using a result of Earle and Hamilton [4]) that any analytic f (not necessarily an automorphism) has an ε-fixed point.

An example of a continuous map of $\bar{B}$ into itself without an ε-fixed point has been given by W. L. Black. It is not hard then to construct a C^∞ map of $\bar{B}$ into itself, continuous on $\bar{B}$ without ε-fixed points for ε small enough. Do uniformly continuous maps have ε-fixed points?

(v) Is there a condition on convex open cones of a real Hilbert space $H_{\boldsymbol{R}}$ that guarantees that the associated Siegel domains are holomorphically equivalent to bounded domains? A sufficient condition is dim $H_{\boldsymbol{R}} < \infty$. Necessary conditions might be derived from Corollary 1.2.

(vi) Is Corollary 1.2 or some variant of it necessary and sufficient?

(vii) If M is a complex Hilbert manifold, when is $M \approx M \times B$? Is there a bounded domain with this property?

(viii) D^∞ has a transitive group of automorphisms (§6). If f is an automorphism of D^∞ fixing 0 we suspect that $f(z_1, z_2, \ldots) = (\exp(i\theta_1) z_{\sigma_1}, \exp(i\theta)_2 z_{\sigma_2}, \ldots)$ where $\theta_j \in \boldsymbol{R}$ and σ is a permutation of the positive integers.

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