A GEOMETRY FOR $E_7$

BY

JOHN R. FAULKNER

Abstract. A geometry is defined by the 56-dimensional representation $\mathfrak{M}$ of a Lie algebra of type $E_7$. Every collineation is shown to be induced by a semisimilarity of $\mathfrak{M}$, and the image of the automorphism group of $\mathfrak{M}$ in the collineation group is shown to be simple.

Using the 56-dimensional ternary algebra $\mathfrak{M}$ with an alternating bilinear form introduced in [2], we define here a geometry and investigate its collineation group. The objects of the geometry are, in the real case, essentially the planes of the symplectic geometry for $E_7$ introduced by H. Freudenthal [4]. In §1, the notion of semisimilarities of $\mathfrak{M}$ is introduced, some semisimilarities are exhibited, and some identities in the group of semisimilarities are demonstrated. In §2, we define the geometry, show that semisimilarities induce collineations, derive some transitivity results, and prove that every collineation is induced by a semisimilarity. Finally, in §3, we show that the image of the automorphism group of $\mathfrak{M}$ in the collineation group is a simple group.

1. Semisimilarities. If $\mathfrak{J} = \mathfrak{J}(N, 1)$ is a quadratic Jordan algebra over a field $\Phi$ constructed as in [6] from an admissible nondegenerate cubic form $N$ with base-point 1, then $yU_x = T(x, y)x - x^\# \times y$ where $T(\ ,\ )$ and $x \rightarrow x^\#$ are respectively the associated nondegenerate bilinear form and quadratic mapping, and $x \times y = (x + y)^\# - x^\# - y^\#$. As in [2, pp. 399–401], we may construct $\mathfrak{M} = \mathfrak{M}((\mathfrak{J})) = \Phi u_1 \oplus \Phi u_2 \oplus \mathfrak{J}_{12} \oplus \mathfrak{J}_{21}$ with elements

$$x = au_1 + \beta u_2 + a_{12} + b_{21}; \quad a, \beta \in \Phi; \ a, b \in \mathfrak{J};$$

with a nondegenerate alternate bilinear form $\langle \ , \ \rangle$, and with a ternary product $\langle \ , \ , \ \rangle$ defined by

$$\langle x_1, x_2 \rangle = \alpha_1 \beta_2 - \alpha_2 \beta_1 - T(a_1, b_2) + T(a_2, b_1),$$

$$\langle x_1, x_2, x_3 \rangle = \gamma u_1 + \delta u_2 + c_{12} + d_{21},$$

where

$$\gamma = \alpha_1 \beta_2 \alpha_3 + 2\alpha_1 \alpha_2 \beta_3 - \alpha_3 T(a_1, b_2) - \alpha_2 T(a_2, b_3) - \alpha_1 T(a_3, b_3) + T(a_1, a_2 \times a_3),$$

$$c = (\alpha_2 \beta_3 + T(b_2, a_3)) a_1 + (\alpha_3 \beta_2 + T(b_3, a_2)) a_2 + (\alpha_1 \beta_3 + T(b_1, a_3)) a_3$$

$$- \alpha_1 b_2 \times b_3 - \alpha_2 b_1 \times b_3 - \alpha_3 b_1 \times b_2 - \{a_1 b_2 a_3\} - \{a_1 b_3 a_2\} - \{a_2 b_1 a_3\},$$

$$\delta = -\gamma^a, \quad d = -c^a,$$
where \( \sigma \) is the permutation \( \sigma = (\alpha \beta)(ab) \) with \( x_i = a_i u_1 + \beta_i u_2 + (a_i)_{12} + (b_i)_{21} \in M \). In [2, pp. 399–401], it was shown that

\begin{align*}
(T1) \quad & \langle x, y, z \rangle = \langle y, x, z \rangle + \langle x, y \rangle z, \\
(T2) \quad & \langle x, y, z \rangle = \langle x, z, y \rangle + \langle y, z \rangle x, \\
(T3) \quad & \langle \langle x, y, z \rangle, w \rangle = \langle \langle x, y, w \rangle, z \rangle + \langle x, y \rangle \langle z, w \rangle, \\
(T4) \quad & \langle \langle x, y, z \rangle, \langle v, x, z \rangle \rangle = \langle \langle x, z, y \rangle + \langle y, z \rangle, x \rangle, \\
\end{align*}

for \( x, y, z, w \in M \). We also wish to recall that we have a nondegenerate four-linear form \( q(x_1, x_2, x_3, x_4) = \langle x_1, x_2, x_3, x_4 \rangle \) for \( x_i \in M \).

If \( \mathbb{Z} = \mathbb{Z}(N', 1') \) where \( N' \) is an admissible nondegenerate cubic form with base-point \( 1' \) on \( \mathbb{Z} \) over a field \( \Phi' \), if \( \mathbb{M}' = \mathbb{M}(\mathbb{Z}') \), and if \( s \) is an isomorphism of \( \Phi \) onto \( \Phi' \), then an \( s \)-semilinear mapping \( W \) of \( \mathbb{M} \) onto \( \mathbb{M}' \) satisfying

\[ q'(x_1 W, x_2 W, x_3 W, x_4 W) = \rho q(x_1, x_2, x_3, x_4), \quad x_i \in \mathbb{M}, \]

for a fixed \( 0 \neq \rho \in \Phi' \) is called an \( s \)-semisimilarity of \( \mathbb{M} \) to \( \mathbb{M}' \) with multiplier \( \rho \). If \( s = 1 \), \( W \) is a similarity. If \( s = 1 \) and \( \rho = 1 \), then \( W \) is a form preserving map. If \( \mathbb{M} = \mathbb{M}' \), we denote the group of semisimilarities (respectively, similarities, form preserving maps) by \( \Gamma = \Gamma(\mathbb{M}) \) (respectively, \( G = G(\mathbb{M}) \), \( S = S(\mathbb{M}) \)).

**Lemma 1.** An \( s \)-semilinear map \( W \) of \( \mathbb{M} \) onto \( \mathbb{M}' \) is an \( s \)-semisimilarity with multiplier \( \rho \) if and only if \( x_1, x_2, x_3, x_4 \) are \( s \)-semilinear maps of \( \mathbb{M} \) onto \( \mathbb{M}' \) satisfying

\[ q'(x_1 W, x_2 W, x_3 W, x_4 W) = \rho q(x_1, x_2, x_3, x_4), \quad x_i \in \mathbb{M}, \]

for a fixed \( 0 \neq \rho \in \Phi' \) is an \( s \)-semilinear map of \( \mathbb{M} \) onto \( \mathbb{M}' \) by

\[ \langle x W, y W' \rangle = \langle x, y W' \rangle, \quad x \in \mathbb{M}, y \in \mathbb{M}'. \]

If \( W \) satisfies (1.4), then \( \langle x_1 W, x_2 W, x_3 W \rangle W' = \rho^{-1} \langle x_1, x_2, x_3 \rangle \) for \( x_i \in \mathbb{M} \). By (T1) we have \( \langle x_1 W, x_2 W \rangle W' = \rho^{-1} \langle x_1, x_2, x_3 \rangle \). Hence, \( x W W' = \lambda^{-1} x \) where \( \lambda \) is given by \( \langle x_1, x_2, x_3 \rangle \). Now \( \lambda(x_1 W, x_2 W) = \rho^{-1} \langle x_1, x_2, x_3 \rangle \), \( x_i \in \mathbb{M} \). If \( \lambda(x_1, x_2, x_3) = \rho^{-1} \langle x_1, x_2, x_3 \rangle \), \( x_i \in \mathbb{M} \).

Conversely, if \( \lambda(x_1, x_2, x_3) = \rho \langle x_1, x_2, x_3 \rangle \), then (T1) yields

\[ \langle x_1, x_2 \rangle \rangle (x_3 W) = \lambda(x_1, x_2) \langle x_3 W \rangle \]

and

\[ \langle x_1 W, x_2 W \rangle = \lambda(x_1, x_2). \]

Clearly,

\[ q'(x_1 W, x_2 W, x_3 W, x_4 W) = (\langle x_1 W, x_2 W, x_3 W \rangle, x_4 W) \]

for \( x_i \in \mathbb{M} \) as desired.
It is now clear that if $W$ is an automorphism of $\mathfrak{M}$ (i.e. $\langle x_1 W, x_2 W, x_3 W \rangle = \langle x_1, x_2, x_3 \rangle$, $x_i \in \mathfrak{M}$), then $W \in S(\mathfrak{M})$. We denote the group of automorphisms by $\text{Aut}(\mathfrak{M})$.

We shall now exhibit some semisimilarities of $\mathfrak{M}$ to $\mathfrak{M}$. If $W$ is an $s$-semisimilarity of $\mathfrak{M}$ to $\mathfrak{M}'$ (with respect to $N$ and $N'$; see [3, p.10]) with multiplier $\lambda$, then we define $\bar{W}$ by

$$x\bar{W} = ax_1 + bx_2 + (ax_1 + bx_2)^2 + (ax_1 + bx_2)2,$$

where $\bar{W} = W^{* -1}$ and $T(x' W^*, y) = T(x', y W^*)$, $x' \in \mathfrak{M}'$, $y \in \mathfrak{M}$. An easy calculation using (1.22) and (1.23) of [3] shows

$$\langle x_1 \bar{W}, x_2 \bar{W}, x_3 \bar{W} \rangle' = \lambda \langle x_1, x_2, x_3 \rangle,$$

so $\bar{W}$ is an $s$-semisimilarity with multiplier $\lambda^2$ by Lemma 1.

If $\mathfrak{M} = \mathfrak{M}'$, we may define $e$ by

$$xe = \beta u_1 - au_2 - b_{12} + a_{21},$$

Clearly $e^2 = -1$ and one checks that $e$ is an automorphism of $\mathfrak{M}$.

If $c \in \mathfrak{M} = \mathfrak{M}'$, we may define $t_c$ by

$$xt_c = ax_1 + (\beta + T(b, c) + T(a, c))u_2 + (a + ac)_{12} + (b + a \times c + ac^2)_{21},$$

for $x$ as in (1.1). We shall show that $t_c \in \text{Aut}(\mathfrak{M})$, but first we introduce $S_c$ defined by

$$xS_c = T(b, c)u_2 + (ac)_{12} + (axc)_{21},$$

for $x$ as in (1.1). Clearly, $S_c^2 = 0$, and if $\Phi$ is not of characteristic two or three, then $t_c = \exp(S_c) = 1 + S_c + \frac{1}{2}S_c^2 + \frac{1}{3}S_c^3$. One checks that $\langle x, u_1, -c_{21} \rangle = xS_c$ and $\langle u_1, -c_{21} \rangle = 0$ for $x \in \mathfrak{M}$. Thus by [2, p. 404], we see that $S_c$ is an inner derivation of $\mathfrak{M}$. It is now clear that $t_c$ is an automorphism of $\mathfrak{M}$, if $\Phi$ is of characteristic zero.

A lengthy calculation would verify that $t_c$ is an automorphism for arbitrary fields. However, we shall be content to show this for $\mathfrak{M}$ a 27 dimensional exceptional simple Jordan algebra by the following trick. By extending $\Phi$, we may assume that $\mathfrak{M} = \mathfrak{S}(\mathfrak{O}_3)$ where $\mathfrak{O}$ is the split octonion (Cayley-Dickson) algebra. $\mathfrak{O}$ has a basis $x = e_i, e_{ij}, e_{i}l, (e_{ij})l, i = 1, 2$, with the involution given by $e_1 = e_2, j = -j, l = -l$, and multiplication given by $e_i^2 = e_i, j^2 = l^2 = 1 = e_1 + e_2, al = l\bar{a}, a(b)l = (ba)l, (al)b = (ab)l, (al)(bl) = ba$, for $a, b$ either $e_i$ or $e_{ij}, i = 1, 2$. $\mathfrak{M}$ has a basis $u_i, a_{ij}, i, j = 1, 2, i \neq j, a$ as above, and $i < j = 1, 2, 3$. $\mathfrak{M}$ has a basis $u_i, a_{ij}, i, j = 1, 2, i \neq j, a$ as above. Using (1.3), one sees that the multiplication table for $\mathfrak{M}$ relative to this basis is integral. The action of $t_c$ on $\mathfrak{M}$ given by (1.9) is also integral for $c$ belonging to the basis for $\mathfrak{M}$. For such a $c$, the automorphism condition for $t_c$ follows from the one in which $\mathfrak{O}$ is the split octonion algebra over the integers, which follows in turn from the one in which $\mathfrak{O}$
is the split octonion algebra over the reals. Thus, $t_c$ is an automorphism, if $c$ belongs to the basis for $\mathfrak{3}$. Easy calculations show

\begin{align}
(1.11) \quad & t_c t_d = t_{c+d} \quad \text{for } c, d \in \mathfrak{3}, \\
(1.12) \quad & t_c \bar{W} = \bar{W} t_c \quad \text{for } c \in \mathfrak{3}, \ W \in \Gamma(\mathfrak{3}),
\end{align}

where $\Gamma(\mathfrak{3})$ is the group of semisimilarities of $\mathfrak{3}$. Since $W$ may be taken to be $a_1$, $0 \neq a \in \Phi$, we see that $t_c, t_d \in \text{Aut}(\mathfrak{M})$ imply $t_{ac+bd} \in \text{Aut}(\mathfrak{M})$ for $a, b \in \Phi$. Thus, $t_c \in \text{Aut}(\mathfrak{M})$ for all $c \in \mathfrak{3}$.

We now list four identities which may be checked directly. The second is an analogue of Hua’s identity (see [5, p. 144]).

\begin{align}
(1.13) \quad & \varepsilon \bar{W} = (\bar{W})^{-1} \varepsilon, \quad \text{for } W \in \Gamma(\mathfrak{M}) \text{ with multiplier } \lambda.
\end{align}

\begin{align}
(1.14) \quad & \varepsilon t_c \varepsilon^{-1} t_c = -N(c)^{-1}(U_c)^{-1}, \quad \text{for } c \in \mathfrak{3} \text{ with } N(c) \neq 0.
\end{align}

2. The geometry and collineations. We denote by $\Pi(\mathfrak{M})$ the set of $0 \neq x \in \mathfrak{M}$, $x$ as in (1.1) with

\begin{align}
(2.1) \quad & a\# = ab, \quad b\# = \beta a, \quad N(a) = a^2\beta, \quad N(b) = a\beta^2, \\
& T(a, b) = 3a\beta, \quad V_{a,b} = 2a\beta.
\end{align}

We say $x \in \Pi(\mathfrak{M})$ is an element of rank one.

**Lemma 2.** If $x \in \Pi(\mathfrak{M})$ and $\varphi = t_c, \bar{W}, \varepsilon, \text{ or } \lambda_1$, where $c \in \mathfrak{3}$, $W \in \Gamma(\mathfrak{3})$, $0 \neq \lambda \in \Phi$, then $x^\varphi \in \Pi(\mathfrak{M})$.

**Proof.** We may assume that the field $\Phi$ is infinite. The set $S = \{x \in \mathfrak{M} \mid x \text{ as in (1.1) with } a \neq 0 \neq \beta \in \Phi, \ a \neq 0 \neq b \in \mathfrak{3}\}$ is open in the Zariski topology on $\mathfrak{M}$ and $\Pi \cup \{0\}$ is closed. Hence $\Pi^\Phi \cup \{0\}$ is also closed and $\Pi^\Phi \cap S$ is dense in $\Pi^\Phi \cup \{0\}$. Thus, we need only show $\Pi^\Phi \cap S \subseteq \Pi \cup \{0\}$. If $x \in S$ with $a\# = ab$ and $b\# = \beta a$, then $x \in \Pi$, since $u\# = N(u)u, T(u, u\#) = 3N(u), V_{u,u\#} = 2N(u)1$ for $u \neq x$. Thus, if $x' = a'_{12} + b'_{12} + b'_{21} = x^\varphi$ for $x \in \Pi$, we need only show $(a')\# = a'b', (b')\# = \beta a'$. These follow by direct calculation from the definitions and (1.1), (1.1a), (1.1b), (1.21), and (1.22) of [3].

If $\Gamma'(\mathfrak{M})$ (respectively, $G'(\mathfrak{M}); \text{Aut'}(\mathfrak{M})$) denotes the group generated by $t_c, \bar{W}, \varepsilon, \lambda_1$ where $c \in \mathfrak{3}, W \in \Gamma(\mathfrak{3}), 0 \neq \lambda \in \Phi$ (respectively, $W \in G(\mathfrak{3}); W \in S(\mathfrak{3}), \lambda = 1$), then $\Gamma'(\mathfrak{M}) \subseteq \Gamma(\mathfrak{M}), \ G'(\mathfrak{M}) \subseteq G(\mathfrak{M})$, and $\text{Aut'}(\mathfrak{M}) \subseteq \text{Aut}(\mathfrak{M})$ by (1.7). From now on, we shall assume $\mathfrak{3} = \mathfrak{J}(\mathfrak{3}, \gamma)$, a reduced exceptional simple Jordan algebra (see [6]).

**Lemma 3.** If $x \in \mathfrak{3}$, then the following are equivalent:

(a) $x$ is of rank one.
(b) $\langle \mathfrak{M}, x, x \rangle = 0$ and dim $\mathfrak{M}_x = \dim \mathfrak{M}_{u_1}$ where $\mathfrak{M}_x = \{y \mid \langle \mathfrak{M}, y, x \rangle = 0, \ y \in \mathfrak{M}\}$.
(c) $x^\varphi = au_1$ for some $\varphi \in \text{Aut'}(\mathfrak{M})$, $0 \neq a \in \Phi$.

**Proof.** Since $u_1$ is of rank one, we have (c) implies (a) by Lemma 2. Since $\langle \mathfrak{M}, u_1, u_1 \rangle = 0$ and $\text{Aut'}(\mathfrak{M}) \subseteq \text{Aut}(\mathfrak{M})$, we see that (c) implies (b).

We shall next show that if $0 \neq x \in \mathfrak{M}$ with $x$ as in (1.1), then after replacing $x$ by
for some $\varphi \in \text{Aut}'(\mathcal{M})$, we may assume $\alpha \neq 0$ and $a = 0$. First, we shall get $\alpha \neq 0$.
If $\alpha = 0$ and $\beta \neq 0$, apply $e$. If $\alpha = \beta = 0$, then one of $a$ or $b$ is nonzero, and by applying $e$, we may assume $b \neq 0$. Since the elements $0 \neq c \in \mathcal{Z}$ with $e^c = 0$ span $\mathcal{Z}$, we may find such a $c$ with $T(b, c) \neq 0$ and replace $x$ by $xt$ to get $\alpha \neq 0$. If $\alpha \neq 0$, replacing $x$ by $xt\cdot a^{-1}c$ allows us to assume $\alpha \neq 0$ and $a = 0$ as desired.

If (a) holds, we may normalize $x$ as above so $\alpha \neq 0$ and $a = 0$. Since $T(a, b) = 3a\beta$ and $V(a, b) = 2a\beta 1$, we see $\beta = 0$. Also, $a^* = ab$ implies $b = 0$, so $x = au_1$ and (c) holds.

If (b) holds, we again normalize $x$ so $\alpha \neq 0$ and $a = 0$. The condition $\langle y, x, x \rangle = 0$ for all $y = c_{12}$, $c \in \mathcal{Z}$, yields $(a\beta + T(a, b))1 = 2V_{b,a}$. Hence, $\beta = 0$.

If $y = \rho u_1 + \eta u_2 + r_{12} + s_{21}$, $\rho, \eta \in \Phi$; $r, s \in \mathcal{Z}$, then using (1.3) one checks that $y \in \mathcal{M}_1$ if and only if $\gamma x - T(r, b) = 0$, $s + pb = 0$, $s \times b = 0$, $V_{b,r} = 0$ (since $V_{b,r} = 0$ implies $V_{r,b} = 0$ and $2T(b, r) = T(1, b, r) = 0$; and $(s\beta) = T(s, c)b + T(c, b)s - (s \times b) \times c$, $c \in \mathcal{Z}$). Thus, $\mathcal{M}_{u_1} = \{\rho u_1 + r_{12} \mid \rho \in \Phi, r \in \mathcal{Z}\}$ and dim $\mathcal{M}_{u_1} = 28$. If $b \neq 0$, then $y \in \mathcal{M}_1$ implies $\gamma = \alpha^{-1}T(r, b)$, $s = -\alpha^{-1}pb$, where $V_{b,r} = 0$ and $2pb^\# = 0$. Since $y$ depends linearly on the choice of $\rho$ and $r$, and since dim $\mathcal{M}_{x} = 28$, all $r \in \mathcal{Z}$ are possible for $y$; but $V_{b,r} = 0$ for all $r \in \mathcal{Z}$ implies $b = 0$ (use (1.35) of [3]), a contradiction. Thus, $x = au_1$.

We are now in a position to define a "geometry" from $\mathcal{M}$. If $x \in \Pi(\mathcal{M})$, let $\hat{x} = \{ax \mid 0 \neq a \in \Phi\}$ and let $\mathcal{P}(\mathcal{M}) = \{x \mid x \in \Pi(\mathcal{M})\}$. Define $\hat{x}$ incident to $\hat{y}$ (denoted $\hat{x} \hat{y}$) if $R(x, y) = 0$, where $zR(u, v) = \langle z, u, v \rangle$, $u, v, z \in \mathcal{M}$; and define $\hat{x}$ connected to $\hat{y}$ (denoted $\hat{x} \hat{z} \hat{y}$) if $\langle x, y \rangle = 0$. Since $\langle x, y \rangle l = R(x, y) - R(y, x)$ and since $\langle uR(x, y), v \rangle + \langle u, vR(y, x) \rangle = 0$ (see (2.7) of [2]), we see that $\hat{x} \hat{y}$ implies $\hat{y} \hat{x}$ and $\hat{x} \hat{z} \hat{y}$.

If $\mathcal{Y} = \mathcal{E}(\mathcal{E}_3, \gamma')$, if $x \in \Pi(\mathcal{M})$, and if $W$ is a semisimilarity of $\mathcal{M}$ onto $\mathcal{M}' = \mathcal{M}((\mathcal{Y}))$, then $xW$ satisfies condition (b) of Lemma 3, so $xW \in \Pi(\mathcal{M}')$. Thus, we may define a map $\gamma W$ of $\mathcal{P}(\mathcal{M})$ onto $\mathcal{P}(\mathcal{M})$ by $xW = (xW)^\gamma$. It is clear that $W$ is a collineation in the sense $\hat{x} \hat{y}$ if and only if $\hat{x} \hat{y}'$ $\gamma W$ and $\hat{x} \hat{z} \hat{y}$ if and only if $\hat{x} \hat{z} \hat{y}'$ $\gamma W$. If $H$ is a subgroup of $\Gamma(\mathcal{M})$, then we denote the image of $H$ in the collineation group of $\mathcal{P}(\mathcal{M})$ under $W \rightarrow \gamma W$ by $PH$. The kernel of $W \rightarrow \gamma W$ in $\Gamma(\mathcal{M})$ is easily seen to be $\{a1 \mid 0 \neq a \in \Phi\}$.

One checks immediately from (1.3) that $\hat{x} \hat{u}_1$ if and only if $x = au_1 + a_{12} \in \Pi(\mathcal{M})$, $a \in \Phi$, $a \in \mathcal{Z}$. Hence, $\hat{x} \hat{u}_1$ and $\hat{x} \hat{u}_2$ if and only if $x = a_{12} + a \in \Pi(\mathcal{Z})$, where $\Pi(\mathcal{Z}) = \{0 \neq a \in \mathcal{Z} \mid a^\# = 0\}$. Similarly, $\hat{x} \hat{u}_2$ and $\hat{x} \hat{u}_1$ if and only if $x = b_{21}$ and $b \in \Pi(\mathcal{Z})$. If $a \in \Pi(\mathcal{Z})$, we shall set $a_* = (a_{12})^\gamma$ and $a^* = (a_{21})^\gamma$. Using (1.1) and (1.2), we see $a_* \simeq b_*$ and $a^* \simeq b^*$ always hold, $a_* \simeq b_*$ holds if and only if $T(a, b) = 0$; each of $a_* | b_*$ and $a^* | b^*$ hold if and only if $a \times b = 0$; while $a_\# | b_\#$ if and only if $V_{b,a} = 0$ (since $V_{b,a} = 0$ implies $V_{a,b} = 0$ and $T(a, b) = 0$), for $a, b \in \Pi(\mathcal{Z})$. Hence, $\{a_\#, b_\# a, b \in \Pi(\mathcal{Z})\}$ may be identified with $\mathcal{P}(\mathcal{Z})$ as defined in [3, p. 32] (with a slight change of notation). Moreover, if $W \in \Gamma(\mathcal{Z})$ so $W \in \Gamma(\mathcal{M})$, we see that if we abuse notation and set $\gamma W = (aW)_*$ and $a^* \gamma W = (aW)^\gamma$ for $a \in \Pi(\mathcal{Z})$, which agrees with the action of $\Gamma(\mathcal{Z})$ on $\mathcal{P}(\mathcal{Z})$ given in [3, p. 32].
Lemma 4. \( P \text{ Aut'}(\mathcal{M}) \) is transitive on
(a) \( x \in \mathcal{P}(\mathcal{M}) \).
(b) \( x, y \in \mathcal{P}(\mathcal{M}) \) with \( x \sim y \).
(c) \( x, y \in \mathcal{P}(\mathcal{M}) \) with \( x \sim y, x \downarrow y \).
(d) \( x, y \in \mathcal{P}(\mathcal{M}) \) with \( x \downarrow y, x \neq y \).

Proof. Lemma 3 yields (a). In the remaining cases we may assume \( y = u_2 \) and \( x \) is as in (1.1). In case (b), \( \langle x, y \rangle = 0 \) implies \( x \neq 0 \), and we may assume \( x = 1 \). The condition \( x \in \Pi(\mathcal{M}) \) yields \( a^* = b \) and \( N(a) = \beta \). Hence, \( x = u_1 t_a \). Since \( u_2 t_a = u_2 \), we are done in this case. In case (c), \( x = 0 \) and \( a \neq 0 \). If \( \beta \neq 0 \), then \( b^* = \beta a \) implies \( b \neq 0 \). One may choose \( c \in \Pi(\mathcal{M}) \) with \( T(b, c) = -\beta \). Replacing \( x \) by \( x t_c \), we may assume \( \alpha = 0 \) and \( a = 0 \). If \( b \neq 0 \), then \( a, b \in \Pi(\mathcal{M}) \) and \( a^* b^* \) since \( V_{a, b} = 0 \). By [3, Lemmas 3.6 and 3.3] we know that there exists \( c^* b^* \) with \( c^* a^* \), and that we may choose \( c \) such that \( x = c = -b \). Replacing \( x \) by \( x t_c \), we may assume \( \alpha = 0 \) and \( a = 0 \). If \( b \neq 0 \) (since \( V_{a, b} = 0 \) implies \( T(b, c) = 0 \)). Since \( PS(\mathcal{M}) \) is transitive on points of \( \mathcal{P}(\mathcal{M}) \), we may choose \( W \in S(\mathcal{M}) \) such that \( x^* W^* = a^* W^* = e^* \), where \( e \in \Pi(\mathcal{M}) \) is fixed. Since \( u_2 W = u_2 \), we are done in this case. In case (d), we have \( x = 0 \) and \( a = 0 \). Since \( \dot{y} \neq x \), we see \( b \neq 0 \). We may choose \( c \in \Pi(\mathcal{M}) \) with \( T(b, c) = -\beta \) and replace \( y \) by \( y t_c \) to assume \( \alpha = 0 = a, b \neq 0 \). Since \( PS(\mathcal{M}) \) is transitive on points of \( \mathcal{P}(\mathcal{M}) \), we may choose \( W \in S(\mathcal{M}) \) with \( \dot{y}^* W^* = b^* W^* = e^* \), where \( e \in \Pi(\mathcal{M}) \) is fixed. This completes the proof of the lemma.

We shall need the following result about \( g(\mathcal{S}, \gamma) \).

Lemma 5. If \( W \) is an \( s \)-semilinear map of \( g = g(\mathcal{S}, \gamma) \) to itself such that for \( a \in \Pi(\mathcal{S}) \) there is \( 0 \neq \lambda_a \in \Phi \) with \( a W = \lambda_a a \), then \( W = \lambda 1 \) for some \( 0 \neq \lambda \in \Phi \).

Proof. We may assume \( x = 1(\mathcal{S}) \) and \( \lambda e_1 = 1 \). If \( u \in \mathcal{S} \) and \( 0 \neq \mu \in \Phi \), then \( x = \mu e_1 + \mu^{-1} n(u)e_2 + u[12] \in \Pi(\mathcal{S}) \). Since \( e_1 W = e_1 \) and \( e_2 W = e_2 e_2 \), we see that \( u[12] W = f_1 e_1 + f_2 e_2 + \lambda u[12] \) where \( 0 \neq \lambda \neq \lambda_x \) is independent of \( u \) and
\[
\lambda_\mu = \mu^s + f_1 \quad \text{for } 0 \neq \mu \in \Phi,
\]
(2.2)
\[
\lambda_\mu^{-1} n(u) = (\mu^{-1} n(u))^s \lambda e_2 + f_2 \quad \text{for } 0 \neq u \in \Phi.
\]
If \( \Phi \) has two elements, then \( \lambda_x = 1 \) for all \( a \in \Pi(\mathcal{S}) \). Since \( \Pi(\mathcal{S}) \) generates \( \mathcal{S} \) under addition, \( W = 1 \). If \( 0 \neq \mu_1, \mu_2 \in \Phi \), with \( \mu_1 \neq \mu_2 \), then (2.2), with \( \mu = \mu_1, \mu_2, \mu_1 - \mu_2 \), gives \( f_1 = 0 \). Thus, \( \mu = 1 \) yields \( s = 1 \) and hence \( s = 1 \). Similarly, (2.3) yields \( f_2 = 0 \) and \( \lambda e_2 = 1 \). Thus, \( W \) is linear and \( e_2 W = e_2, u[12] W = u[12] \) for \( u \in \mathcal{S} \). Similarly, \( e_i W = e_i, u[ij] W = u[ij] \), \( u \in \mathcal{S}, i 
eq j = 1, 2, 3 \) and \( W = 1 \).

Theorem 1. If \( \mathcal{S} = g(\mathcal{S}, \gamma) \) and \( \mathcal{M} = \mathcal{M}(\mathcal{S}) \), then \( \Gamma(M) \) (respectively, \( G(\mathcal{M}) \); \( \text{Aut}(\mathcal{M}) \)) is generated by \( t_c, W, e, \lambda 1 \) where \( c \in \mathcal{S}, W \in \Gamma(\mathcal{S}), 0 \neq \lambda \in \Phi \) (respectively, \( W \in G(\mathcal{S}); W \in S(\mathcal{S}), \lambda = 1 \)).

Proof. Using the notation preceding Lemma 3, we need only show that \( \Gamma(M) \) \( \subseteq \Gamma(M) \), \( G(M) \subseteq G(M) \), and \( \text{Aut}(M) \subseteq \text{Aut}'(M) \). If \( W \in \Gamma(M) \), then \( W^\gamma \) is a collineation of \( \mathcal{P}(\mathcal{M}) \), and by Lemma 4(b) there is \( W_2 \in \text{Aut}'(M) \) with \( W_2^\gamma \)
1972

A GEOMETRY FOR E₇

55

Since \( rW W^{-1} \) induces a collineation of \( \mathcal{P}(3) \), we may apply the fundamental theorem of octonion planes (see [3, p. 40]) to find \( W_3 \in \Gamma(3) \) such that \( rW W^{-1} \) agrees with \( rW W^{-1} \) on \( \mathcal{P}(3) \). Since \( u_1 W W^{-1} W^{-1} = u_1 \) and, by Lemma 5, \( a_1 W W^{-1} W^{-1} = (\eta a)_{12} \), for some \( 0 \neq \lambda, \eta \in \Phi \), we may set \( W_1 = \lambda^{-1} \eta W_3 \) to get \( W' = W W^{-1} W^{-1} \lambda^{-1} \) satisfying \( xW' = u_1 + \rho \beta u_2 + a_1 + (\mu b)_{21} \) for some fixed \( 0 \neq \rho, \mu \in \Phi, x \) as in (1.1). (Note: \( W' \) is linear on \( 3_{12} \) and hence on all of \( \mathcal{W} \).) If \( a = \beta = 1 \) and \( a = b = 1 \), then \( xW' \in \Pi(\mathcal{W}) \) which implies \( \rho = \mu = 1 \) by (2.1). Hence, \( W' = 1 \) and \( W = \lambda W W_2, 0 \neq \lambda \in \Phi, W_2 \in \text{Aut}'(\mathcal{W}) \). Clearly, \( \Gamma(\mathcal{W}) \subseteq \Gamma'(\mathcal{W}) \).

If \( W \in G(\mathcal{W}) \), then \( W_1 \) must be linear so \( G(\mathcal{W}) \subseteq G'(\mathcal{W}) \). If \( W \in \text{Aut}(\mathcal{W}) \), then \( \lambda W_1 \in \text{Aut}(\mathcal{W}) \). If \( W_1 \in G(\mathcal{W}) \) has multiplier \( \rho \), then \( 1 = \langle \lambda u_1 W_1, \lambda u_2 W_1 \rangle = \lambda^2 \rho \). Set \( c = (\lambda - 1)e + 1 \in \mathfrak{I} \) where \( e \in \mathfrak{I} \) is a primitive idempotent. Since \( c^{\#} = \lambda c \), we see that \( N(c) = \lambda \) and \( U_c \in G(\mathcal{W}) \) with multiplier \( \lambda^2 = \rho^{-1} \). By (1.14), we see that \( \lambda^{-1}(U_c) \in \text{Aut}'(\mathcal{W}) \). But \( \lambda W_1 \lambda^{-1}(U_c) = (W_1 U_c) \in \text{Aut}'(\mathcal{W}) \) since \( W_1 U_c \in S(3) \). Thus, \( \lambda W_1 \in \text{Aut}'(\mathcal{W}) \) and \( \text{Aut}(\mathcal{W}) \subseteq \text{Aut}'(\mathcal{W}) \).

We shall need the following result on the plane \( \mathcal{P}(3) \).

**Lemma 6.** If \( x_\ast, y_\ast \in \mathcal{P}(\mathcal{W}) \) and \( x_\ast \sim z_\ast \) if and only if \( y_\ast \sim z_\ast \), then \( x_\ast = y_\ast \).

**Proof.** Since \( P(\mathcal{W}) \) is transitive on points of \( \mathcal{P}(3) \), we may assume \( x = e_1 \), where \( e_1, e_2, e_3 \) are pairwise orthogonal primitive idempotents for \( \mathfrak{I} \). If \( \mathfrak{I} \) is split, then there is a basis for \( \mathfrak{I} \) of elements of rank one of the form \( z = e_{i j}, a_i e + b_i, w(a_i) = 0 \). The condition \( T(y, z) = 0 \) if and only if \( T(e_1, z) = 0 \) yields \( y \in \Phi e_1 \), as desired. If \( \mathfrak{I} \) is not split, then \( \mathcal{P}(3) \) is a projective plane and \( u_\ast \sim v_\ast \) if and only if \( u_\ast | v_\ast \) (see [3, p. 50]). Thus, \( y_\ast | e_2^\# \) and \( y_\ast | e_3^\# \) implies \( y_\ast = e_1^\ast \).

We shall eventually show that every collineation of \( \mathcal{P}(\mathcal{W}) \) is in \( P(\mathcal{W}) \), but first we must demonstrate the following two characterizations of the identity collineation.

**Lemma 7.** If \( \sigma \) is a collineation of \( \mathcal{P}(\mathcal{W}) \) such that \( \sigma \) fixes \( u_\ast \) and all points incident to \( u_\ast \), then \( \sigma \) is the identity.

**Proof.** We have \( (a u_1 + a_{12})^\# \) and \( u_\ast \) fixed by \( \sigma \), for \( a \in \Phi, a \in \Pi(3) \). Since by (1.3) \( \tilde{u}_1 \) is the unique point of \( \mathcal{P}(\mathcal{W}) \) incident to all \( a_\ast, a \in \Pi(3), \tilde{u}_1 = \tilde{u}_1 \). Since \( \sigma \) stabilizes \( \mathcal{P}(3) \) and fixes the points \( a_\ast, a \in \Pi(3), \sigma \) also fixes \( a_\ast \). Let \( y = y u_1 + y u_2 + c_{12} + d_{21} \in \Pi(\mathcal{W}) \) and let \( y' = y' u_1 + y' u_2 + c'_{12} + d'_{21} \). If \( \delta \neq 0 \), then \( y' \sim u_1 \) implies \( \delta' = 0 \). In this case, \( d, d' \in \Pi(3) \). The condition \( y \sim a_\ast \) if and only if \( y' \sim a_\ast \) implies \( d' = \lambda d \) for some \( 0 \neq \lambda \in \Phi \) by Lemma 6. If \( \delta = 0 \), then \( \delta' \neq 0 \), and we may assume \( \delta = \delta' = 1 \). Then \( y' \sim (a u_1 + a_{12})^\# \) if and only if \( y' \sim (a u_1 + a_{12})^\# \). Using the above argument, for some \( 0 \neq \rho \in \Phi \), one sees \( y = (y u_1 + \rho y u_1 + c_{12} + d_{21})^\# \). Since \( c^\# = \gamma d = \rho y d \) and since \( c = 0 \) implies \( \gamma = 0 \) or \( y = y u_1 \) (and \( \sigma \) fixes \( y \) in either case), we may assume \( c \in \Pi(3) \). Similarly, we may also assume \( d \in \Pi(3) \) so \( \gamma = \delta = 0 \) and \( c_\ast | d_\ast \). If \( b \in \mathfrak{I} \) is such that \( T(d, b) = T(c, b^\#) \)
\( \neq 0 \), then \( \hat{y} \hat{z} = \hat{w} \) implies \( \hat{y}^* \approx \hat{w}^* = \hat{w} \) where \( w = u_1 + N(b)u_2 + b_{12} + (b^\#)_{21} \). Thus, 
\[ \rho T(d, b) = T(c, b^\#) \] and \( \rho = 1 \). To show such a b exists, we choose \( c_2 \approx d^* \) and \( c_3 \approx c_3 \) such that \( c = c_1, c_2, c_3 \) form a three-point (see [3, p. 33]). If \( b = c_1 + c_2 + c_3 \), then 
\[ T(c_1, d) = 0, \ i = 1, 2, \text{ and } T(c_3, d) \neq 0 \] since \( d^* = (c_1 \times c_3)^* \). Thus, \( T(b, d) \neq 0 \). Also, 
\[ T(c_1 \times c_1, c_1) = 0, \ i = 2, 3, \text{ and } T(c_2 \times c_3, c_1) \neq 0 \], so \( T(b^\#, c) \neq 0 \). Replacing \( b \) by 
\[ T(b, d)T(b^\#, c)^{-1}b \], we get \( b \) as desired.

**Lemma 8.** If \( \sigma \) is a collineation of \( \mathcal{P}(\mathbb{R}) \) fixing \( a_\# \), \( a \in \Pi(3) \), and \( (u_1 + e_{12})^{-1} \) for some \( e \in \Pi(3) \), then \( \sigma \) is the identity.

**Proof.** As in the proof of Lemma 7, we see that \( \hat{u}_1 \) is fixed by \( \sigma \). The condition, 
\[ \hat{u}_2 \hat{z} \hat{u}_1 \] implies 
\[ \hat{u}_2^* = (au_1 + u_2 + a_{12} + b_{21})^{-1} \] for some \( a \in \Phi \); \( a, b \in \Sigma \). Since \( \hat{u}_2^* \approx c_\#, \ c \in \Pi(3) \), we see \( b = 0 \) and \( a = 0 \), \( a = 0 \), by (2.1). Thus, \( \hat{u}_2^* = \hat{u}_2 \) and \( a^* = a^* \) for \( a \in \Pi(3) \), since \( \sigma \) stabilizes \( \mathcal{P}(\mathbb{R}) \) and fixes its points. Since \( \hat{x}^\# \approx \hat{u}_1 \) and \( \hat{x}^\# \approx a^* \) if and only if \( \hat{x} \approx a^* \), \( a \in \Pi(3) \), for \( x = u_1 + c_{12}, \ c \in \Pi(3) \), we see that \( \hat{x}^\# = (u_1 + \rho(c)c_{12})^* \) for some \( 0 \neq \rho(c) \in \Phi \). Similarly, \( (u_2 + d_{21})^{-\#} = (u_2 + \lambda(d)d_{21})^{-1} \) for \( d \in \Pi(3) \), \( 0 \neq \lambda(d) \in \Phi \). Since there is a norm similarity \( W \) of \( \mathcal{P}' = \mathcal{P}(\mathbb{R}) \) onto \( \mathcal{P}' = \mathcal{P}(\mathbb{R}) \), \( (u_2 + d_{21})^{-\#} = (u_2 + \lambda(d)d_{21})^{-1} \) for \( d \in \Pi(3) \), \( 0 \neq \lambda(d) \in \Phi \).

**Theorem 2.** If \( \mathcal{Z} = \mathcal{Z}(\mathcal{O}, \gamma) \) and \( \mathcal{Z}' = \mathcal{Z}(\mathcal{O}', \gamma) \), then \( \sigma \) is a collineation of \( \mathcal{P}(\mathbb{R}(\mathcal{Z})) \) onto \( \mathcal{P}(\mathbb{R}(\mathcal{Z}')) \) if and only if \( \sigma = \mathcal{W}^{-1} \) for some semisimilarity \( W \) of \( \mathbb{R}(\mathcal{Z}) \) onto \( \mathbb{R}(\mathcal{Z}') \).

**Proof.** By Lemma 4(b), we may assume that \( \hat{u}_i^\# = \hat{u}_i, \ i = 1, 2 \). Thus, \( \sigma \) induces a collineation of \( \mathcal{P}(\mathcal{Z}) \) onto \( \mathcal{P}(\mathcal{Z}') \). By the fundamental theorem of octonion planes (see [3, p. 40]), \( \sigma \) agrees with \( \mathcal{W}^{-1} \) on \( \mathcal{P}(\mathcal{Z}) \) for some semisimilarity \( W \) of \( \mathcal{Z} \) onto \( \mathcal{Z}' \). Replacing \( \sigma \) by \( \sigma^* \mathcal{W}^{-1} \), we may assume \( \mathcal{Z} = \mathcal{Z}', \ \hat{u}_i^\# = \hat{u}_i, \ i = 1, 2, \ a_\#^* = a_\#, \ a^* = a^* \), \( a \in \Pi(3) \). If \( e \in \Pi(3) \), then \( (u_1 + e_{12})^* \hat{u}_1 \) and \( (u_1 + e_{12})^* \approx a^* \) if and only if \( (u_1 + e_{12})^* \approx a^* \) imply by Lemma 6 that \( (u_1 + e_{12})^* = (u_1 + \rho e_{12})^* \) for some \( 0 \neq \rho \in \Phi \). Replacing \( \sigma \) by \( \sigma^* (\rho^{-1})^{-1} \), we may assume that \( \sigma \) fixes \( a_\#, a \in \Pi(3) \) and \( (u_1 + e_{12})^* \). By Lemma 8, \( \sigma \) is the identity.
3. Simplicity of \( P \text{ Aut} (\mathfrak{M}) \). The purpose of this section is to prove the following

**Theorem 3.** \( P \text{ Aut} (\mathfrak{M}) \) is a simple group.

We shall first establish some facts about \( P \text{ Aut} (\mathfrak{M}) \).

**Lemma 9.** \( P \text{ Aut} (\mathfrak{M}) \) is a primitive permutation group of \( \mathfrak{P}(\mathfrak{M}) \).

**Proof.** By Lemma 4(a), \( P \text{ Aut} (\mathfrak{M}) \) is transitive. Suppose \( M_1, M_2, \ldots \) is a system of imprimitivity for \( P \text{ Aut} (\mathfrak{M}) \) with \( u_1 \in M_1 \). If \( \tilde{x} \neq x \in M_1 \), then either \( \tilde{x} \neq x \) or \( \tilde{x} = x \). By Lemma 4, we may assume \( \tilde{x} = x \). We shall first establish some facts about \( P \text{ Aut} (\mathfrak{M}) \).

**Lemma 10.** \( \Pi(\mathfrak{M}) \) is a normal abelian subgroup of the subgroup \( H \) of \( P \text{ Aut} (\mathfrak{M}) \) fixing \( u_2 \).

**Proof.** \( \Pi(\mathfrak{M}) \) is abelian by (1.11) and fixes \( u_2 \) by (1.9). If \( \sigma \in H \), write \( \sigma = (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) + \sigma_5 \) for \( \sigma \in \Pi(\mathfrak{M}) \). Replacing \( \sigma \) by \( \sigma' \), we may assume \( \tilde{x} = x \). By the proof of Theorem 2, we have \( \sigma \in \mathfrak{F}(\mathfrak{M}) \). We see \( \sigma^{-1} \tau_{c, e} \sigma = \tau c^{-1} \mathfrak{F} c^{-1} \tau_{c', e} \sigma \) implies \( \sigma \in H \). If \( \sigma = e \), then \( \sigma = e \) implies \( \tilde{x} = x \). We shall first establish some facts about \( P \text{ Aut} (\mathfrak{M}) \).

**Lemma 11.** \( \text{Aut} (\mathfrak{M}) \) is generated by conjugates of \( \Pi(\mathfrak{M}) \) in \( \text{Aut} (\mathfrak{M}) \).

**Proof.** If \( G \) is the group generated by conjugates of \( \Pi(\mathfrak{M}) \) in \( \text{Aut} (\mathfrak{M}) \), then by (1.14) with \( c = 1 \), we see \( e = - (t_1 t_2 t_3)^{-1} = (t_1 e t_1 e t_3)^{-1} \in G \). Since \( v_c \in G \) for \( c \in \mathfrak{M} \), (3.2) shows that \( (t_{u(12), e})^c \in G \). Theorem 4.7 of [3] implies that conjugates of \( T(\mathfrak{M}) \) generate \( S(\mathfrak{M}) \). By Theorem 1, we see \( G = \text{Aut} (\mathfrak{M}) \).

**Lemma 12.** \( P \text{ Aut} (\mathfrak{M}) = \mathfrak{F}(\text{Aut} (\mathfrak{M})) \), the derived group.

**Proof.** By Lemma 11, we need only show \( \gamma T_{a, e} \gamma = \mathfrak{F}(\text{Aut} (\mathfrak{M})) \), \( a \in \mathfrak{M} \). By (1.11), (1.12), the fact that \( \Pi(\mathfrak{M}) \) spans \( \mathfrak{M} \) and the transitivity of \( \Pi(\mathfrak{M}) \) on points of \( \mathfrak{M}(\mathfrak{M}) \), we need only show \( t_{u(12), e} \in \mathfrak{F}(\text{Aut} (\mathfrak{M})) \) by Lemma 4.6 of [3], we see by (3.2) that \( v_{u(12), e} \in \mathfrak{F}(\text{Aut} (\mathfrak{M})) \), \( u \in \mathfrak{M} \). Hence, \( t_{u(12), e} \in \mathfrak{F}(\text{Aut} (\mathfrak{M})) \).
If $\alpha \in \Phi$, there exist $u, v \in \mathbb{O}$ with $n(u, v) - n(u + v) - n(u) - n(v) = \alpha$. Thus, by (1.11), $t_{u+v} \in D$, as desired.

**Proof of Theorem 3.** This follows immediately from Lemmas 9, 10, 11, 12 and Lemma 4, p. 39 of [1].

**REFERENCES**


**Department of Mathematics, University of Virginia, Charlottesville, Virginia 22904**