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TEMPERATURES IN SEVERAL VARIABLES: KERNEL FUNCTIONS, REPRESENTATIONS, AND PARABOLIC BOUNDARY VALUES

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Abstract. This work develops the notion of a kernel function for the heat equation in certain regions of $n+1$-dimensional Euclidean space and applies that notion to the study of the boundary behavior of nonnegative temperatures. The regions in question are bounded between spacelike hyperplanes and satisfy a parabolic Lipschitz condition at points on the lateral boundary.

Kernel functions (normalized, nonnegative temperatures which vanish on the parabolic boundary except at a single point) are shown to exist uniquely. A representation theorem for nonnegative temperatures is obtained and used to establish the existence of finite parabolic limits at the boundary (except for a set of heat-related measure zero).

0. **Introduction.** The notion of a kernel function has been developed in the case of the Laplace operator by Hunt and Wheeden [4], who prove existence and uniqueness for such functions in Lipschitz domains of n-dimensional Euclidean space $\boldsymbol{R}^n$ and use these functions to study the nontangential boundary behavior of harmonic functions which have a one-sided bound in the domain. In [7] we reported analogous results for the heat equation in certain regions of the plane. These results had been obtained in [6]. There, difficulties in the application of the techniques of [4] to the heat equation were overcome and some simplification of those techniques was achieved, notably in the proof of uniqueness of kernel functions. However, technical problems prevented treatment of the heat equation in more than one space dimension. The present work extends those results to the case of several space variables.

In §1 existence and uniqueness of kernel functions is established and a representation theorem is obtained for temperatures with a one-sided bound in a region satisfying a certain mixed-Lipschitz condition (conditions L1 and L2 below). The main result of §2 is the existence almost everywhere (with respect to caloric measure) on the parabolic boundary of finite parabolic limits for temperatures with a one-sided bound in the region. This generalizes the work of Jones and Tu [5] and Hattemer [2], who considered regions less general than those dealt with here.

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1. **Kernel functions and a representation theorem.** We denote by (x, t) a point in $\boldsymbol{R}^{n+1}$, where $x=(x_1, x_2, \ldots, x_n)=(x', x_n)$ are the space variables and t the time variable. For a domain D in $\boldsymbol{R}^{n+1}$, we let $\partial_p D$ be the parabolic boundary of D; i.e. $\partial_p D$ is the set of points on the boundary of D which can be connected to some interior point of D by a closed curve having strictly increasing t-coordinate. D is a regular domain for the Dirichlet problem for the heat equation if that problem is solvable in the Weiner-Perron sense for any (Borel) integrable boundary values. For regular domains we have the following:

DEFINITION. If $(x, t) \in D$ and $Z \subset \partial D$ is a Borel measurable set, the *caloric measure at* (x, t) *of* Z, denoted $\omega_D^{(x,t)}(Z)$, is the value at (x, t) of the unique solution of the Dirichlet problem for the heat equation in D with boundary data given by the characteristic function of Z. (When there is no possibility of confusion, we shall suppress the subscript D, writing $\omega^{(x,t)}(Z)$.)

Throughout our discussion (X, T) will denote a fixed point in D. If $t \leqq \inf\{s : \exists y \text{ with } (y, s) \in Z\}$, we must have $\omega^{(x,t)}(Z)=0$ by the maximum principle. Furthermore, if $(x, t) \in D$ can be joined to (X, T) by a closed curve in D with strictly increasing t-coordinate, then Harnack's inequality and Besicovitch's general theory of differentiation [1] imply that the Radon-Nikodym derivative, $\partial\omega^{(x,t)}/\partial\omega^{(X,T)}$, which exists in $L^1(\omega^{(X,T)})$, is given a.e. $(\omega^{(X,T)})$ by

$$\operatorname*{limit}_{\Delta_n \to (y,s)} \frac{\omega^{(x,t)}(\Delta_n)}{\omega^{(X,T)}(\Delta_n)},$$

where Δ_n is any sequence of closed sets in $\partial_p D$ which contain (y, s) and satisfy $\inf_n (\omega^{(X,T)}(\Delta_n)/\omega^{(X,T)}(B_n))>0$, where B_n is the intersection of $\partial_p D$ with the smallest sphere centered at (y, s) and containing Δ_n.

We will be concerned with the following concept of a kernel function:

DEFINITION. If $(Y, S) \in \partial_p D$ with $S<T$, a function $K(x, t)$ defined in D is a *kernel function at* (Y, S) *for the heat equation in D with respect to* (X, T) if

(i) $K(x, t) \geqq 0$ for $(x, t) \in D$,

(ii) $K(x, t)$ satisfies the heat equation, $\Delta K=K_t$, in D,

(iii) $\operatorname{limit}_{(x,t)\to(y,s);\,(x,t)\in D} K(x, t)=0$ for $(y, s) \in \partial_p D-\{(Y, S)\}$,

(iv) $K(X, T)=1$ (normalization condition).

(If $S \geqq T$, we shall take the kernel function at (Y, S) with respect to (X, T) to be identically zero.)

Suppose that D is a regular domain for the heat equation, and let $D_T = D \cap \{(x, t) : t \leqq T\}$. It is clear that if $S<T$ and a function $K(x, t)$ satisfies (i)–(iv) in D_T, then $K(x, t)$ can be extended to a kernel function in D by solving a Dirichlet problem in $D-D_{S+\varepsilon}$. Conversely, if $K(x, t)$ is a kernel function at (Y, S) in D, then its restriction to D_T is a kernel function at (Y, S) in D_T.

To obtain existence and uniqueness of kernel functions at a point $(Y, S) \in \partial_p D$, we require additional restrictions on $\partial_p D$ in a neighborhood of (Y, S). We allow two conditions:

Condition L1. $\partial_p D$ is given locally at (Y, S) by $t = S$.

Condition L2. $\partial_p D$ is given locally at (Y, S) by a function satisfying a mixed-Lipschitz condition with exponent 1 in the space variables and $\frac{1}{2}$ in the time variable.

Specifically, for condition L2 to hold, there is a sphere $\mathcal{O}$ with center (Y, S), local space coordinates x, and a function $f(x', t)$ defined in $\mathcal{O}' = \{(x', t) : \exists x_n$ with $(x', x_n, t) \in \mathcal{O}\}$ satisfying

$$|f(x', t) - f(x'_0, t_0)| \leqq C(|x' - x'_0| + |t - t_0|^{1/2})$$

for (x', t) and (x'_0, t_0) in $\mathcal{O}'$, such that

$$\mathcal{O} \cap \partial_p D \cap \{(x, t) : t > S\} = \mathcal{O} \cap \{(x, t) : x_n = f(x', t)\} \cap \{t > S\},$$
$$\mathcal{O} \cap D \cap \{(x, t) : t > S\} = \mathcal{O} \cap \{(x, t) : x_n > f(x', t)\} \cap \{t > S\}.$$

Thus, L2 describes the "side" points of $\partial_p D$. Similarly, condition L1 describes the "bottom" points of $\partial_p D$. In the case of a "bottom corner" point (Y, S), condition L2 applies. In fact, by extending the domain, we may assume that $\partial_p D$ is given by the mixed-Lipschitz function $f(x', t)$ in a complete neighborhood of (Y, S). We also note that any point $(Y, S) \in \partial_p D$ satisfying either condition L1 or L2 is a regular boundary point for the heat equation by a theorem of Petrovski [8].

The proofs of existence and uniqueness of kernel functions at points satisfying either condition are similar. We shall concentrate on points satisfying condition L2, pointing out any essential changes which would be required in the proofs for points satisfying L1.

We begin by establishing the following notation: if (Y, S) is a point satisfying condition L2 with a mixed-Lipschitz constant C, we fix a constant $d < 2C$ and define, for sufficiently small r,

$$\Psi((Y, S), r) = D \cap \{(x, t) : |x' - Y'| < r, |t - S| < r^2, |x_n - Y_n| < rd\},$$

with $\Delta((Y, S), r) = \partial D \cap \{(x, t) : |x' - Y'| < r, |t - S| < r^2\}$, and $A((Y, S), r) = (Y', Y_n + rd, S + (1+\mu)r^2)$, where μ is small and depends only on D. (μ is chosen so that $A((Y, S), r) \in D$ for small r.) In the case of a point satisfying condition L1, we have

$$\Psi((Y, S), r) = D \cap \{(x, t) : |x - Y| < r, |t - S| < r^2\},$$

with $\Delta((Y, S), r) = \partial D \cap \{(x, t) : |x - Y| < r\}$ and $A((Y, S), r) = (Y, S + (1+\mu)r^2)$.

LEMMA 1.1. *Suppose that D is a regular domain for the heat equation which satisfies condition* L2 *at $(Y, S) \in \partial_p D$. If $\gamma \in (0, 1)$, then there is a constant C, depending only on γ and the mixed-Lipschitz constant, such that $\omega^{(x,t)}(\Delta((Y, S), r)) \geqq C$ for $(x, t) \in \Psi((Y, S), \gamma r)$ as long as r is sufficiently small.*

Proof. Let $G = G((Y, S), r) = \{(x, t) : |t - S| < r^2, |x' - Y'| < r, |x_n - Y_n| < rd\}$ and $h(x, t) = \omega_G^{(x,t)}(\{(x, t) : |x' - Y'| < r,\ |t - S| < r^2,\ x_n = Y_n - rd\})$. For small r, $G \cap D$

$=\Psi((Y, S), r)$. By the maximum principle in that set, $\omega_D^{(x,t)}(\Delta((Y, S), r)) \geqq h(x, t)$ for $(x, t) \in \Psi((Y, S), r)$. Since $\Psi((Y, S), \gamma r) \subset G((Y, S), \gamma r)$ and

$$C = \inf_{(x,t)\in G((Y,S),\gamma r)} h(x, t) > 0,$$

we have

$$\omega_D^{(x,t)}(\Delta((Y, S), r)) \geqq C > 0 \quad \text{for } (x, t) \in \Psi((Y, S), \gamma r). \qquad \text{Q.E.D.}$$

LEMMA 1.2. *Suppose that D is a regular domain for the heat equation which satisfies condition* L2 *at $(Y, S) \in \partial_p D$. Then there is a constant $C>0$, depending only on the mixed-Lipschitz constant, such that, for $r' \in (0, r)$, we have*

$$(*) \qquad \omega^{(x,t)}(\Delta((Y, S), r')) \leqq C\omega^{A((Y,S),r)}(\Delta((Y, S), r'))$$

for $(x, t) \in D-\Psi((Y, S), r)$ if r is sufficiently small.

Proof. (For convenience, let $\Delta=\Delta((Y, S), r)$ and $\Delta'=\Delta((Y, S), r')$.) Since $\omega^{(x,t)}(\Delta')=0$ for $(x, t) \in \partial_p D-\Delta$, it suffices, by the maximum principle, to prove (*) for $(x, t) \in D \cap \partial\Psi((Y, S), r)$.

We define sets $\Psi_k=\Psi((Y, S), 2^{k-1}r')$, with $\Delta_k=\Delta((Y, S), 2^{k-1}r')$ and $A_k =A((Y, S), 2^{k-1}r')$ for $k=1, 2, \dots, L$, where $2^{L-1}r'<3r/4<2^L r'$. By Harnack's inequality there is a positive constant C_1, independent of k, such that

$$\text{(a)} \qquad \omega^{A_k}(\Delta') \leqq C_1\omega^{A_{k+1}}(\Delta') \quad \text{for } k = 1, 2, \dots, L.$$

The main part of the proof will be a demonstration of statements

$$\mathrm{S}_k: \qquad \omega^{(x,t)}(\Delta') \leqq C\omega^{A_k}(\Delta') \quad \text{for } (x, t) \in D-\Psi_k$$

for $k=1, 2, \dots, L-1$. Once this has been done, the conclusion of the lemma will follow from statement S_{L-1} and Harnack's inequality.

If $B_1=(Y', Y_n+rd, S)$, Lemma 1.1 (with $\gamma=\frac{1}{2}$ and the parameter $2d$ in place of d) establishes the existence of a positive constant C_2' such that $\omega^{B_1}(\Delta')\geqq C_2'$ if r is sufficiently small. There is another constant $C_2''>0$ such that $\omega^{A_1}(\Delta')\geqq C_2''\omega^{B_1}(\Delta')$, and, setting $C_2=C_2'\cdot C_2''$, $\omega^{A_1}(\Delta')\geqq C_2$. It follows that $\omega^{(x,t)}(\Delta')\leqq 1 \leqq (1/C_2)\omega^{A_1}(\Delta')$ for $(x, t) \in D$ and, in particular, for $(x, t) \in D-\Psi_1$. This establishes statement S_1.

Next, for each point (y, s) in the ring $(\partial_p\Psi_2 \cap \partial_p D)-\Delta_2$, we construct an auxiliary function as follows: let

$$D(y, s) = \{(x, t) : x_n - Y_n > d_0 r' \text{ or } |t-S| > r'^2 \text{ or } |x'-Y'| > r'\}$$
$$\cap \{(x, t) : x_n - y_n > -M(|t-s|^{1/2}+|x'-y'|)\},$$

where d_0 and M are positive constants to be chosen in due course. Let $D_N(y, s) =D(y, s) \cap \{(x, t) : |x-Y|<N, |t-S|<N\}$, and take $h_N(x, t)$ to be the caloric measure in $D_N(y, s)$ of that part of $\partial_p D_N(y, s)$ lying on the boundary of the removed rectangle, $\{(x, t) : x_n-Y_n\leqq d_0 r', |t-S|\leqq r'^2, |x'-Y'|\leqq r'\}$. By the maximum principle, h_N increases in $D(y, s)$ as N increases. Since $h_N\leqq 1$ for each N,

there exists a temperature in $D(y, s)$, $h(x, t) = \text{limit}_{N\to\infty} h_N(x, t)$. Because the h_N's are uniformly bounded and vanish on a common boundary neighborhood of (y, s), a regular point, we have $\text{limit}_{(x,t)\to(y,s);(x,t)\in D(y,s)} h(x, t)=0$. We claim that, for each $(y, s) \in (\partial_p\Psi_2 \cap \partial_p D)-\Delta_2$, the rate at which the corresponding $h(x, t)$ tends to zero at (y, s) is the same. To see this, construct

$$\bar{D}(y, s) = \{(x, t) : x_n - y_n > -M(|t-s|^{1/2}+|x'-y'|),\\ |x'-y'| < r', |t-s| < r'^2\},$$

and let $g(x, t)$ be the caloric measure in $\bar{D}(y, s)$ of that part of the boundary of $\bar{D}(y, s)$ not on the parabolic surface. Clearly, $\bar{D}(y, s) \subset D(y, s)$ and, by the maximum principle in $\bar{D}(y, s)$, $h(x, t) \leqq g(x, t)$. Since the regions $\bar{D}(y, s)$ are identical, except for a translation, we have the desired uniform estimate on the functions $h(x, t)$. Accordingly, choose a positive number δ such that $h(x, t) \leqq 1/C_1$ whenever $(x, t) \in D(y, s)$ with $|(x, t)-(y, s)| < \delta$. Define the set $\beta_2 = \{(x, t) : (x, t) \in \partial\Psi_2 - \partial D$ and $\text{dist}((x, t), (\partial\Psi_2 \cap \partial_p D)-\Delta_2) < \delta\}$, where $\text{dist}(P, S)$ is the distance from the point P to the set S. Let $\alpha_2 = \partial\Psi_2 - \beta_2$. By Harnack's inequality, there is a constant $C_3 > 0$ such that

(b) $$\omega^{(x,t)}(\Delta') < C_3\omega^{A_2}(\Delta') \quad \text{for } (x, t) \in \alpha_2.$$

If $C_4 = \max(C_3, 1/C_2)$, we have already seen that $\omega^{(x,t)}(\Delta') \leqq C_4\omega^{A_1}(\Delta')$ for $(x, t) \in D-\Psi_1$. Furthermore, we now see that $\omega^{(x,t)}(\Delta') \leqq C_4\omega^{A_2}(\Delta')$ for $(x, t) \in \alpha_2$, by (b). Therefore, to prove $\omega^{(x,t)}(\Delta') \leqq C_4\omega^{A_2}(\Delta')$ in all of $D-\Psi_2$, it suffices, by the maximum principle, to prove this estimate for $(x, t) \in \beta_2$. Provided that the constants M and d_0 have been chosen sufficiently large that Ψ_1 is contained in the complement of $D_1(y, s)$ and the parabolic cone in the complement of $D(y, s)$ lies in the complement of D, which is made possible by the Lipschitz nature of $\partial_p D$, we may proceed. Since $\omega^{(x,t)}(\Delta') \leqq C_4\omega^{A_1}(\Delta')$ for $(x, t) \in D \cap \partial D_1(y, s)$, the maximum principle in $D \cap D_1(y, s)$ implies that $\omega^{(x,t)}(\Delta') \leqq C_4\omega^{A_1}(\Delta')h_1(x, t)$ in that set. Recalling (a) and our previous estimate on $h(x, t)$ for $(x, t) \in \beta_2$, we have $\omega^{(x,t)}(\Delta') \leqq C_4\omega^{A_2}(\Delta')$ for $(x, t) \in \beta_2$ and, therefore, in $D-\Psi_2$, establishing statement S_2. Continuing inductively, the statements $S_3, S_4, \ldots, S_L$ are established, each with the same constant $C=C_4$. Q.E.D.

The next lemma is of fundamental importance.

Lemma 1.3. *Suppose that D_T is a regular domain for the heat equation which satisfies condition* L2 *at $(Y, S) \in \partial_p D_T$. Then there is a positive constant C, depending only on the mixed-Lipschitz constant, such that for each neighborhood N of (Y, S) in $\Psi((Y, S), r/4)$ for which $D_T - N$ is regular and points of N also satisfy condition* L2 *with the same mixed-Lipschitz constant as $\partial_p D_T$ at (Y, S), if $u(x, t)$ is a nonnegative temperature in $D_T - N$ which is continuous in the closure of that set and vanishes on $\partial_p D_T - \bar{N}$, then*

$$u(x, t) \leqq Cu(A((Y, S), r))\omega^{(x,t)}(\Delta((Y, S), r))$$

for $(x, t) \in D_T - \Psi((Y, S), (1+\mu)^{1/2}r/4)$ if r is sufficiently small, depending on (Y, S).

Proof. Fix $\lambda \in (0,1)$ and let $\Delta' = \Delta((\bar{Y}, \bar{S}), r')$ for $(\bar{Y}, \bar{S}) \in \Delta((Y, S), r)$ and $r' < \lambda r$. By Lemma 1.2, if r is small,

$$\omega^{(x,t)}(\Delta') \leqq C\omega^{A_\lambda}(\Delta') \quad \text{for } (x, t) \in D - \Psi((\bar{Y}, \bar{S}), \lambda r)$$

where $A_\lambda = (\bar{Y}', \bar{Y}_n + rd, \bar{S} + \lambda^2 r^2(1+\mu))$. In particular,

$$\omega^{(x,t)}(\Delta') \leqq C\omega^{A_\lambda}(\Delta') \quad \text{for } (x, t) \in D - \Psi((Y, S), (1+\lambda)r).$$

If λ is small, $\bar{S} + \lambda^2 r^2(1+\mu) \leqq S + r^2 + \lambda^2 r^2 (1+\mu) < S + (1+\mu/2)r^2$, and Harnack's inequality applies to show that

$$\omega^{A_\lambda}(\Delta') \leqq C\omega^{A}(\Delta'), \quad \text{where } A = A((Y, S), r).$$

If we also require that $\lambda < (1+\mu)^{1/2} - 1$, we have

$$\omega^{(x,t)}(\Delta') \leqq C\omega^{A}(\Delta') \quad \text{for } (x, t) \in D - \Psi((Y, S), (1+\mu)^{1/2} r).$$

Next, let $(\bar{x}, \bar{t}) \in D_T - \Psi((Y, S), (1+\mu)^{1/2} r)$. By Besicovitch's results on the differentiation of measures [1],

$$\frac{d\omega^{(\bar{x},\bar{t})}}{d\omega^{(X,T)}}(\bar{Y}, \bar{S}) = \lim_{\Delta' \to ((\bar{Y}, \bar{S}))} \frac{\omega^{(\bar{x},\bar{t})}(\Delta')}{\omega^{(X,T)}(\Delta')}$$

for a.e. $(\omega^{(X,T)})(\bar{Y}, \bar{S}) \in \Delta((Y, S), r)$. Therefore,

$$(d\omega^{(\bar{x},\bar{t})}/d\omega^{(X,T)})(\bar{Y}, \bar{S}) \leqq C(d\omega^{A}/d\omega^{(X,T)})(\bar{Y}, \bar{S})$$

for a.e. $(\omega^{(X,T)})(\bar{Y}, \bar{S}) \in \Delta((Y, S), r)$, where C is independent of $(\bar{Y}, \bar{S}) \in \Delta((Y, S), r)$.

If $u(x, t)$ is a nonnegative temperature in D_T, continuous in $\bar{D}_T$, and vanishing on $\partial_p D_T - \Delta((Y, S), r)$, we have

$$u(\bar{x}, \bar{t}) = \int_{\Delta((Y,S),r)} u(\bar{Y}, \bar{S})\, d\omega^{(\bar{x},\bar{t})}(\bar{Y}, \bar{S}).$$

Since $d\omega^{(\bar{x},\bar{t})} = (d\omega^{(\bar{x},\bar{t})}/d\omega^{(X,T)})\, d\omega^{(X,T)}$, it follows that

$$u(\bar{x}, \bar{t}) \leqq Cu(A) \quad \text{for } (\bar{x}, \bar{t}) \in D_T - \Psi((Y, S), (1+\mu)^{1/2} r).$$

Applying this argument to the function $u(x, t)$ given in the hypotheses and the region $D_T - N$, with r replaced by $r/4$, and making use of Lemma 1.1 and Harnack's inequality, we obtain

$$u(x, t) \leqq Cu(A((Y, S), r))\omega^{(x,t)}(\Delta((Y, S), r))$$
$$\text{for } (x, t) \in D_T - \Psi((Y, S), (1+\mu)^{1/2} r/4). \qquad \text{Q.E.D.}$$

We will most often use the following form of Lemma 1.3:

LEMMA 1.4. *Suppose that D is a regular domain for the heat equation which satisfies condition* L2 *at $(Y, S) \in \partial_p D_T$. Then there is a positive constant C, depending only on D and the mixed-Lipschitz constant at (Y, S), such that, if $u(x, t)$ is a nonnegative temperature in D which vanishes on $\partial_p D - \Delta((Y, S), r/8)$ for sufficiently small r, we then have $u(x, t) \leqq Cu(A((Y, S), r))\omega^{(x,t)}(\Delta((Y, S), r))$ for $(x, t) \in D - \Psi((Y, S), (1+\mu)^{1/2} r/4)$.*

Proof. Let r_0 satisfy the requirements of Lemma 1.3. For $r<r_0$, let N be a neighborhood of (Y, S) in $\Psi((Y, S), r/4)$ such that $\Delta((Y, S), r/8)\subset\partial N\cap\partial D$ and $\partial_p(D_T-N)$ satisfies condition L2 with a mixed-Lipschitz constant at points of N which is no bigger than the mixed-Lipschitz constant for D at (Y, S). Application of Lemma 1.3 and the maximum principle yield the desired result. Q.E.D.

We can now prove the existence of kernel functions for a domain D if each point $(Y, S)\in\partial_p D$ satisfies condition L1 or conditon L2. Let (X, T) be the normalization point for the kernel functions with $T>S$ and consider a sequence of positive numbers r_n which decrease to zero. Let $\Delta_n=\Delta((Y, S), r_n)$ and set

$$v_n(x, t) = \omega^{(x,t)}(\Delta_n)/\omega^{(X,T)}(\Delta_n).$$

Each v_n is a nonnegative temperature in D and $v_n(y, s)=0$ for $(y, s)\in\partial_p D-\Delta_n$. If r is small enough for Lemma 1.4 to hold, there is a number n_0 such that v_n satisfies the hypotheses of that lemma for $n>n_0$. Therefore, if $A=A((Y, S), r)$ and $\Delta=\Delta((Y, S), r)$, we have

$$(*)\qquad v_n(x, t) \leqq Cv_n(A)\omega^{(x,t)}(\Delta) \quad \text{for } (x, t)\in D-\Psi((Y, S), (1+\mu)^{1/2}r/4).$$

By Harnack's inequality, $v_n(A)\leqq Cv_n(X, T)=C$. Furthermore, since $\omega^{(x,t)}(\Delta)\leqq 1$, we have $v_n(x, t)\leqq C$ for $(x, t)\in D-\Psi_0$, where $\Psi_0=\Psi((Y, S), (1+\mu)^{1/2}r/4)$. The Ascoli theorem assures the existence of a subsequence of the functions v_n which converge uniformly on compact subsets of D. By Harnack's convergence theorem, the limit function $K(x, t)$ must be a temperature in D. Clearly, $K(x, t)\geqq 0$ and $K(X, T)=1$. Finally, to see that $K(x, t)$ vanishes on $\partial_p D-\{(Y, S)\}$, we need only let n tend to infinity in (*). Since r may be chosen arbitrarily small, it follows that $K(x, t)$ is a kernel function for the heat equation in D at $(Y, S)\in\partial_p D$.

At this point we remark that Lemmas 1.1 through 1.4 can be similarly proven for points $(Y, S)\in\partial_p D$ which satisfy condition L1. The specific changes required are that

$$D(y, s) = \{(x, t) : |x-Y| > r' \text{ or } t > d_0 r'^{1/2}\}$$
$$\cap \{(x, t) : t > -M|x-y|^2\}$$

in Lemma 1.2 and that $A=(\overline{Y}, S+\lambda^2r^2(1+\mu))$ in Lemma 1.3.

Our next goal is to establish the uniqueness of kernel functions. Before proceeding with the actual proof, however, we will be able to simplify the form of the domain with which we must deal. Again, we consider kernel functions at a point $(Y, S)\in\partial_p D$ with respect to a fixed point $(X, T)\in D$ with $T>S$. Generally, D is a regular domain for the Dirichlet problem for the heat equation which satisfies condition L1 or condition L2 at (Y, S).

LEMMA 1.5. *Suppose that $D^*\subset D$ are both domains which are regular for the heat equation with $(Y, S)\in\partial_p D^*\cap\partial_p D$ and N a neighborhood of (Y, S) such that $N\cap\partial_p D^*=N\cap\partial_p D$. If there is at most one kernel function in D^* at (Y, S) with respect to $(X^*, T^*)\in D^*$ with $T^*>S$, then there is at most one kernel function in D at (Y, S) with respect to $(X, T)\in D$ with $T>S$.*

Proof. Suppose that u and v are both kernel functions at (Y, S) in D. Define Pu to be the solution of the Dirichlet problem for the heat equation in D^* with boundary values

$$\begin{aligned} Pu(y, s) &= u(y, s) \quad \text{for } (y, s) \in \partial_p D^* \cap D, \\ &= 0 \qquad\quad \text{for } (y, s) \in \partial_p D^* \cap \partial_p D. \end{aligned}$$

By the maximum principle, $Pu \leqq u$ in D^*, so that $u - Pu$ is a nonnegative temperature in D^* with zero boundary values on $\partial_p D^* - \{(Y, S)\}$. Consequently, the normalized function $(u(x, t) - Pu(x, t))/(u(X^*, T^*) - Pu(X^*, T^*))$ is a kernel function in D^* at (Y, S) with respect to (X^*, T^*). By the same reasoning, the analogous function with u replaced everywhere by v is also a kernel function in D^* at (Y, S) with respect to (X^*, T^*). Since the uniqueness of kernel functions in D^* is assumed, we must have

$$u(x, t) - Pu(x, t) = C(v(x, t) - Pv(x, t)) \quad \text{in } D^*,$$

where $C = (u(X^*, T^*) - Pu(X^*, T^*))/(v(X^*, T^*) - Pv(X^*, T^*))$. It follows that $u - Cv = Pu - CPv = P(u - Cv)$ in D^*, where $P(u - Cv)$ is defined in the obvious manner, vanishing on $\partial_p D^* \cap \partial_p D$. Because it is also true that $u - Cv$ vanishes on $\partial_p D - \partial_p D^*$, the maximum principle implies that $u - Cv \equiv 0$ in D. Evaluation at (X, T) then reveals that $C = 1$. Q.E.D.

This result enables us to reduce the question of uniqueness for kernel functions at (Y, S) in D, a regular domain for the heat equation which satisfies condition L1 or condition L2 at (Y, S), to a question of uniqueness for kernel functions at (Y, S) in the domain $D^* = \Psi((Y, S), r)$, with arbitrarily small positive r. In this case, $\partial_p D^* - \partial_p D$ has a simplified form; in particular, one "end" of D^* lies in the hyperplane $x_n = Y_n + rd$. We shall next show that D^* satisfies an additional condition.

Definition. A bounded region D_T in R^{n+1} is *parabolically starlike at* (X, T) if, for each $(y, s) \in \partial_p D_T$, there exists a finite parabolic ray with vertex (y, s) and endpoint (X, T) which is contained in D_T. (We allow the degenerate case in which the parabolic ray becomes a vertical line segment.)

We also require the notion of a parabolic cone.

Definition. Γ is a *parabolic cone* with vertex (y, s) if one of the following holds for some triple of positive constants, C, C', and C'', and for some choice of local space coordinates:

$$\Gamma = \{(x, t) : C > x_n - y_n > C'|x' - y'| + C''|t - s|^{1/2}\}$$

or

$$\Gamma = \{(x, t) : C > t - s > C'|x - y|^2\}.$$

It is clear that, for each point $(y, s) \in \partial_p D_T$ satisfying condition L1 or condition L2, there exists a parabolic cone Γ with vertex (y, s) which lies in D_T. Furthermore, if (Y, S) is such a point, the intersection of $\Psi((Y, S), r)$ with the union of parabolic cones with vertices in $\Delta((Y, S), r)$ is a parabolically starlike region. As we have seen, it suffices to prove the uniqueness of kernel functions at (Y, S) in such a region.

LEMMA 1.6. *Suppose that conditon* L2 *is satisfied at* $(Y, S)\in\partial_p D$ *and that* $\Psi=\Psi((Y, S), r_0)$ *is parabolically starlike at* (X, T). *Then there is a positive constant* C *such that, if* $u(x, t)$ *is any kernel function at* (Y, S) *in* Ψ *with respect to* (X, T), *we have*

$$u(x, t) \geqq CK(x, t) \quad \text{for } (x, t)\in\Psi,$$

where $K(x, t)$ *is the kernel function at* (Y, S) *in* Ψ *with respect to* (X, T) *given by* $\text{limit}_{n\to\infty}\, \omega^{(x,t)}(\Delta_n)/\omega^{(X,T)}(\Delta_n)$, *with* ω *denoting caloric measure in* Ψ *and* $\Delta_n=\Delta((Y, S), 1/n)$.

Proof. Since the result is trivially true for $t<S$, we will assume that $S=0$, so that (Y, S) is a point on the "bottom edge" of D. For convenience, we will also assume that $Y=0$, and, by a suitable choice of space coordinates, that the parabolic arc from $(0, 0)$ to (X, T) is defined by $x_n=\gamma t^{1/2}$, with $x'=0$, for some positive constant γ.

For $r\in(0, r_0)$, define $\beta=\beta(r)=1-(3(1+\mu))^{1/2}\gamma^{-1/2}r$, and set $u_r(x, t)=u(Q_r(x, t))=u(\beta x', \beta x_n+(1-\beta)+Br^{1/2}, \beta^2 t+\gamma(1-\beta)^2)$, with B to be chosen. Note that u_r is a temperature in $Q_r^{-1}(D)$. Next, define $\Psi^r=\Psi\cap\{(x, t) : (x', x_n+(1-\beta)+Br^{1/2}, t)\in\Psi\}$. For small r, we claim that $Q_r(\Psi^r)\subset\Psi$, which is evident away from (Y, S). Thus, it suffices to show that $Q_r(\Psi^r)\subset\{(x, t) : x_n>f(x', t)\}$, where $f(x', t)$ is the mixed-Lipschitz function defining the boundary of D near (Y, S). Specifically, we are requiring that

$$\beta f(y', s)+(1-\beta)+Br^{1/2} > f(\beta y', \beta^2 s+\gamma(1-\beta)^2)$$

for (y', s) near $(0, 0)$. It is already known that there is a positive constant C such that

$$f(y', s)-f(\beta y', \beta^2 s+\gamma(1-\beta)^2) \geqq -C((1-\beta)|y'|+((1-\beta^2)s+\gamma(1+\beta)^2)^{1/2}).$$

Consequently, it is enough to show that

$$(\beta-1)f(y', s)+(1-\beta)+Br^{1/2}-C((1-\beta)|y'|+((1-\beta^2)s+\gamma(1-\beta)^2)^{1/2}) > 0.$$

For small r, if we make use of the definition of β,

$$C((1-\beta)|y'|+((1-\beta^2)s+(1-\beta)^2\gamma)^{1/2}) \leqq C_1 r^{1/2}$$

and

$$(1-\beta)f(y', s) \leqq (3(1+\mu)/\gamma)^{1/2} r^{1/2}\cdot C(T+L),$$

where C is here the mixed-Lipschitz constant at (Y, S) and L is the (space) diameter of Ψ. Thus, if B is chosen sufficiently large, we do have $Q_r(\Psi^r)\subset\Psi$, and it follows that u_r is a temperature in Ψ^r. Furthermore, u_r is continuous in the closure of Ψ^r, so that

$$u_r(x, t) = \int_{\partial_p \Psi^r} u_r(y, s)\, d\omega_r^{(x,t)}(y, s) \quad \text{for } (x, t) \in \Psi^r,$$

where ω_r denotes caloric measure in Ψ^r. It follows that

$$u_r(x, t) \geqq \inf_{(y,s)\in\Delta_r} u_r(y, s)\cdot \omega_r^{(x,t)}(\Delta_r) \quad \text{for } (x, t) \in \Psi^r,$$

where $\Delta_r = \Delta((Y, S), r)$.

For $(y, s) \in \Delta_r$, $Q_r(y, s)$ has t-coordinate

$$\beta^2 s + \gamma(1-\beta)^2 \geqq -\beta^2 r^2 + 3(1+\mu)r^2 \geqq (2+3\mu)r^2.$$

Recalling that the point $A_r = A((Y, S), r)$ has t-coordinate $(1+\mu)r^2$, we see that Harnack's inequality will imply $\inf_{(y,s)\in\Delta_r} u_r(y, s) \geqq Cu(A_r)$, with a constant C independent of r, provided only that $Q_r(y, s)$ is bounded away from $\partial\Psi$ in the space variables by a fixed multiple of r for points $(y, s) \in \Delta_r$. Since $Q_r(y, s) = Q_r(y', f(y', s), s) = (\beta y', \beta f(y', s) + (1-\beta) + Br^{1/2}, \beta^2 s + \gamma(1-\beta)^2)$, this requirement is satisfied by our choice of B above. We then have

$$(*) \qquad u_r(x, t) \geqq Cu(A_r)\omega_r^{(x,t)}(\Delta_r) \quad \text{for } (x, t) \in \Psi^r.$$

By Lemma 1.4, there is a constant $C_0 > 0$ such that

$$1 = u(X, T) \leqq C_0 u(A_r)\omega^{(x,t)}(\Delta_r),$$

ω denoting caloric measure in Ψ. Thus,

$$u_r(x, t) \geqq C\omega_r^{(x,t)}(\Delta_r)/\omega^{(X,T)}(\Delta_r) \quad \text{for } (x, t) \in \Psi^r.$$

We allow r to range through the sequence $1/n$, $n = 1, 2, \ldots$. According to the maximum principle,

$$\omega_n^{(x,t)}(\Delta_n) \geqq \omega^{(x,t)}(\Delta_n) - \sup_{(z,u)\in\partial\Psi^n\cap\Psi} \omega^{(z,u)}(\Delta_n) \quad \text{for } (x, t) \in \Psi^n,$$

where the "n" notation refers in each case to $r = 1/n$. Combining this with our estimate for u_r above, we have

$$u_n(x, t) \geqq C\omega^{(x,t)}(\Delta_n)/\omega^{(X,T)}(\Delta_n) - \sup_{(z,u)\in\partial\Psi^n\cap\Psi} \omega^{(z,u)}(\Delta_n)/\omega^{(X,T)}(\Delta_n)$$

for $(x, t) \in \Psi^n$. As n tends to infinity, Ψ^n tends to Ψ and u_n tends to u. Thus, to conclude that $u \geqq CK$ in Ψ, we need only show that

$$\sup_{(z,u)\in\partial\Psi^n\cap\Psi} \omega^{(z,u)}(\Delta_n)/\omega^{(X,T)}(\Delta_n)$$

tends to zero as n tends to infinity. This follows from Lemma 1.4 and another application of Harnack's inequality. Q.E.D.

THEOREM 1.7. *Suppose that D is a regular domain for the heat equation which satisfies condition* L2 *at* $(Y, S) \in \partial_p D$. *If* $(X, T) \in D$ *with* $T > S$, *then there is a unique kernel function at* (Y, S) *in* D *with respect to* (X, T).

Proof. We have previously established the existence of a kernel function at (Y, S) in D with respect to (X, T) in the form of a limit of normalized caloric measures. We will continue to denote this kernel function by $K(x, t)$. Our present task is then reduced to a proof of the uniqueness of $K(x, t)$. According to Lemma 1.5, we may assume that D is a set of the form $\Psi((Y, S), r)$. Then, by Lemma 1.6, if $u(x, t)$ is any kernel function at (Y, S) in D with respect to (X, T), then $u(x, t) \geqq CK(x, t)$ for $(x, t) \in D$, where C is a constant independent of $u(x, t)$.

Let

$$C_0 = \sup\{C : u(x, t) \geqq CK(x, t) \text{ for } (x, t) \in D, \text{ for every kernel function } u(x, t) \text{ at } (Y, S) \text{ in } D \text{ with respect to } (X, T)\}.$$

Clearly, $u \geqq C_0 K$ for every kernel function u at (Y, S) in D with respect to (X, T). Furthermore, $C_0 \leqq 1$. If $C_0 = 1$, the strong maximum principle implies that $u = K$. Assuming $C_0 < 1$, $u_0 = (u - C_0 K)/(1 - C_0)$ is another kernel function at (Y, S) in D with respect to (X, T), in which case $u_0 \geqq C_0 K$ in D. This leads to a new inequality for u, $u \geqq (2C_0 - C_0^2)K$ in D. However, $2C_0 - C_0^2 > C_0$, contradicting our assumption that C_0 is the maximal constant satisfying $u \geqq CK$ in D for every kernel function u at (Y, S) in D with respect to (X, T). Q.E.D.

REMARK 1.8. We have given proofs of the existence and uniqueness of kernel functions at a point (Y, S) of the parabolic boundary of D only in the case that condition L2 is satisfied at (Y, S). For points satisfying condition L1 ("bottom" points of D), existence of a kernel function is established in corresponding manner. In this case, the uniqueness of the kernel function follows from the uniqueness of the kernel function at the center base point of a cylinder, which can be verified along lines similar to those of Lemma 1.6. Assuming that the point in question is $(0, 0)$, and that $u(x, t)$ is an arbitrary kernel function at that point, the approximating functions are defined to be

$$u_r(x, t) = u(\beta x, \beta^2 t + (1-\beta)^2), \quad \text{with } \beta = 1 - (3(1+\mu))^{1/2} r.$$

Only the details of the proof differ from the previous one.

REMARK 1.9. As an easy consequence of Theorem 1.7, we see that the kernel function at a point $(Y, S) \in \partial_p D$ with respect to (X, T) is equal to the Radon-Nikodym derivative, $d\omega^{(x,t)}/d\omega^{(X,T)}$, evaluated at (Y, S). Furthermore, if we now emphasize the dependence of the kernel function on the point (Y, S) by denoting the kernel function at (Y, S) in D with respect to (X, T) by $K(x, t, Y, S)$, it is easily seen that this dependence is continuous for (Y, S) on the parabolic boundary.

With the notion of kernel functions well established, we can now prove an important representation theorem for nonnegative temperatures.

THEOREM 1.10. *Suppose that D is a domain which satisfies condition* L1 *or condition* L2 *at each point of $\partial_p D$. If $u(x, t)$ is a nonnegative temperature in $D_T = D \cap \{(x, t) : t < T\}$, then there is a unique Borel measure ν on $\partial_p D_T$ such that*

$$u(x, t) = \int_{\partial_p D_T} K(x, t, y, s)\, d\nu(y, s),$$

where $K(x, t, y, s)$ is the kernel function at (y, s) in D with respect to a fixed point $(X, T) \in D$.

Proof. For any relatively closed subset B of D_T, define, for $(x, t) \in D_T$, $R_u^B(x, t) = \inf\{v(x, t) : v$ is a nonnegative supercaloric function in D_T with $v \geqq u$ on $B\}$. (A supercaloric function is a super-solution of the heat equation.) For $(x, t) \in B$, $R_u^B(x, t) = u(x, t)$, and, for $(x, t) \in D_T - B$, $R_u^B(x, t)$ is equal to the Wiener solution of the Dirichlet problem for the heat equation with boundary values u on $\partial B \cap D_T$ and zero on the closure of $\partial_p D_T - B$.

Next, for $(x, t) \in D_T$ and F a closed subset of $\partial_p D_T$, define

$$\bar{\nu}^{(x,t)}(F) = \inf\{R_u^{\bar{U} \cap D}(x, t) : U \text{ is an open subset of } \boldsymbol{R}^{n+1}, F \subset U\}.$$

For any sequence of open sets U_i which decrease to F, we have

$$\bar{\nu}^{(x,t)}(F) = \lim_{i \to \infty} R_u^{U_i \cap D}(x, t).$$

By Harnack's monotone convergence theorem, $\bar{\nu}^{(x,t)}(F)$ is a nonnegative temperature in D_T.

For fixed $(x, t) \in D_T$, $\bar{\nu}^{(x,t)}(F)$ is a nonnegative, monotone, subadditive function on the closed subsets of $\partial_p D_T$ which is additive on disjoint closed sets. Moreover, $\bar{\nu}^{(x,t)}(F)$ is regular:

$$\bar{\nu}^{(x,t)}(F) = \inf\{\bar{\nu}^{(x,t)}(G) : G \text{ is a closed subset of } \partial_p D_T, F \subset G^\circ\}.$$

(Here, G° denotes the interior of G.) A standard result implies that $\bar{\nu}^{(x,t)}$ can be extended to a regular Borel measure $\nu^{(x,t)}(\cdot)$ on $\partial_p D_T$.

By Harnack's inequality, $\nu^{(x,t)}$ is absolutely continuous with respect to $\nu^{(X,T)}$ if $(x, t) \in D_T$. Taking $\Delta_r = \Delta((y, s), r)$ for $(y, s) \in \partial_p D_T$, it follows from Lemma 1.4 and the uniqueness of kernel functions that

$$K(x, t, y, s) = \lim_{r \to 0} \nu^{(x,t)}(\Delta_r)/\nu^{(X,T)}(\Delta_r) = \frac{d\nu^{(x,t)}}{d\nu^{(X,T)}}(y, s).$$

We then have

$$u(x, t) = \nu^{(x,t)}(\partial_p D_T) = \int_{\partial_p D_T} d\nu^{(x,t)}(y, s) = \int_{\partial_p D_T} K(x, t, y, s)\, d\nu^{(X,T)}(y, s).$$

To prove uniqueness for the representing measure, let η be another regular Borel measure on $\partial_p D_T$ such that

$$u(x, t) = \int_{\partial_p D_T} K(x, t, y, s)\, d\eta(y, s).$$

For a closed subset F of $\partial_p D_T$, choose a sequence of open sets $G_k \subset \mathbf{R}^{n+1}$ such that $F = \bigcap G_k$ and $\nu^{(X,T)}(F) = \text{limit}_{k\to\infty} R_u^{\bar{G}_k \cap D}(X, T)$. Let ω_k denote the caloric measure in $D - G_k$ and let $H_k = \partial_p(D_T - \bar{G}_k)$. For $(x, t) \in D_T - \bar{G}_k$,

$$
\begin{aligned}
R_u^{\bar{G}_k \cap D}(x, t) &= \int_{D \cap H_k} u(y, s)\, d\omega_k^{(x,t)}(y, s) \\
&= \int_{D \cap H_k} \left(\int_{\partial_p D_T} K(y, s, z, u)\, d\eta(z, u) \right) d\omega_k^{(x,t)}(y, s) \\
&= \int_{\partial_p D_T} \left(\int_{D \cap H_k} K(y, s, z, u)\, d\omega_k^{(x,t)}(y, s) \right) d\eta(z, u).
\end{aligned}
$$

For $(z, u) \in F$, $K(y, s, z, u)$ is a temperature (y, s) in $D_T - \bar{G}_k$ which is continuous in the closure of that set. For such (z, u),

$$\int_{D \cap H_k} K(y, s, z, u)\, d\omega_k^{(x,t)}(y, s) = K(x, t, z, u).$$

For $(z, u) \in \partial_p D_T - F$,

$$\operatorname*{limit}_{k\to\infty} \int_{D \cap H_k} K(y, s, z, u)\, d\omega_k^{(x,t)}(y, s) = 0.$$

By the maximum principle,

$$\int_{D \cap H_k} K(y, s, z, u)\, d\omega_k^{(x,t)}(y, s) \leqq K(x, t, z, u) \quad \text{for } (x, t) \in D_T - \bar{G}_k$$

and for each value of k. Lebesgue's theorem then implies that

$$\nu^{(X,T)}(F) = \operatorname*{limit}_{k\to\infty} R_u^{\bar{G}_k \cap D}(X, T) = \int_F K(X, T, z, u)\, d\eta(z, u) = \eta(F).$$

Since both of the measures are regular, uniqueness is established. Q.E.D.

2. **Existence of parabolic limits at the boundary.** In this section we will make use of Theorem 1.10 in proving the existence of parabolic limits almost everywhere $(\omega^{(X,T)})$ on $\partial_p D_T$ for nonnegative temperatures in D_T.

Definition. A function $u(x, t)$ defined in D_T has *parabolic limit* L at a point $(y, s) \in \partial_p D_T$ if, for each parabolic cone $V \subset D$ with vertex (y, s) which opens away from ∂D and satisfies $\bar{V} \cap \partial_p D_T = \{(y, s)\}$, we have $\text{limit}_{(x,t)\to(y,s);\,(x,t)\in V}\, u(x, t) = L$. (For $(y, s) \in \partial_p D_T$ satisfying condition L2, V opens away from ∂D if $V \subset \{(x, t) : x_n > y_n\}$.)

We begin with several lemmas concerning the uniform behavior of kernel functions. We continue to assume condition L1 or condition L2 on $\partial_p D$.

Lemma 2.1. *Let $(Y, S) \in \partial_p D$ with $\Delta = \Delta((Y, S), r)$. Then, for sufficiently small r,*

$$\sup_{(y,s)\in\partial_p D - \Delta} K(x, t, y, s) \to 0 \quad \text{as } (x, t) \to (Y, S) \text{ in } D.$$

Proof. (For condition L2 holding at (Y, S).)

Since $K(x, t, y, s)\equiv 0$ for $s\geqq T$, we need only consider points (y, s) in $\partial_p D_T-\Delta$. For a large positive number M, let $\Sigma=\{(x, t) : x_n> -M(|t|^{1/2}+|x'|), |x|<1, |t|<1\}$, and let $h(x, t)$ denote the caloric measure in Σ of that part of $\partial_p\Sigma$ satisfying $|x|=1$ or $|t|=1$. Shrink Σ parabolically by the map $(x, t)\to(rx/2, r^2t/4)$ and translate this region in such a way that the origin is moved to the point (Y, S) and the orientation coincides with the local coordinates at (Y, S). If M is large, the cone $\{(x, t) : x_n> -M(|t|^{1/2}+|x'|)\}$, after the shrinking and translation, lies in the complement of D.

For $s\in(S, T)$, $K(x, t, y, s)=0$ in a neighborhood of (Y, S), so we need only prove the lemma for $(y, s)\in U$, where $U\subset\partial_p D_T-\Delta$ is bounded away from $\{(x, t) : t=T\}$. By Lemma 1.4, if $(Y_0, S_0)\in U$ and $\Delta_0=\Delta((Y_0, S_0), r_0)$, with $r_0<r/4$, we have

$$\omega^{(x,t)}(\Delta_0) \leqq C\omega^{A((Y_0,S_0),r)}(\Delta_0) \quad \text{for } (x, t)\in D-\Psi((Y_0, S_0), (1+\mu)^{1/2}r/4).$$

Furthermore, a careful examination of the proof of Lemma 1.4 reveals that the constant may be chosen to hold for all $(Y_0, S_0)\in U$. Applying Harnack's inequality,

$$\omega^{(x,t)}(\Delta_0) \leqq C\omega^{(X,T)}(\Delta_0) \quad \text{for } (x, t)\in D-\Psi((Y_0, S_0), (1+\mu)^{1/2}r/4),$$

with C now depending on r as well as on D and U. If Σ_r denotes the transformed region Σ, then $D\cap\Sigma_r\subset D-\Psi((Y_0, S_0), (1+\mu)^{1/2}r/4)$. By the maximum principle in $D\cap\Sigma_r$,

$$\omega^{(x,t)}(\Delta_0) \leqq C\omega^{(X,T)}(\Delta_0)h_r(x, t) \quad \text{for } (x, t)\in D\cap\Sigma_r,$$

where h_r is the caloric measure in Σ_r corresponding to h in Σ. Since $h_r(x, t)$ tends to zero as (x, t) tends to (Y, S), we see that

$$\omega^{(x,t)}(\Delta_0)/\omega^{(X,T)}(\Delta_0)\to 0 \quad \text{as } (x, t)\to(Y, S),$$

independent of (Y_0, S_0) in U and $r_0<r/4$. It follows that

$$\sup_{(y,s)\in U} K(x, t, y, s)\to 0 \quad \text{as } (x, t)\to(Y, S). \qquad \text{Q.E.D.}$$

In the several lemmas that follow, attention will be restricted to domains D_T which satisfy an additional condition:

(Z) There is a hyperplane $H=\{(x, t) : \langle x, \alpha\rangle=\theta\}$ such that $\partial_p D_T$ intersects H in a simply connected set B, which is open in H, with $\sup_{(x,t)\in B} t=T$ and $\inf_{(x,t)\in B} t=0$.

In particular, (Z) is satisfied by domains $\Psi((Y, S), r)$ with an appropriate choice of local coordinates. In connection with condition (Z), we shall also assume that $\theta=0$ and $D_T\subset\{(x, t) : \langle x, \alpha\rangle<0\}$. (Here, α is an n-vector and $\langle\cdot, \cdot\rangle$ denotes the inner product in $\mathbf{R}^n$.)

LEMMA 2.2. *Let D_T satisfy* (Z) *and let $U\subset\partial_p D_T$ be bounded away from the set B. Then*

$$\sup_{\{(x,t,y,s):\langle x,\alpha\rangle>-\varepsilon, t\leqq T, (y,s)\in U\}} K(x, t, y, s)\to 0 \quad \text{as } \varepsilon\to 0.$$

Proof. Let $D_{\varepsilon_0} = D_T \cap \{(x, t) : \langle x, \alpha\rangle < -\varepsilon_0\}$, where $0 < \varepsilon_0 < \inf_{(y,s)\in U} -\langle y, \alpha\rangle$. By Lemma 2.1 and Harnack's inequality,

$$\sup_{(z,u)\in D_T\cap\partial_p D_{\varepsilon_0}} K(z, u, y, s) \leqq CK(X, T+\varepsilon_0, y, s) \quad \text{for } (y, s) \in U.$$

Since $K(X, T+\varepsilon_0, y, s)$ depends continuously on (y, s) in $\partial_p D_T$, there is a positive constant M such that $\sup_{(y,s)\in U} K(X, T+\varepsilon_0, y, s) \leqq M$. Therefore,

$$\sup_{(y,s)\in U;\,(z,u)\in D_T\cap\partial_p D_{\varepsilon_0}} K(z, u, y, s) \leqq CM.$$

Since $K(z, u, y, s) \to 0$ as $(z, u) \to \partial_p D_T - U$ for $(y, s) \in U$, the maximum principle implies that

$$K(z, u, y, s) \leqq M(z, u) \quad \text{for } (z, u) \in D_T - D_{\varepsilon_0},$$

where $M(z, u)$ is the unique bounded temperature in $D_T - D_{\varepsilon_0}$ with boundary values equal to CM for $(z, u) \in D_T \cap \partial_p D_{\varepsilon_0}$ and equal to zero for $(z, u) \in \partial_p D_T \cap \partial_p D_{\varepsilon_0}$. Clearly,

$$\sup_{(z,u)\in D_T - D_\varepsilon} M(z, u) \to 0 \quad \text{as } \varepsilon \to 0. \quad \text{Q.E.D.}$$

LEMMA 2.3. *Again let D_T satisfy* (Z). *Let $U_0 \in \partial_p D_T$ be bounded away from the set B and let $D_{T,r} = D_T \cap \{(x, t) : \langle x, \alpha\rangle < -r\}$. Then there is a positive constant C such that, if r is sufficiently small, $\omega_{T,r}^{(X,T)}(U) \geqq C\omega^{(X,T)}(U)$ for each measurable set $U \subset U_0$, where the caloric measures are taken in $D_{T,r}$ and D, respectively.*

Proof. For sufficiently small r, $(X, T) \in D_{T,r}$, and we have

$$\omega_{T,r}^{(X,T)}(U) = \int_U W_r(y, s)\, d\omega^{(X,T)}(y, s),$$

where $W_r(y, s) = 1 - \int_{D_T\cap\partial_p D_{T,r}} K(z, u, y, s)\, d\omega_{T,r}^{(X,T)}(z, u)$, K denoting the kernel functions in D. To complete the proof, we will show that $W_r(y, s) \geqq C > 0$ for small r and for $(y, s) \in U_0$. Clearly, $W_r(y, s) \geqq 1 - \sup_{(z,u)\in D_T\cap\partial_p D_{T,r};\,(y,s)\in U_0} K(z, u, y, s)$. By Lemma 2.2, there is a constant $r_0 > 0$ such that

$$\sup_{(z,u)\in D_T\cap\partial_p D_{T,r};\,(y,s)\in U_0} K(z, u, y, s) \leqq \tfrac{1}{2} \quad \text{for } r < r_0. \qquad \text{Q.E.D.}$$

LEMMA 2.4. *Suppose that D_T is a set of the form $\Psi((Y_0, S_0), r_0)$, where condition* L2 *is satisfied at (Y_0, S_0) for some larger domain. Then there is a positive constant C such that, if $(Y, S) \in \Delta((Y_0, S_0), r_0/2)$ and r is sufficiently small, we have*

$$K(A, y, s) \leqq C/\omega^{(X,T)}(\Delta((Y, S), r)) \quad \textit{for } (y, s) \in \Delta((Y, S), r),$$

where $A = A((Y, S), r)$.

Proof. From equation (*) in the proof of Lemma 1.6,

$$K_r(x, t, y, s) \geqq CK(A, y, s)\omega_r^{(x,t)}(\Delta((Y, S), r))$$

for $(x, t) \in D^r$ and $(y, s) \in \Delta((Y_0, S_0), r_0/2)$, where the "$r$" notation corresponds to that lemma. However, each set D^r (Lemma 1.6) is equal to a set $D_{T,r'}$ (Lemma 2.3) and r, r' tend to zero together. Applying Lemma 2.3 with $(x, t)=(X, T)$, we have

$$K(A, y, s) \leqq CK_r(X, T, y, s)/\omega^{(X,T)}(\Delta((Y, S), r))$$

for $r<r_1$ and $(y, s) \in \Delta((Y_0, S_0), r_0/2)$, where $r_1<r_0/2$. Repeating an argument in the proof of Lemma 2.2, there is a constant $M>0$ such that

$$\sup_{(y,s)\in\Delta((Y_0,S_0),r_0/2)} K_r(X, T, y, s) \leqq M. \quad \text{Q.E.D.}$$

Again, a similar result can be proven when condition L1 is satisfied at a point $(Y, 0) \in \partial_p D$. In this case one considers $K_r(x, t, y, s)=K(x, t+(2+\mu)r^2, y, s)$. By the maximum principle and Harnack's inequality,

$$K_r(X, T, y, s) \geqq CK(A, y, s)\omega^{(X,T)}(\Delta((Y, 0), r)) \quad \text{for small } r.$$

We can now prove the final essential lemma.

LEMMA 2.5. *Again suppose that D_T is a set of the form $\Psi((Y_0, S_0), r_0)$, where condition* L2 *is satisfied at (Y_0, S_0) for some larger domain. Let V be a parabolic cone in D_T with vertex $(Y, S) \in \Delta((Y_0, S_0), r_0)$, with V opening away from ∂D. Let $(\bar{x}, \bar{t}) \in V$ satisfy $\bar{x}_n - Y_n=rd$, where d is the fixed parameter in the definition of $\Psi((Y, S), r)$, and let $\Delta_j=\Delta((Y, S), 2^j r)$, with $R_0=\Delta_0$ and $R_j=\Delta_j-\Delta_{j-1}$, $j=1, 2, \ldots$. For sufficiently small r, we then have*

$$\sup_{(y,s)\in R_j} K(\bar{x}, \bar{t}, y, s) \leqq CC_j/\omega^{(X,T)}(\Delta_j), \qquad j = 1, 2, \ldots, N,$$

where N is the smallest integer such that $2^N r>r_0$. The positive constant C depends only on D_T and V and the positive constants C_j satisfy $\sum_{j=0}^{\infty} C_j \leqq C'=C'(D_T)$.

Proof. Let $A_j=A((Y, S), 2^j r)$. By Harnack's inequality, if u is any nonnegative temperature in D_T, $u(\bar{x}, \bar{t}) \leqq Cu(A_0)$, with the constant depending only on D_T and V. In particular,

$$K(\bar{x}, \bar{t}, y, s) \leqq CK(A_0, y, s) \quad \text{for } (y, s) \in \partial_p D_T.$$

By Lemma 2.4,

$$\sup_{(y,s)\in\Delta_j} K(A_j, y, s) \leqq C/\omega^{(X,T)}(\Delta_j),$$

with C independent of j. Moreover, for $j=1, 2, 3, 4$, say, Harnack's inequality establishes the existence of constants $C_j=C_j(D_T)$ such that $K(A_0, y, s) \leqq C_j K(A_j, y, s)$. Therefore,

$$\sup_{(y,s)\in R_j} K(A_0, y, s) \leqq CC_j/\omega^{(X,T)}(\Delta_j) \quad \text{for } j = 1, 2, 3, 4.$$

Now, let $(y_j, s_j) \in R_j$ for $j>4$ and suppose $\Delta \subset \Delta'_j = \Delta((y_j, s_j), r_j)$, where $r_j = 2^{j-4}r$. Let $A'_j = A((y_j, s_j), r_j)$. By Lemma 1.4,

$$K(x, t, y_j, s_j) \leqq CK(A'_j, y_j, s_j) \quad \text{for } (x, t) \in D_T - \Psi_j,$$

where $\Psi_j = \Psi((y_j, s_j), (1+\mu)^{1/2} r_j/4)$. Since

$$(2^{j+1}r)^2(1+\mu) - ((2^j r)^2 + (2^{j-4}r)^2(1+\mu)) \geqq r^2(2^{2j+2} - 2^{2j+1}) = 2^{2j+1}r^2,$$

Harnack's inequality can be applied to show $K(A'_j, y_j, s_j) \leqq CK(A_{j+1}, y_j, s_j)$. For $(x, t) \in D - \Psi_j$, combining this with Lemma 2.4, we have

$$K(x, t, y_j, s_j) \leqq CK(A_{j+1}, y_j, s_j) \leqq C/\omega^{(X,T)}(\Delta_{j+1}) \leqq C/\omega^{(X,T)}(\Delta_j).$$

Next, let

$$\begin{aligned} \Sigma = \{(x, t) : |x-Y| < 1, |t-S| < 1\} \\ \cap \{(x, t) : x_n - Y_n > -B(|t-S|^{1/2} + |x'-Y'|)\}, \end{aligned}$$

where B is chosen so that $\{(x, t) : x_n - Y_n \leqq -B(|t-S|^{1/2} + |x'-Y'|)\}$ is contained in the complement of D_T. Let Σ_j be the region formed by parabolically shrinking Σ by the factor $2^{j-3}r$ and let h_j denote the caloric measure in Σ_j of that part of $\partial_p \Sigma_j$ which does not lie on the cone. The maximum principle in $D_T \cap \Sigma_j$, which is contained in $D_T - \Psi_j$, implies that

$$K(x, t, y_j, s_j) \leqq Ch_j(x, t)/\omega^{(X,T)}(\Delta_j) \quad \text{for } (x, t) \in D_T \cap \Sigma_j.$$

Setting $(x, t) = A_0$, $K(A_0, y_j, s_j) \leqq Ch_j(A_0)/\omega^{(X,T)}(\Delta_j)$, and it follows that

$$\sup_{(y,s)\in R_j} K(\bar{x}, \bar{t}, y, s) \leqq Ch_j(A_0)/\omega^{(X,T)}(\Delta_j).$$

To complete the proof we must show that $\sum_{j=5}^{\infty} h_j(A_0) < \infty$. If h denotes the caloric measure in Σ corresponding to h_j in Σ_j, we have

$$\begin{aligned} h_j(A_0) &= h_j(Y', Y_n + rd, S + (1+\mu)r^2) \\ &= h(Y', Y_n + 2^{3-j}d, S + 2^{6-2j}(1+\mu)). \end{aligned}$$

If $m_0 = \max\{h(x, t) : (x, t) \in \Sigma, |x-Y| \leqq \frac{1}{2}, |t-S| \leqq \frac{1}{4}\}$, it is clear that $0 < m_0 < 1$. By the maximum principle,

$$h(Y + (x-Y)/2, S + (t-S)/4) \leqq m_0 h(x, t) \quad \text{for } (x, t) \in \Sigma.$$

Taking $(x, t) = (Y', Y_n + 2^{3-j}d, S + 2^{6-2j}(1+\mu))$, we have $h_{j+1}(A_0) \leqq m_0 h_j(A_0)$. Thus, by the ratio test, $\sum_{j=5}^{\infty} h_j(A_0) < \infty$. Q.E.D.

As usual, the proof for a point (Y, S) at which condition L1 is satisfied is similar. We omit the details. Using Lemma 2.5, we can now prove the almost everywhere existence of parabolic limits.

THEOREM 2.6. *Let $u(x, t)$ be a nonnegative temperature in D_T, which is assumed to satisfy condition* L1 *or condition* L2 *at each point on its parabolic boundary $\partial_p D_T$. Then $u(x, t)$ has finite parabolic limits at each point $(y, s) \in \partial_p D_T$, except for a set of zero caloric measure.*

Proof. We shall first prove the theorem in certain simple situations (Cases 1a and 1b, below), then combine these for the general result (Case 2, below).

Case 1a. Here D_T is assumed to be a set of the form $\Psi((Y_0, S_0), r_0)$. By Theorem 1.10, there is a unique regular Borel measure ν on $\delta_p D_T$ such that

$$u(x, t) = \int_{\partial_p D_T} K(x, t, y, s)\, d\nu(y, s).$$

Furthermore, we have $d\nu(y, s) = f(y, s)\, d\omega^{(X,T)}(y, s) + d\sigma(y, s)$, where $f(y, s) \in L^1(\omega^{(X,T)})$ and σ is singular with respect to $\omega^{(X,T)}$. In particular, $f(y, s) < \infty$ for $(y, s) \in \partial_p D_T$ except on a set of zero $(\omega^{(X,T)})$ measure. We will show that $u(x, t)$ has parabolic limits equal to $f(y, s)$ almost everywhere $(\omega^{(X,T)})$ on $\Delta((Y_0, S_0), r_0)$.

Let $(Y, S) \in \Delta((Y_0, S_0), r_0)$ with $f(Y, S) < \infty$. Then,

$$\begin{aligned} u(x, t) - f(Y, S) &= \int_{\partial_p D_T} K(x, t, y, s)(f(y, s)\, d\omega^{(X,T)}(y, s) + d\sigma(y, s)) - f(Y, S) \\ &= \int_{\partial_p D_T} K(x, t, y, s)((f(y, s) - f(Y, S))\, d\omega^{(X,T)}(y, s) + d\sigma(y, s)). \end{aligned}$$

Let V be a parabolic cone in D_T with vertex (Y, S) which opens away from ∂D, and let $(x, t) \in V$. Define Δ_j, R_j, and A_j as in Lemma 2.5 for $j = 0, 1, 2, \dots, N$, where $\Delta_{N-1} \subset \Delta = \Delta((Y, S), r_1) \subset \Delta_N$ for some small positive r_1. Then,

$$\begin{aligned} &|u(x, t) - f(Y, S)| \\ (*) \qquad &\leqq \left| \int_{\partial_p D_T - \Delta} K(x, t, y, s)((f(y, s) - f(Y, S))\, d\omega^{(X,T)}(y, s) + d\sigma(y, s)) \right| \\ &\quad + \sum_{j=0}^{N} \int_{R_j} K(x, t, y, s)(|f(y, s) - f(Y, S)|\, d\omega^{(X,T)}(y, s) + d\sigma(y, s)). \end{aligned}$$

The second term on the right is dominated by

$$\sum_{j=0}^{N} \sup_{(y,s) \in R_j} K(x, t, y, s) \left(\int_{\Delta_j} |f(y, s) - f(Y, S)|\, d\omega^{(X,T)}(y, s) + \sigma(\Delta_j) \right).$$

From Besicovitch's general theory of differentiation of measures,

$$\int_{\Delta_j} |f(y, s) - f(Y, S)|\, d\omega^{(X,T)}(y, s) = o(\omega^{(X,T)}(\Delta_j))$$

and

$$\sigma(\Delta_j) = o(\omega^{(X,T)}(\Delta_j)) \quad \text{as } \Delta_j \to \{(Y, S)\} \quad \text{for a.e } (\omega^{(X,T)})(Y, S) \in \Delta((Y_0, S_0), r_0).$$

Since $\sup_{(y,s) \in R_j} K(x, t, y, s) \leqq C\, C_j / \omega^{(X,T)}(\Delta_j)$ with $\sum_{j=0}^{\infty} C_j < \infty$ by the previous lemma, the final term in (*) can be made small uniformly for $(x, t) \in V$ if Δ (and, hence, Δ_j, $j = 0, 1, 2, \dots, N$) is sufficiently small.

Since the first term on the right of (*) can be dominated by

$$\sup_{(y,s) \in \partial_p D_T - \Delta} K(x, t, y, s)(\|\nu\| + f(Y, S)),$$

an application of Lemma 2.1 completes the proof in this case.

Case 1b. Here $D_T = \{(x, t) : |x| < 1, 0 < t < 1\}$, and $u(x, t)$ is shown to have finite parabolic limits almost everywhere ($\omega^{(X,T)}$) on that part of $\partial_p D_T$ where $t=0$. The proof of Case 1a can be repeated in this situation, taking into account the remarks made previously concerning boundary points at which condition L1 is satisfied.

Case 2. In this case we consider general domains D_T which satisfy condition L1 or condition L2 at each point of $\partial_p D_T$. Let (y_i, s_i), $i=1, 2, \ldots$, be a countable dense subset of $\partial_p D_T$. For each i there is a neighborhood N_i of (y_i, s_i) in $\partial_p D_T$ which satisfies one of the following conditions:

(I) If (y_i, s_i) is a point of $\partial_p D_T$ at which condition L1 is satisfied, then N_i is the base of a cylinder R_i which is contained in D_T and has axis in the t-direction.

(II) If (y_i, s_i) is a point of $\partial_p D_T$ at which condition L2 is satisfied, then there is a positive number r_i such that $N_i = \Delta((y_i, s_i), r_i)$. In this case we take $R_i = \Psi((y_i, s_i), r_i)$.

For each i, $N_i \subset \partial_p R_i$, and we can choose another neighborhood N_i' of (y_i, s_i) in $\partial_p D_T$ with $\bar{N}_i' \subset N_i$. Conditions (I) and (II) guarantee that $u(x, t)$ has finite parabolic limits on N_i', except possibly for a set of zero caloric measure in R_i, according to the results of Cases 1a and 1b. To complete the proof, it suffices to show that if $Z_i \subset N_i'$ is the set of zero caloric measure in R_i where $u(x, t)$ fails to have finite parabolic limits, then Z_i must also be a set of zero caloric measure in D_T. With this aim, let $(x, t) \in R_i$ with $\omega_i^{(x,t)}(Z_i) = 0$, where ω_i denotes caloric measure in R_i. If ω denotes caloric measure in D_T, then $\omega^{(x,t)}(Z_i) = P_i(x, t)$, where P_i is the solution of the Dirichlet problem for the heat equation in R_i with boundary values equal to $\omega^{(z,u)}(Z_i)$ for $(z, u) \in \partial_p R_i - \partial_p D_T$ and equal to zero for $(z, u) \in \partial_p R_i \cap \partial_p D_T$. Since $N_i \subset \partial_p R_i \cap \partial_p D_T$, we have

$$\lim_{(x,t)\to(y,s);\,(x,t)\in R_i} \omega^{(x,t)}(Z_i) = 0 \quad \text{for } (y, s) \in N_i.$$

It follows that $\lim_{(x,t)\to(y,s);\,(x,t)\in D_T} \omega^{(x,t)}(Z_i) = 0$. Since we also have

$$\lim_{(x,t)\to(y,s);\,(x,t)\in D_T} \omega^{(x,t)}(Z_i) = 0 \quad \text{for } (y, s) \in \partial_p D_T - N_i,$$

the maximum principle implies that $\omega^{(x,t)}(Z_i) = 0$ for $(x, t) \in D_T$. Q.E.D.

The following result is a simple corollary of the proof.

COROLLARY 2.7. (I) *If* $f(y, s) \in L^1(\partial_p D_T)$ *with respect to caloric measure and* $f(y, s) \geq 0$, *then the nonnegative temperature*

$$u(x, t) = \int_{\partial_p D_T} f(y, s)\, d\omega^{(x,t)}(y, s)$$

has parabolic limits equal to $f(y, s)$ *for almost every* $(\omega^{(X,T)})(y, s)$ *in* $\partial_p D_T$.

(II) $\omega^{(x,t)}(E)$ *has parabolic limit equal to* 1 *almost everywhere* $(\omega^{(X,T)})$ *on* E, *where* E *is a measurable subset of* $\partial_p D_T$.

REMARKS. Following the arguments of Hunt and Wheeden [4] in the case of harmonic functions, these results can be extended. Specifically, we need only require that the temperature $u(x, t)$ have a one-sided bound in some parabolic cone at each point of the parabolic boundary in order that the parabolic limits of $u(x, t)$ exist almost everywhere ($\omega^{(X,T)}$) on the parabolic boundary.

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