ZEROS OF PARTIAL SUMS AND REMAINDERS OF POWER SERIES

BY

J. D. BUCKHOLTZ AND J. K. SHAW

Abstract. For a power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ let $s_n(f)$ denote the maximum modulus of the zeros of the $n$th partial sum of $f$ and let $r_n(f)$ denote the smallest modulus of a zero of the $n$th normalized remainder $\sum_{k=n}^{\infty} a_k z^{k-n}$. The present paper investigates the relationships between the growth of the analytic function $f$ and the behavior of the sequences $(s_n(f))$ and $(r_n(f))$. The principal growth measure used is that of $R$-type: if $R = \{R_n\}$ is a nondecreasing sequence of positive numbers such that $\lim (R_{n+1}/R_n) = 1$, then the $R$-type of $f$ is $\tau(f) = \limsup |a_n R_1 R_2 \cdots R_n|^{1/n}$. We prove that there is a constant $P$ such that

$$\tau_n(f) \liminf (s_n(f)/R_n) \leq P \quad \text{and} \quad \tau_n(f) \limsup (r_n(f)/R_n) \leq (1/P)$$

for functions $f$ of positive finite $R$-type. The constant $P$ cannot be replaced by a smaller number in either inequality; $P$ is called the power series constant.

1. Introduction. The following theorem is a consequence of results of the first author [3] and J. L. Frank [4].

**Theorem A.** Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ have radius of convergence $c(f)$, $0 < c(f) < \infty$. There exists an absolute constant $P$ such that, if $\varepsilon > 0$, then

(i) infinitely many of the partial sums

$$S_n(f; z) = \sum_{k=0}^{n} a_k z^k \quad (n = 1, 2, 3, \ldots)$$

have all their zeros in the disc $|z| \leq c(f)(P + \varepsilon)$;

(ii) infinitely many of the normalized remainders

$$\mathcal{S}_n(f) = \sum_{k=n}^{\infty} a_k z^{k-n} \quad (n = 0, 1, 2, \ldots)$$

have no zero in the disc $|z| \leq c(f)(P + \varepsilon)^{-1}$;

(iii) the constant $P$ cannot be replaced by a smaller number in either (i) or (ii).

In view of (iii), the constant $P$ is uniquely determined by Theorem A. We call $P$ the power series constant; its numerical value is known to lie between 1.7818 and

Presented to the Society, January 23, 1970 under the title Partial sums and remainders of power series; received by the editors May 4, 1971.

AMS 1970 subject classifications. Primary 30A08, 30A10; Secondary 30A06.

Key words and phrases. The power series constant, zeros of partial sums, zeros of remainders, $R$-type, entire functions, extremal functions.

Copyright © 1972, American Mathematical Society

269
Our object in the present paper is to give a simpler proof of Theorem A, to investigate the extremal functions associated with it, and to obtain corresponding results for various classes of entire functions.

For \( f(z) = \sum_{k=0}^{\infty} a_k z^k \), let \( s_n = s_n(f) \) denote the largest of the moduli of the zeros of \( S_n(f; z) = \sum_{k=0}^{n} a_k z^k \) \((n = 1, 2, 3, \ldots)\) with the convention that \( s_n = \infty \) if \( a_n = 0 \).

Let \( r_n = r_n(f) \) denote the supremum of numbers \( r \) such that \( \sum_{n=0}^{\infty} a_k z^{k-r} \) is analytic and has no zero in the disc \( |z| < r \). Theorem A is equivalent to the estimates

\[
\liminf_{n \to \infty} s_n(f) \leq c(f) P,
\]

\[
\limsup_{n \to \infty} r_n(f) \geq \frac{c(f)}{P},
\]

for \( 0 < c(f) < \infty \), together with the assertion that the constant \( P \) is best possible in both cases.

Okada [6] has shown that \( \limsup_{n \to \infty} s_n(f) = \infty \) if and only if \( f \) is entire. For entire \( f \), M. Tsuji [6] proved the surprising result that

\[
\log n \leq \limsup_{n \to \infty} s_n(f)
\]

is always equal to the order of \( f \). For functions of positive finite order and type, we are able to sharpen Tsuji's theorem considerably.

**Theorem B.** Suppose the entire function \( f \) is of order \( \rho \) and type \( \tau \), \( 0 < \rho, \tau < \infty \). Then

\[
\limsup_{n \to \infty} \left( \frac{\rho \tau}{n} \right)^{1/\rho} r_n(f) \geq \frac{1}{P}
\]

and

\[
e^{-1/\rho} \leq \liminf_{n \to \infty} \left( \frac{\rho \tau}{n} \right)^{1/\rho} s_n(f) \leq P.
\]

Furthermore, for each of the three inequalities, there exists an \( f \) of order \( \rho \) and type \( \tau \) for which equality is assumed.

Both Theorem A and Theorem B are special cases of a result involving a more general measure of growth for analytic functions. Let \( R = (R_n)_{n=1}^{\infty} \) be a non-decreasing sequence of positive numbers such that \( \lim_{n \to \infty} R_{n+1}/R_n = 1 \). The \( R \)-type, \( \tau_R(f) \), of an analytic function \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) is defined to be

\[
\tau_R(f) = \limsup_{n \to \infty} |a_n R_1 R_2 \cdots R_n|^{1/n}.
\]

If \( R_n \to \infty \) as \( n \to \infty \), \( R \)-type can be related to the growth of the maximum modulus of \( f \) [1, p. 6]. It follows easily from the expression for the type of an entire function in terms of its coefficients that \( f \) is of order \( \rho \) and type \( \tau \), \( 0 < \rho, \tau < \infty \), if and only if
Our principal result is the following.

**Theorem C.** If \( 0 < \tau_R(f) < \infty \), then

\[
\liminf_{n \to \infty} \left( \frac{R_1 R_2 \cdots R_n}{R_n} \right)^{1/n} \leq \tau_R(f) \liminf_{n \to \infty} \frac{r_n(f)}{R_n} \leq P
\]

and

\[
\tau_R(f) \limsup_{n \to \infty} \frac{r_n(f)}{R_n} \geq \frac{1}{P}.
\]

Furthermore, for each of the three inequalities, there exists a function of \( R \)-type 1 for which equality is assumed.

If one takes \( R_n = 1 \), Theorem C reduces to Theorem A. If one takes \( R_n = (n/\rho)^{1/\sigma} \), then Theorem C reduces to Theorem B.

Suppose \( 0 < c(f) < \infty \) and \( \varepsilon > 0 \). In 1906, M. B. Porter [5] proved that an infinite sequence of the partial sums of \( f \) tends uniformly to \( \infty \) outside the disc \( |z| \leq c(f)(2 + \varepsilon) \). In view of Theorem A, the constant 2 in Porter’s theorem cannot be replaced by a number less than \( P \). We prove in §2 that the best possible constant for Porter’s theorem is \( P \). This follows fairly easily from a theorem on the partial sums of polynomials which is of some interest in itself.

**Theorem D.** Let \( Q(z) = a_0 + a_1 z + \cdots + a_n z^n \) be a polynomial of degree \( n \). Then for at least one integer \( k \), \( 0 \leq k \leq n \), we have

\[
|a_0 + a_1 z + \cdots + a_k z^k| \geq |a_n| |z|^k/(n+1)
\]

for all \( |z| \geq P \).

Theorem D guarantees that the partial sum \( a_0 + a_1 z + \cdots + a_k z^k \) has all its zeros in the disc \( |z| \leq P \). Since (1.7) holds for large \( |z| \), we must have

\[
|a_k| \geq |a_n|/(n+1).
\]

In applications, this yields information about the value of \( k \) for which (1.7) holds.

2. **The remainder polynomials.** The treatment of the power series constant in [3] and [4] involves the *remainder polynomials* \( B_n(z; z_0, z_1, \ldots, z_{n-1}) \), defined recursively by

\[
B_0(z) = 1,
\]

\[
B_n(z; z_0, z_1, \ldots, z_{n-1}) = z^n - \sum_{k=0}^{n-1} z_k^{n-k} B_k(z; z_0, z_1, \ldots, z_{k-1}).
\]

Let

\[
H_n = \max |B_n(0; z_0, \ldots, z_{n-1})|,
\]
where the maximum is taken over all sequences \{z_k\}_{k=0}^n whose terms lie on \(|z|=1\). Buckholtz [3] proved that

\[ P = \lim_{n \to \infty} H_n^{1/n} = \sup_{1 \leq n < \infty} H_n^{1/n}. \]

For a power series \( f(z) = \sum_{k=0}^\infty a_k z^k \), we write (2.1) in the form

\[ z^n = \sum_{k=0}^n z_k^{-n-k} B_k(z; z_0, \ldots, z_{k-1}) \]

and substitute this expression into the power series for \( f \). We obtain the formal expansion

\[
\begin{aligned}
f(z) &= \sum_{n=0}^\infty a_n z^n = \sum_{n=0}^\infty a_n \left\{ \sum_{k=0}^n z_k^{-n-k} B_k(z; z_0, \ldots, z_{k-1}) \right\} \\
&= \sum_{k=0}^\infty B_k(z; z_0, \ldots, z_{k-1}) \sum_{n=k}^\infty a_n z_k^{-n-k} = \sum_{k=0}^\infty \mathcal{S}^k f(z_k) B_k(z; z_0, \ldots, z_{k-1}),
\end{aligned}
\]

which holds whenever the interchange in the order of summation can be justified. In particular, (2.2) holds if \( f \) is a polynomial and yields considerable information when \( f \) is taken to be a remainder polynomial. In the latter case, an easy induction argument establishes the identity

\[ B_n(z; z_0, \ldots, z_{n-1}) = B_{n-k}(z; z_k, \ldots, z_{n-1}), \quad 0 \leq k \leq n, \]

for the remainder polynomials also satisfy the following properties:

\begin{align*}
(2.4) & \quad B_k(\lambda z; \lambda z_0, \ldots, \lambda z_{n-1}) = \lambda^n B_k(z; z_0, \ldots, z_{n-1}), \\
(2.5) & \quad B_k(z_0, z_0, \ldots, z_{n-1}) = 0, \\
(2.6) & \quad z^n B_n(1/z; z_0, \ldots, z_1) = \sum_{k=0}^n B_k(0; z_k, \ldots, z_1) z^k, \\
(2.7) & \quad B_n(z; z_n, \ldots, z_1) = \sum_{k=0}^n B_k(0; z_k, \ldots, z_1) B_{n-k}(z; z_n, \ldots, z_{n+1}, 0, \ldots, 0) \text{ for } 0 \leq n_1 \leq n, \\
(2.8) & \quad H_{m+n} = H_m H_n \text{ for nonnegative integers } m \text{ and } n.
\end{align*}

The proofs of these identities may be found in [3].

We are now ready to prove Theorem D. Thus let \( Q(z) = a_0 + a_1 z + \cdots + a_n z^n \) be a polynomial of degree \( n \). Define \( f(z) = z^n Q(1/z) = b_0 + b_1 z + \cdots + b_n z^n \); note that \( b_{n-k} = a_k, 0 \leq k \leq n \). Let \( \{z_j\}_{j=0}^\infty \) be a sequence of complex numbers satisfying

\[ |S^j f(z_j)| = \min_{|z| \leq 1/P} |S^j f(z)|, \quad 0 \leq j \leq n. \]

From (2.2),

\[ |f(0)| \leq \sum_{k=0}^n |S^k f(z_k)| |B_k(0; z_0, \ldots, z_{k-1})|. \]

Setting \( w_k = P z_k, 0 \leq k \leq n \), we have \(|w_k| \leq 1|P| \) and, by (2.4),

\[
\begin{aligned}
|B_k(0; z_0, \ldots, z_{k-1})| &= |B_k(0; w_0/P, \ldots, w_{k-1}/P)| \\
&= (1/P^k) |B_k(0; w_0, \ldots, w_{k-1})| \leq (1/P^k) H_k \leq 1,
\end{aligned}
\]
for $0 \leq k \leq n$. Hence $|f(0)| \leq \sum_{k=0}^{\infty} |S^k f(z_k)|$ and so $|f(0)| \leq (n+1)|S^n f(z_m)|$ for some $m$, $0 \leq m \leq n$. Since $f(0) = b_0$, we have $|S^n f(z)| \leq |b_0|/(n+1)$ for all $|z| \leq 1/P$. Now

$$S^n f(z) = b_m + b_{m+1}z + \cdots + b_n z^{n-m}$$

and therefore, replacing $z$ by $1/z$, we obtain

$$|b_m z^{n-m} + b_{m+1} z^{n-m-1} + \cdots + b_n| \geq |z|^{n-m}|b_0|/(n+1)$$

for all $|z| \geq P$. Letting $p = n - m$, this inequality is equivalent to

$$|a_0 + a_1 z + \cdots + a_p z^p| \geq |z|^p|a_n|/(n+1)$$

for all $|z| \geq P$, and this completes the proof.

Corollary 1. Suppose that the power series $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence less than 1. Then there are infinitely many integers $k$ such that

$$|\sum_{j=0}^{k} a_j z^j| \geq |z|^k$$

for all $|z| \geq P$.

Proof. For each positive integer $n$ such that $a_n \neq 0$, let $k(n)$ denote the least positive integer $k$ for which (1.7) holds. The condition $\limsup |a_n|^{1/n} > 1$ implies that there is an infinite set $I$ of positive integers such that $|a_n|/(n+1) > n$ for all $n \in I$. For each $n \in I$ we therefore have $|\sum_{j=0}^{k(n)} a_j z^j| \geq |z|^{k(n)}$ and, by (1.8), $|a_{k(n)}| \geq |a_n|/(n+1) > n$. The latter condition guarantees that $k(n)$ assumes infinitely many values as $n$ ranges over $I$, and this completes the proof.

Suppose $f$ has radius of convergence $t$, $0 < t < \infty$. Let $\epsilon > 0$ and define $g(z) = f(tz/(1 - \epsilon))$. Then $c(g) < 1$ and (2.9) implies that $s_n(g) \leq P$ for infinitely many integers $n$. Thus $\liminf_{n \to \infty} s_n(g) \leq P$. But $s_n(g) = ((1 - \epsilon)/t)s_n(f)$ and therefore $\liminf_{n \to \infty} s_n(f) \leq tP/(1 - \epsilon)$. It follows that $\liminf_{n \to \infty} s_n(f) \leq c(f)P$ and this proves (1.1).

Lemma 1. If $n$ is a nonnegative integer, then

$$1 \leq P^n/H_n \leq 17.$$ 

This will be proved in §3.

Let $m$ be a positive integer and suppose $z_0, z_1, \ldots, z_{m-1}$ lie on $|z| = 1$. If $k \geq m$, then (2.1) implies

$$B_k(0; z_0, \ldots, z_{m-1}, 0, \ldots, 0) = - \sum_{j=0}^{m-1} z_j^{-k} B_j(0; z_0, \ldots, z_{j-1}).$$

It follows that

$$|B_k(0; z_0, \ldots, z_{m-1}, 0, \ldots, 0)| \leq \sum_{j=0}^{m-1} H_j \leq \sum_{j=0}^{m-1} P^j < \frac{P^m}{P-1}.$$
The assertion that the constant $P$ is best possible in (1.1) depends on the existence of a function $f$ such that $\lim \inf s_n(f) = c(f)P$. It suffices to construct such an $f$ satisfying $c(f) = 1$.

**Lemma 2.** There exists a power series $\sum_{k=0}^{\infty} A_k z^k$, with radius of convergence 1, such that each partial sum $\sum_{k=0}^{m} A_k z^k$ has a zero of modulus $P$.

**Proof.** For each nonnegative integer $n$, let $\{z^{(n)}_j\}_{j=1}^{n_1}$ be a sequence of complex numbers of modulus $1/P$ such that

$$|B_n(0; z^{(n)}_1, z^{(n)}_2, \ldots, z^{(n)}_n)| = H_n/P^n.$$  

Here, we have used (2.4). If $n, n_1$ and $j$ are positive integers such that $j \leq n_1 \leq n$, then (2.7) implies

$$|B_n(0; z^{(n)}_1, \ldots, z^{(n)}_n)| \leq H_k/P^k \leq 1$$

and (2.11) implies

$$|B_{n-k}(0; z^{(n)}_1, \ldots, z^{(n)}_{n_1+1}, 0, \ldots, 0)| \leq P^{n-n_1}/(P^{n-k}(P-1)).$$

The first sum on the right of (2.13) therefore does not exceed

$$\sum_{k=n_1-j+1}^{n_1-j} \frac{P^{n-n_1}}{(P-1)} = \frac{P^{-j+1} - P^{-n_1}}{(P-1)^2} < \frac{1}{P^{j-1}(P-1)^2}.$$  

If $n_1-j+1 \leq k \leq n_1$, then

$$|B_{n-k}(0; z^{(n)}_1, \ldots, z^{(n)}_{n_1+1}, 0, \ldots, 0)| \leq H_{n-k}/P^{n-k} \leq 1.$$  

In view of (2.10), (2.13) now yields

$$\sum_{k=n_1-j+1}^{n_1} |B_k(0; z^{(n)}_1, \ldots, z^{(n)}_n)| \geq \frac{1}{17} \frac{1}{P^{j-1}(P-1)^2}.$$  

Taking $j=7$ and using the bound $P > 1.78$, we have $\sum_{k=n_1-j+1}^{n_1} |B_k(0; z^{(n)}_1, \ldots, z^{(n)}_n)| > 1/1000$. Therefore $|B_k(0; z^{(n)}_1, \ldots, z^{(n)}_n)| > 1/7000$ for at least one integer $k$, $n_1-6 \leq k \leq n_1$. Moreover, $|B_k(0; z^{(n)}_1, \ldots, z^{(n)}_n)| \leq 1$ for all $n$ and $k$. Now define

$$P_n(z) = z^n B_n(1/z; z^{(n)}_1, \ldots, z^{(n)}_n), \quad n = 0, 1, 2, \ldots.$$  

Since (2.6) implies $P_n(z) = \sum_{k=0}^{\infty} B_k(0; z^{(n)}_1, \ldots, z^{(n)}_n)z^k$, it follows that the coefficients of $P_n$ are bounded by 1 and that in a set of 7 consecutive coefficients, at least one
coefficient has modulus greater than 1/7000. The sequence \( \{P_n\} \) is uniformly bounded on compact subsets of the unit disc. Extract a uniformly convergent subsequence of \( \{P_n\} \) and let \( F \) denote the limit function. Writing \( F(z) = \sum_{k=0}^\infty A_k z^k \), it follows that \( |A_k| \leq 1 \), \( 0 \leq k < \infty \), and that in a set of 7 consecutive coefficients \( A_k \), at least one coefficient has modulus greater than 1/7000. Hence \( c(F) = 1 \). If \( m < n \), then (2.6) implies that the \( m \)th partial sum of \( P_n \) is given by

\[
S_m(P_n; z) = z^m B_m(1/z; z_1^{(m)}, \ldots, z_1^{(n)}).
\]

By (2.5), \( S_m(P_n; 1/z^{(n)}') = 0 \). Since \( S_m(F; z) \) is the uniform limit of a subsequence of \( \{S_m(P_n; z)\} \), it follows that \( S_m(F; z) \) has a zero of modulus \( P \). This completes the proof of the lemma.

The function \( F \) of the preceding lemma satisfies \( c(F) = 1 \) and \( \lim \inf_{n \to \infty} s_n(F) \geq P \). It follows that the constant \( P \) is best possible in (1.1).

We now show that \( P \) is the sharp constant in Porter's theorem. If \( f(z) = \sum_{k=0}^\infty a_k z^k \) has radius of convergence \( t \), then Corollary 1 implies that there are infinitely many integers \( k \) such that \( |\sum_{j=0}^k a_j z^j| \geq (|z|/t(1+\varepsilon))^k \) for all \( |z| \leq tP(1+\varepsilon) \). The corresponding subsequence of partial sums \( \{S_k(f; z)\} \) therefore tends uniformly to \( \infty \) outside the disc \( |z| \leq c(f)P \). On the other hand, we can, by Lemma 3, construct a function \( F \) such that \( c(F) = t \) and such that each partial sum of \( F \) has a zero in \( |z| \leq c(F)P \).

The inequality (1.2) is a special case of (1.6); the latter will be proved in §4. To show that \( P \) is the sharp constant in (1.2), it suffices to construct a function \( G \) satisfying \( c(G) = 1 \) and \( \lim \sup_{n \to \infty} r_n(G) \leq 1/P \).

**Lemma 3.** There exists a power series \( G(z) = \sum_{k=0}^\infty A_k z^k \), with \( c(G) = 1 \), such that each normalized remainder of \( G \) has a zero of modulus \( 1/P \). In particular, \( \lim \sup_{n \to \infty} r_n(G) \leq 1/P \).

**Proof.** Consider the sequence of complex numbers \( \{B_n(0; z_1^{(n)}, \ldots, z_1^{(n)})\}_{n=1}^\infty \) constructed in Lemma 2. For each \( n \) we have \( |z_1^{(n)}| = 1/P \), for \( 1 \leq j \leq n \), \( |B_j(0; z_1^{(n)}, \ldots, z_1^{(n)})| \leq 1 \), for \( 0 \leq j \leq n \), and \( |B_n(0; z_1^{(n)}, \ldots, z_1^{(n)})| = H_n/P^n \). Furthermore, if \( n_1 \leq n \), then \( |B_n(0; z_1^{(n)}, \ldots, z_1^{(n)})| \geq 1/7000 \) for at least one integer \( k \) such that \( n_1 - 6 \leq k \leq n_1 \). By (2.6),

\[
B_n(z; z_1^{(n)}, \ldots, z_1^{(n)}) = \sum_{k=0}^n B_k(0; z_1^{(n)}, \ldots, z_1^{(n)}) z_1^{n-k}.
\]

The sequence \( \{B_n(z; z_1^{(n)}, \ldots, z_1^{(n)})\}_{n=1}^\infty \) is therefore uniformly bounded on compact subsets of the unit disc. Extract a uniformly convergent subsequence from \( \{B_n\} \) and let \( G \) denote the limit function. If \( G(z) = \sum_{k=0}^\infty A_k z^k \), then \( |A_k| \leq 1 \) for all \( k \) and \( |A_k| \geq 1/7000 \) for infinitely many \( k \); thus \( c(G) = 1 \). The identities

\[
\mathcal{G} B_n(z; z_1^{(n)}, \ldots, z_1^{(n)}) = B_{n-k}(z; z_1^{(n-k)}, \ldots, z_1^{(n)}),
\]

\[
B_{n-k}(z_1^{(n-k)}, \ldots, z_1^{(n)}) = 0,
\]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
for $0 \leq k < n$, show that $B_n$ and each of its first $(n-1)$ normalized remainders have zeros of modulus $1/P$. Furthermore, if $m$ is a nonnegative integer, then $\mathcal{S}^mG(z)$ is the uniform limit of a subsequence of $\{\mathcal{S}^m B_n(z; z^n, \ldots, z^n)\}$ on the compact set $|z| \leq (1/P) + \epsilon < 1$. It follows that $\mathcal{S}^mG(z)$ has a zero of modulus $1/P$.

3. The functions $T_m(\mathcal{U})$. For $m = 1, 2, 3, \ldots$, and $0 \leq \mathcal{U} < 1$, define

$$T_m(\mathcal{U}) = \max_{k=m}^{\infty} |B_k(0; w_0, w_1, \ldots, w_{m-1}, 0, \ldots, 0)|$$

where the maximum is taken over all sequences $(w_k)_{k=0}^{m-1}$ whose terms lie on $|z| = 1$. The functions $T_m(\mathcal{U})$ were characterized by Buckholtz [3]. For each $m$, $T_m$ is increasing; the unique solution to the equation $T_m(\mathcal{U}) = 1$ is denoted by $\mathcal{U}_m$. The most important property of the sequence $\{\mathcal{U}_m\}$ is the determination

$$P = \lim_{m \to \infty} \mathcal{U}_m^{-1} = \inf_{1 \leq m \leq \infty} \mathcal{U}_m^{-1}. \tag{3.1}$$

Since $T_m$ is increasing, (3.1) implies

$$T_m(1/P) > 1, \quad m = 1, 2, 3, \ldots \tag{3.2}$$

**Proof of Lemma 1.** By (2.11) and (3.2), we have

$$1 \leq T_m(1/P) \leq \frac{H_m}{P^m} + \frac{H_{m+1}}{P^{m+1}} + \frac{H_{m+2}}{P^{m+2}} + \sum_{k=m+3}^{\infty} \frac{(1/P)^k}{P-1},$$

for each positive integer $m$.

In view of (2.8), the previous inequality implies

$$1 \leq \left(\frac{H_{m+2}}{P^{m+2}}\right) \left[1 + \frac{P}{H_1 + H_2} + \frac{P^m}{P-1} \frac{P^{-m-3}}{1-(1/P)}\right],$$

therefore,

$$1 \leq \left(\frac{H_{m+2}}{P^{m+2}}\right)[1 + P + P^2/2] + P^{-2} (P-1)^{-2}.$$

Using the bounds $1.78 < P < 1.82$, we obtain $H_{m+2}/P^{m+2} \geq 1/17$. It is easily verified that $H_j/P^j > 1/17$ for $j = 1, 2$. Since $P = \sup_{1 \leq n < \infty} H_n$, we have $1/17 \leq H_n/P^n \leq 1$ for all $n$.

4. Main results. In this section, we prove (1.5) and (1.6).

**Lemma 4.** Let $m$ be a positive integer and $\{A_k\}_{k=0}^{\infty}$ a sequence of complex numbers $(A_0 = 1)$ such that $|A_k| \leq 1$ for $k \geq m$. Then for at least one integer $p$, $0 \leq p \leq m-1$, the function $A_p + A_{p+1}z + A_{p+2}z^2 + \cdots$ has no zero in the disc $|z| < \mathcal{U}_m$.

**Proof.** Let $f(z) = 1 + \sum_{k=0}^{\infty} A_k z^k$. We have to show that for some $p$, $0 \leq p \leq m-1$, $\mathcal{S}^p f(z)$ has no zero in $|z| < \mathcal{U}_m$. Let $(z_k)_{k=0}^{\infty}$ be a sequence of points in $|z| < 1$ such that $z_k = 0$ for $k \geq m$. Then, by (2.1),
\[
\sum_{k=0}^{m-1} \mathcal{S}^k f(z_k) B_k(z; z_0, \ldots, z_{k-1}) \\
= \sum_{j=0}^{m-1} A_j \sum_{k=0}^{j} z_k^{j-k} B_k(z; z_0, \ldots, z_{k-1}) + \sum_{j=m}^{\infty} A_j \sum_{k=0}^{m-1} z_k^{j-k} B_k(z; z_0, \ldots, z_{k-1}) \\
= \sum_{j=0}^{m-1} A_j z_j + \sum_{j=m}^{\infty} A_j [z^j - B_j(z; z_0, \ldots, z_{m-1}, 0, \ldots, 0)] \\
= \sum_{j=0}^{m-1} A_j z_j - \sum_{j=m}^{\infty} A_j B_j(z; z_0, \ldots, z_{m-1}, 0, \ldots, 0).
\]

By transposing, we obtain the important identity

\[(4.1) \quad f(z) = \sum_{k=0}^{m-1} \mathcal{S}^k f(z_k) B_k(z; z_0, \ldots, z_{k-1}) + \sum_{k=m}^{\infty} A_k B_k(z; z_0, \ldots, z_{m-1}, 0, \ldots, 0).\]

Without loss of generality, we may assume that each of \(\mathcal{S}^k f(z)\), \(0 \leq k \leq m-1\), has a zero in \(|z| < 1\). For \(0 \leq k \leq m-1\), let \(w_k\) denote the smallest modulus of a zero of \(y^k f(z)\). It follows from (4.1) that

\[1 = f(0) \leq \sum_{k=m}^{\infty} |B_k(0; w_0, \ldots, w_{m-1}, 0, \ldots, 0)|.\]

If \(\mathcal{U} = \max_{0 \leq k \leq m} |w_k|\), then

\[1 \leq \sum_{k=m}^{\infty} \mathcal{U}^k |B_k(0; w_0/\mathcal{U}, \ldots, w_{m-1}/\mathcal{U}, 0, \ldots, 0)| \leq T_m(\mathcal{U})\]

and therefore \(\mathcal{U} \geq \mathcal{U}_m\). Thus there is an integer \(p\), \(0 \leq p \leq m-1\), such that \(|w_p| \geq \mathcal{U}_m\) and it follows that \(\mathcal{S}^p f(z)\) has no zero in \(|z| < \mathcal{U}_m\).

**Lemma 5.** Let \(m\) be a positive integer and \(a_0 + a_1 z + \cdots + a_n z^n\) a polynomial of degree \(n\), \(n \geq m-1\), such that \(|a_k| \leq |a_n|\), \(0 \leq k \leq n\). Then for at least one integer \(p\), \(n - m + 1 \leq p \leq n\), the polynomial \(a_0 + a_1 z + \cdots + a_p z^n\) has all its zeros in the disc \(|z| \leq \mathcal{U}_m^{-1}\).

**Proof.** Let \(a_k = a_{n-k}/a_n\), \(0 \leq k \leq n\). Lemma 4 implies that there exists an integer \(q\), \(0 \leq q \leq m-1\), such that \(A_q + A_{q+1} z + \cdots + A_n z^{n-q}\) does not vanish in \(|z| < \mathcal{U}_m\). Therefore, the function \((a_{n-q}/a_n) + (a_{n-q-1}/a_n) z + \cdots + (a_0/a_n) z^{n-q}\) has no zero in \(|z| < \mathcal{U}_m\), so the same is true of \((z^{n-q}/a_n)(a_0 + a_1 z + \cdots + a_{n-q} z^{n-q})\). It follows that \((1/a_n z^{n-q})(a_0 + a_1 z + \cdots + a_{n-q} z^{n-q})\) has no zero in the region \(|z| > \mathcal{U}_m^{-1}\), hence \(a_0 + a_1 z + \cdots + a_{n-q} z^{n-q}\) has all its zeros in \(|z| \leq \mathcal{U}_m^{-1}\). Taking \(p = n-q\), we obtain the desired result.

**Lemma 6.** Suppose \(f(z) = \sum_{k=0}^{\infty} A_k z^k\) has R-type greater than 1. Then

\[\liminf_{n \to \infty} \frac{s_n(f)}{R_n} \leq P.\]
Proof. If \( f(z) \) is written
\[
f(z) = \sum_{k=0}^{\infty} (a_k/R_1R_2\cdots R_n)z^k,
\]
then \( \tau(f) = \lim \sup_{n \to \infty} |a_n|^{1/n} \). The condition \( \tau(f) > 1 \) implies that there exists an infinite set \( N \) of positive integers such that \( n \in N \) implies \( |a_n| > |a_k|, \quad 0 \leq k < n \).

Let \( m \) be a positive integer and suppose \( n \in N \) is such that \( n \geq m - 1 \). The \( n \)th partial sum of \( f(Rnz) \) is given by
\[
S_n(f; Rnz) = a_0 + a_1R_n z + a_2R_n z^2 + \cdots + a_nR_n z^n
\]
\[
= \frac{a_nR_n}{R_1R_2\cdots R_n} \left( z^n + \frac{a_{n-1}R_n}{a_nR_n} z^{n-1} + \frac{a_{n-2}R_{n-1}R_n}{a_nR_n^2} z^{n-2} + \cdots + \frac{a_0R_1R_2\cdots R_n}{a_nR_n^m} \right).
\]

For \( n \in N \) and \( n \geq m - 1 \), Lemma 5, applied to the polynomial
\[
z^n + \frac{a_{n-1}R_n}{a_nR_n} z^{n-1} + \frac{a_{n-2}R_{n-1}R_n}{a_nR_n^2} z^{n-2} + \cdots + \frac{a_0R_1R_2\cdots R_n}{a_nR_n^m},
\]
implies that at least one of the partial sums \( S_n(f; Rnz) \), \( S_{n-1}(f; Rnz) \), \ldots, \( S_{n-m+1}(f; Rnz) \) has all its zeros in the disc \( |z| \leq \Theta_m^{-1} \). In view of \( s_n(f(Rnz)) = R_n^{-1}s_n(f) \), for \( n - m + 1 \leq k \leq n \), it follows that
\[
\min \{ s_n(f)/R_n, s_{n-1}(f)/R_n, \ldots, s_{n-m+1}(f)/R_n \} \leq \Theta_m^{-1}
\]
for all \( n \in N \), \( n \geq m - 1 \). If \( n - k(n) \) denotes the subscript for which the minimum in (4.2) is assumed, then
\[
\min \{ s_n(f)/R_n, s_{n-1}(f)/R_n, \ldots, s_{n-m+1}(f)/R_n \} \leq \Theta_m^{-1}
\]
for \( n \in N \), \( n \geq m - 1 \). Since \( \lim_{n \to \infty} (R_{n-m+1}/R_n) = 1 \), then (4.3) implies
\[
\liminf_{j \to \infty} s_j(f)/R_j \leq \Theta_m^{-1}.
\]
Since \( m \) is arbitrary, (3.1) implies \( \liminf_{j \to \infty} s_j(f)/R_j \leq P \), which is the desired result.

For a power series \( f(z) = 1 + \sum_{k=1}^{\infty} a_kz^k \), the estimate
\[
s_n(f) \geq |a_n|^{-1/n} \quad (a_n \neq 0)
\]
follows from the fact that the geometric mean of the moduli of the zeros of \( S_n(f; z) \) does not exceed the maximum modulus of its zeros. The following lemma, whose proof we omit, is an extension of (4.4).

**Lemma 7.** Suppose the power series \( f(z) = \sum_{k=0}^{\infty} a_kz^k \) has positive radius of convergence and is not a polynomial. If \( N = \{ n : a_n \neq 0 \} \), then
\[
\liminf_{n \to \infty} |a_n|^{1/n} s_n(f) \geq 1.
\]
We are now ready to prove (1.5) of Theorem C.
Theorem 1. If $0 < \tau_R(f) < \infty$, then

\[ \liminf_{n \to \infty} \frac{(R_1 R_2 \cdots R_n)^{1/n}}{R_n} \leq \tau_R(f) \liminf_{n \to \infty} \frac{s_n(f)}{R_n} \leq P. \]

Proof. If $f(z) = \sum_{k=0}^{\infty} A_k z^k = \sum_{k=0}^{\infty} (a_k/R_1 R_2 \cdots R_k) z^k$, then

\[ \tau_R(f) = \limsup_{n \to \infty} |a_n|^{1/n} = \limsup_{n \to \infty} |A_n|^{1/n}(R_1 \cdots R_n)^{1/n} \geq R_1 \limsup_{n \to \infty} |A_n|^{1/n} = R_1/c(f) \]

and therefore $c(f) > 0$. Since $\tau_R(f) > 0$, $f$ is not a polynomial. By Lemma 7,

\[ \liminf_{n \to \infty} \frac{|A_n|^{1/n}}{s_n(f)} \geq 1, \]

where $N = \{n : A_n \neq 0\}$. Therefore,

\[ \liminf_{n \to \infty} \frac{(R_1 R_2 \cdots R_n)^{1/n}}{R_n} \leq \liminf_{n \to \infty; n \in N} \frac{(R_1 R_2 \cdots R_n)^{1/n}}{R_n} \liminf_{n \to \infty; n \in N} \frac{|A_n|^{1/n}}{s_n(f)} \]

\[ \leq \limsup_{n \to \infty} \frac{(R_1 R_2 \cdots R_n)^{1/n}}{|A_n|^{1/n}} \liminf_{n \to \infty; n \in N} \frac{s_n(f)}{R_n} \]

\[ = \tau_R(f) \liminf_{n \to \infty} \frac{s_n(f)}{R_n}, \]

which is the left side of (4.6). For the right side of (4.6), suppose $\tau_R(f) = 1$, let $\alpha > 1$ and define $f_1(z) = f(\alpha z)$. Then $\tau_R(f_1) = \alpha$ and Lemma 6 implies $\liminf_{n \to \infty} s_n(f_1)/R_n \leq P$. Since $s_n(f_1) = \alpha^{-1}s_n(f)$, we have $\liminf_{n \to \infty} s_n(f)/R_n \leq Pa$. Letting $\alpha \to 1$, we obtain $\liminf_{n \to \infty} s_n(f)/R_n \leq P$. Now suppose $\tau_R(f) = \ell$ and define $g(z) = f(z/\ell)$. Since $\tau_R(g) = 1$, the previous inequality implies $\liminf_{n \to \infty} s_n(g)/R_n \leq P$. But $s_n(g) = ts_n(f)$ and therefore

\[ \tau_R(f) \liminf_{n \to \infty} \frac{s_n(f)}{R_n} \leq P. \]

This completes the proof of the theorem and establishes (1.5).

For the proof of (1.6), we require the following lemma.

Lemma 8. If $0 < \tau_R(f) < 1$, then

\[ \limsup_{n \to \infty} \frac{r_n(f)}{R_n} \geq 1/P. \]

Proof. Let $f(z) = \sum_{k=0}^{\infty} A_k z^k = \sum_{k=0}^{\infty} (a_k/R_1 R_2 \cdots R_k) z^k$. Since $\tau_R(f) = \limsup |a_n|^{1/n}$ and $0 < \tau_R(f) < 1$, then there is an infinite set $N$ of positive integers such that $n \in N$ implies $|a_n| > |a_k|$ for $k > n$. Let $m$ be a positive integer, let $n \in N$ and suppose $k$ is an integer such that $0 \leq k \leq m - 1$. The expression

\[ \frac{R_n^k(R_1 \cdots R_n)}{a_n} g^{k+1}f(R_n z) = \frac{a_{n+k} R_n^k}{a_n R_{n+1} \cdots R_{n+k}} + \frac{a_{n+k+1} R_n^{k+1}}{a_n R_{n+1} \cdots R_{n+k+1}} z + \cdots \]
is the $k$th normalized remainder of

\[
1 + \frac{a_{n+1}R_n}{a_nR_{n+1}}z + \frac{a_{n+2}R_n}{a_nR_{n+1}R_{n+2}}z^2 + \ldots.
\]

By Lemma 4, there is an integer $k(n)$, $0 \leq k(n) \leq m-1$, such that

\[
\frac{a_{n+k(n)}R_{n+k(n)}}{a_nR_{n+1}\cdots R_{n+k(n)}}z + \ldots
\]

does not vanish in $|z| \leq \mathcal{U}_m$. Therefore, $\mathcal{P}^{n+k(n)}f(R_nz)$ has no zero in $|z| \leq \mathcal{U}_m$, so that $r_{n+k(n)}(f)/R_n \geq \mathcal{U}_m$ for all $n \in N$. It follows that $(r_{n+k(n)}(f))/R_{n+k(n)}(R_n+1/R_n) \geq \mathcal{U}_m$ and, therefore, $\limsup_{n\to\infty} r_n(f)/R_n \geq \mathcal{U}_m$. By (3.1), $\limsup_{n\to\infty} r_n(f)/R_n \geq 1/P$, and this completes the proof.

The proof of (1.6) of Theorem C is contained in the following theorem.

**Theorem 2.** If $\tau_R(f) > 0$, then

\[
\tau_R(f) \limsup_{n \to \infty} \frac{r_n(f)}{R_n} \geq 1/P.
\]

**Proof.** Suppose first that $\tau_R(f) = 1$, let $0 < \alpha < 1$, and define $f_1(z) = f(az)$. Then $r_n(f_1) = \alpha^{-1}r_n(f)$ and $\tau_R(f_1) = \alpha$. By Lemma 8, $\limsup_{n \to \infty} r_n(f_1)/R_n \geq 1/P$. Thus $\limsup_{n \to \infty} r_n(f)/R_n \geq \alpha/P$ and, letting $\alpha \to 1$, we have $\limsup_{n \to \infty} r_n(f)/R_n \geq 1/P$.

Now suppose $\tau_R(f) = t$. If $t = \infty$, there is nothing to prove. For finite $t$, define $g(z) = f(z/t)$. Then $\tau_R(g) = 1$ and $r_n(g) = tr_n(f)$. By the previous inequality, $\tau_R(f) \limsup_{n \to \infty} r_n(f)/R_n \geq 1/P$, which is the desired result.

### 5. Extremal functions.

In this section, we construct extremal functions which show that $P$ is the sharp constant in each of the three inequalities of Theorem C.

**Theorem 3.** There is a function $f$ of $R$-type 1 such that $\liminf_{n \to \infty} s_n(f)/R_n = P$.

**Proof.** Let $F(z) = \sum_{k=0}^{\infty} A_k z^k$ be the function constructed in Lemma 2. Recall that $c(F) = 1$, $s_n(F) \geq P$, $|A_n| \leq 1$ and $\max\{|A_n|, |A_{n+1}|, \ldots, |A_{n+6}|\} \leq 1/7000$ for all $n$. Let

\[
f(z) = \sum_{k=0}^{\infty} (A_k/R_1R_2\cdots R_k)z^k \quad (R_0 = 1)
\]

and

\[
x = \liminf_{n \to \infty} \frac{s_n(f)}{R_n}.
\]

Let $A$ denote an infinite set of positive integers such that $x = \lim_{n \to \infty; n \in A} s_n(f)/R_n$.

For $n \in A$, define

\[
P_n(z) = z^n s_n(f; R_n/z)(R_1R_2\cdots R_n)/R_n^n
\]

and

\[
Q_n(z) = z^n s_n(f; 1/z) = \sum_{k=0}^{n} A_{n-k}z^k.
\]
The bound
\[ |P_n(z) - Q_n(z)| \leq \sum_{k=1}^{n} |z|^k (1 - (R_n R_{n-1} \cdots R_{n-k+1})/R_n^k) \leq (1 - |z|)^{-1} \]
holds for all \( n \in A \) and \( |z| < 1 \). Thus there is an infinite set of integers \( B \subset A \) such that the sequence \( \{P_n - Q_n\}_{n \in B} \) converges uniformly on compact subsets of \( |z| < 1 \) to a function \( g(z) = \sum_{k=0}^{\infty} a_k z^k \) analytic in the unit disc. Since
\[ a_m = \lim_{n \to \infty; n \in B} A_n - m (1 - (R_n R_{n-1} \cdots R_{n-m+1})/R_n^m) = 0, \]
for \( m = 1, 2, 3, \ldots \), and \( a_0 = 0 \), then \( g \equiv 0 \). For \( n \in B \), we also have the bound \( |Q_n(z)| < (1 - |z|)^{-1} \), \( |z| < 1 \). Thus there is an infinite subset \( C \subset B \) such that \( \{Q_n\}_{n \in C} \) converges uniformly on compact subsets of \( |z| < 1 \) to a function \( Q(z) = \sum_{k=0}^{\infty} \beta_k z^k \) analytic in the unit disc. The bound \( \max \{ |\beta_k|, |\beta_{k+1}|, \ldots, |\beta_{k+\alpha}| \} > 1/7000 \) holds for the coefficients of \( Q \); in particular, \( Q \) is not identically zero. The sequence \( \{P_n(1/z)\}_{n \in C} \) converges uniformly to \( Q(1/z) \) in \( |z| < 1/p \) for all \( p < 1 \). Moreover, if \( \Gamma_n \) denotes the maximum modulus of the zeros of \( P_n(1/z) \), then \( \Gamma_n \geq P - \epsilon \) for \( n \in C \) sufficiently large. Since \( \Gamma_n = R_n^{-1} s_n(f) \), then \( s_n(f)/R_n \geq P - \epsilon \) for large \( n \in C \). Therefore
\[ x = \lim_{n \to \infty; n \in C} \frac{s_n(f)}{R_n} \geq P - \epsilon; \]
letting \( \epsilon \to 0 \), we obtain the desired result.

**Theorem 4.** There is a function \( g \) of \( R \)-type 1 such that \( \lim \sup_{n \to \infty} r_n(g)/R_n = 1/P \).

**Proof.** Let \( G(z) = \sum_{k=0}^{\infty} A_k z^k \) denote the function constructed in Lemma 3. We have \( c(G) = 1 \), \( |A_k| \leq 1 \) and \( \max \{ |A_n|, |A_{n+1}|, \ldots, |A_{n+\alpha}| \} \geq 1/7000 \) for all \( n \). Let
\[ g(z) = \sum_{k=0}^{N} (A_k/R_1 R_2 \cdots R_k) z^k \quad (R_0 = 1), \]
and
\[ x = \lim_{n \to \infty} \frac{r_n(g)}{R_n}. \]
Let \( A \) denote an infinite set of positive integers for which \( x = \lim_{n \to \infty; n \in A} r_n(g)/R_n \). For \( m \in A \), define
\[ E_m(z) = s_m G(z) - (R_1 R_2 \cdots R_m|R_m)|G(R_m z) \]
and let \( 0 < \alpha < 1 \). If \( m \in A \), \( |z| \leq \alpha \) and \( N \) is a positive integer, then
\[ |E_m(z)| \leq \sum_{k=1}^{\infty} \left| A_{m+k} \right| \left| (1 - R_m^k/(R_{m+1} \cdots R_{m+k})) \right| \leq \sum_{k=1}^{N} \left| (1 - R_m^k/(R_{m+1} \cdots R_{m+k})) \right| + \sum_{k=N+1}^{\infty} \alpha^k \leq (1 - R_m^N(R_{m+1} \cdots R_{m+N})) \leq 0 \leq (1 - \alpha)^{-1} + \alpha^{N+1}(1 - \alpha)^{-1}. \]
Let \( \epsilon > 0 \) and choose \( N \) so that \( \alpha^{N+1}(1 - \alpha)^{-1} < \epsilon/2 \). Let \( m_0 \in A \) be a positive integer...
such that \( m \geq m_0 \) implies \((1 - R_m/(R_{m+1} \cdots R_{m+m}))((1 - \alpha)^{-1} < \epsilon/2. \) Then the conditions \( m \geq m_0 \) and \(|z| \leq \alpha \) imply \(|E_m(z)| < \epsilon. \) Thus \( \{E_m\}_{m \in A} \) converges uniformly to zero on compact subsets of \(|z| < 1. \) For \( m \in A, \) we also have \(|\mathcal{S}^m G(Z)| \leq (1 - |z|)^{-1}. \) Thus there is an infinite subset \( B \subset A \) of integers such that \( \{\mathcal{S}^m G(Z)\}_{m \in B} \) converges uniformly on compact subsets of \(|z| < 1 \) to a function \( S(z) = \sum_{k=0}^\infty b_k z^k. \) The relation

\[
|b_k| + |b_{k+1}| + \cdots + |b_{k+\epsilon}| \geq 1/1000
\]

holds for all \( k; \) in particular \( S \neq 0. \) Since \( \mathcal{S}^m G(z) \) has a zero of modulus \( 1/P \) for all \( m \in B, \) then \( S(z) \) has a zero of modulus \( 1/P. \) Moreover, \( S(z) \) is the uniform limit of the sequence \( \{(R_1 R_2 \cdots R_m/R_m)^m \mathcal{S}^m g(R_m z)\}_{m \in B} \) and it follows from Hurwitz’s Theorem that, if \( \epsilon > 0, \) then \( \mathcal{S}^m g(R_m z) \) has a zero of modulus at most \((1/P) + \epsilon \) for \( m \in B \) sufficiently large. Therefore \( r_m(g)/R_m \leq (1/P) + \epsilon \) for large \( m \in B, \) and it follows that

\[
x = \lim_{m \to \infty; m \in B} \frac{r_m(g)}{R_m} \leq (1/P) + \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, we obtain \( \lim \sup_{n \to \infty} r_n(g)/R_n \leq 1/P \) and this completes the proof.

For the left-hand side of (1.5), we begin by considering the infinite matrix \((a_{mn}),\)

\[
amn = \frac{2(m-n+1)/m^2}{m = n = m},
\]

\[
= 0, \quad m < n.
\]

It is easily verified that

1. \( \lim_{m \to \infty} a_{mn} = 0, \quad n = 1, 2, 3, \ldots, \)
2. \( \sup_m \sum_{n=1}^\infty |a_{mn}| = 2, \)
3. \( \lim_{m \to \infty} \sum_{n=1}^\infty a_{mn} = 1. \)

Thus \((a_{mn})\) provides a regular method of summability. If \( \{R_n\}_{n=1}^\infty \) is a non-decreasing sequence of positive numbers \((R_0 = 1)\) such that \( R_{n+1}/R_n \to 1, \) then \((a_{mn})\) transforms the sequence \( \{\log(R_n/R_{n-1})\}_{n=1}^\infty \) into \( \{2 \log(R_1 R_2 \cdots R_n)^{1/n^2}\}_{n=1}^\infty. \) Therefore

\[
\lim_{n \to \infty} [2 \log(R_1 R_2 \cdots R_n)^{1/n^2}] = \lim_{n \to \infty} [\log(R_n/R_{n-1})] = 0,
\]

or

\[
(5.1) \quad \lim_{n \to \infty} (R_1 R_2 \cdots R_n)^{1/n^2} = 1.
\]

We use this result to prove the following lemma.

**Lemma 9.** Let \( \{R_n\}_{n=1}^\infty \) \((R_0 = 1)\) be a nondecreasing sequence of positive numbers such that \((R_1 R_2 \cdots R_n)^{1/n} \to \infty \) and \( R_{n+1}/R_n \to 1, \) as \( n \to \infty. \) For each pair of positive integers \( m \) and \( p, \) let \( x_{mp} \) be the largest root of the equation

\[
x^m + p \quad R_1 \cdots R_{m+p} = x^m \quad R_1 \cdots R_m + \frac{x^{m-1}}{R_1 \cdots R_{m-1}} + \cdots + \frac{X}{R_1} + 1.
\]
Then
\[ \lim_{p \to \infty} \frac{x_{mp}}{(R_1 \cdots R_{m+p})^{1/(m+p)}} = 1 \]
for \( m = 1, 2, 3, \ldots \).

**Proof.** For all \( m \) and \( p \) we have \( x_{mp} \leq (R_1 \cdots R_{m+p})^{1/(m+p)} \), and therefore \( x_{mp} \to \infty \) as \( p \to \infty \), \( m = 1, 2, 3, \ldots \). Let \( m \) be a positive integer and choose \( p \) so large that
\[ \frac{x_{mp}^m}{(R_1 \cdots R_m)^{m^2/k}} \leq (R_1 \cdots R_{m-k})^{1/(m-k)} \]
for \( 0 \leq k \leq m \). For such integers \( p \) we have \( x_{mp}^{m+p}/(R_1 \cdots R_{m+p}) \leq (m+1)x_{mp}/(R_1 \cdots R_m) \) and hence
\[ x_{mp} \leq (m+1)^{1/p}(R_{m+1} \cdots R_{m+p})^{1/p}. \]
Thus
\[ 1 \leq \frac{x_{mp}}{(R_1 \cdots R_{m+p})^{1/(m+p)}} \leq \frac{(m+1)^{1/p}}{(R_1 \cdots R_m)^{1/(m+p)}} \left( \frac{R_{m+1} \cdots R_{m+p}}{R_1 \cdots R_m} \right)^{(1/p)-(1/(m+p))}. \]
Since each of \((m+1)^{1/p}\) and \((R_1 \cdots R_m)^{1/(m+p)}\) tends to 1 as \( p \to \infty \), it is sufficient to show that
\[ (R_{m+1} \cdots R_{m+p})^{1/(m+p)-(1/(m+p))} = (R_1 \cdots R_{m+p})^{1/(m+p+1)} \to 1, \quad p \to \infty. \]
Since \((R_1 \cdots R_m)^{m/(m+p)} \to 1\), it is sufficient to show that \((R_1 \cdots R_{m+p})^{1/(m+p)} \to 1\).

Now
\[ (R_1 \cdots R_{m+p})^{1/(m+p+1)} = (R_1 \cdots R_{m+p})^{1/(m+p+1)} [(R_1 \cdots R_{m+p})^{1/(m+p+1)}]^{m/p}, \]
and we know that \((R_1 \cdots R_{m+p})^{1/(m+p+1)} \to 1, \quad p \to \infty. \) Thus \((R_1 \cdots R_{m+p})^{1/(m+p+1)} \to 1\), and this completes the proof.

**Theorem 5.** There is a function \( \varphi \) of \( R \)-type 1 such that
\[ \liminf_{n \to \infty} \frac{(R_1 R_2 \cdots R_n)^{1/n}}{R_n} = \liminf_{n \to \infty} \frac{x_\varphi(\varphi)}{R_n}. \]

**Proof.** Let \( \{R_n\}_{n=1}^{\infty} \) (\( R_0 = 1 \)) and \( \{x_{mp}\}_{m,p=1}^{\infty} \) be defined as in Lemma 9. Let \( \{n_k\}_{k=1}^{\infty} \) denote a sequence of positive integers such that
\[ \lim_{n \to \infty} \frac{(R_1 R_2 \cdots R_n)^{1/n}}{R_n} = \lim_{k \to \infty} \frac{(R_1 \cdots R_{n_k})^{1/n_k}}{R_{n_k}}. \]
Let \( m_1 = n_1 \), choose an integer \( p_1 \) such that \( m_1 + p_1 \in \{n_j\} \) and
\[ \frac{x_{mp_1}}{(R_1 \cdots R_{m_1+p_1})^{1/(m_1+p_1)}} < 1 + \frac{1}{2}, \]
and let \( m_2 = m_1 + p_1 \). If \( m_k = m_{k-1} + p_{k-1} \in \{n_j\} \) has been chosen, choose the integer \( p_k \) such that \( m_k + p_k \in \{n_j\} \) and
\[ x_{mp_k}/(R_1 \cdots R_{m_k+p_k})^{1/(m_k+p_k)} < 1 + 1/(k+1). \]
and let $m_{k+1} = m_k + p_k$. Thus we inductively obtain the sequence \{\{m_j\} \subset \{n_j\} such that (5.3) holds for $k = 1, 2, 3, \ldots$. Now let

$$\varphi(z) = 1 + z^{m_1}/(R_1 \cdots R_{m_1}) + z^{m_2}/(R_1 \cdots R_{m_2}) + \cdots.$$  

Note that

$$|S_{m_j}(\varphi; z)| \geq \frac{|z|^{m_j}}{R_1 \cdots R_{m_j}} - \frac{|z|^{m_j-1}}{R_1 \cdots R_{m_j-1}} - \cdots - \frac{|z|^{m_1}}{R_1 \cdots R_{m_1}} - 1,$$

for $j = 1, 2, 3, \ldots$. Moreover, if $x > x_{mp}$, then

$$\frac{x^{m+p}}{R_1 \cdots R_{m+p}} > \frac{x^m}{R_1 \cdots R_{m}} + \cdots + \frac{x}{R_{1}} + 1,$$

since $x_{mp}$ is the largest positive root of (5.2). Thus if $|z| = x > x_{mp-1}$, then $|S_{m_j}(\varphi; z)| > 0$. Therefore $s_{m_j}(\varphi) \leq x_{m_j-1}$. From (5.3) we have

$$s_{m_j}(\varphi)/(R_1 \cdots R_{m_j})^{1/m_j} \leq 1 + 1/j \text{ for } j = 1, 2, 3, \ldots.$$

Since $s_n(\varphi) = \infty$ for integers $n \notin \{m_j\}$, then

$$\liminf_{n \to \infty} \frac{s_n(\varphi)}{R_{n}} = \liminf_{n \to \infty} \frac{s_{m_j}(\varphi)}{R_{m_j}} \leq \liminf_{n \to \infty} \left[ \left(\frac{R_1 \cdots R_{m_j}}{R_{m_j}} \right)^{1/m_j} \left(1 + \frac{1}{j} \right) \right] = \lim_{k \to \infty} \left(\frac{R_1 \cdots R_{n_k}}{R_{n_k}} \right)^{1/n_n} = \liminf_{n \to \infty} \left(\frac{R_1 \cdots R_{n}}{R_{n}} \right)^{1/n}.$$

and this completes the proof.

REFERENCES