Abstract. Let $A$ be a Banach $*$-algebra. $A$ is locally $B^*$-equivalent if, for every selfadjoint element $t \in A$, the closed $*$-subalgebra of $A$ generated by $t$ is $*$-isomorphic to a $B^*$-algebra. In this paper it is shown that when $A$ is locally $B^*$-equivalent, and in addition every selfadjoint element in $A$ has at most countable spectrum, then $A$ is $*$-isomorphic to a $B^*$-algebra.

1. Introduction. Assume that $A$ is a Banach $*$-algebra. We say that $A$ is $B^*$-equivalent if there exists a $*$-isomorphism of $A$ onto a $B^*$-algebra. When $t \in A$ is selfadjoint, $C(t)$ denotes the closed $*$-subalgebra of $A$ generated by $t$. We say that $A$ is locally $B^*$-equivalent if $C(t)$ is $B^*$-equivalent for every selfadjoint element $t$ in $A$. It does not seem unlikely that every locally $B^*$-equivalent Banach $*$-algebra $A$ is $B^*$-equivalent. This is true when $A$ is commutative; see Proposition 2.2. In this paper we prove this for certain noncommutative algebras. Specifically we prove (Theorem 4.1) that when $A$ is a Banach $*$-algebra which is locally $B^*$-equivalent, and in addition the property that every selfadjoint element has at most a countable spectrum, then $A$ is $B^*$-equivalent.

This paper is motivated in part by some theorems of Y. Katznelson which hold for commutative Banach algebras. Let $A$ be a semisimple Banach $*$-algebra with identity and hermitian involution. For the present assume that $A$ is commutative. When $a \in A$, we denote the Gelfand transform of $a$ by $\hat{a}$. Given a complex function $\phi$ defined on a subset $\mathcal{D}$ of the complex plane, we say $\phi$ operates on $A$ if $\phi \circ \hat{a} \in \hat{A}$ whenever the range of $\hat{a}$ is in $\mathcal{D}$. When $A$ is noncommutative, we say $\phi$ operates on $A$ if $\phi$ operates on $C(t)$ for every selfadjoint element $t$ in $A$. Let $\sqrt{\cdot}$ denote the positive square root function, with domain $\mathcal{D}$ the nonnegative reals. Katznelson proves in [5] that if $A$ is commutative and $\sqrt{\cdot}$ operates on $A$, then $A$ is $B^*$-equivalent. Therefore when $A$ is noncommutative and $\sqrt{\cdot}$ operates on $A$, then $A$ is locally $B^*$-equivalent. If this implies that $A$ is $B^*$-equivalent, then Katznelson's Theorem extends to the noncommutative algebra $A$. In fact the question of whether Katznelson's Theorem holds for noncommutative algebras is equivalent to the question of whether local $B^*$-equivalence implies $B^*$-equivalence.

2. Locally $B^*$-equivalent algebras. In this section we establish those properties of locally $B^*$-equivalent algebras which we need in the subsequent sections.
Throughout this section $A$ is a Banach $*$-algebra. When $A$ has no identity, let $A_1$ denote the usual Banach $*$-algebra formed by adjoining an identity to $A$. Then it is easy to verify that $A$ is $B^*$-equivalent (locally $B^*$-equivalent) if and only if $A_1$ is $B^*$-equivalent (locally $B^*$-equivalent). This means that to show that a particular locally $B^*$-equivalent algebra $A$ is $B^*$-equivalent, we may assume without loss of generality that $A$ has an identity.

Now we establish some basic properties of locally $B^*$-equivalent algebras.

**Proposition 2.1.** Assume that $A$ is a Banach $*$-algebra which is locally $B^*$-equivalent. Then

1. $A$ is semisimple.
2. $*$ is continuous on $A$.
3. $*$ is symmetric on $A$.
4. There is a unique norm $|\cdot|$ on $A$ which has the $B^*$-property, $|a*a| = |a|^2$ for all $a \in A$.

**Proof.** The radical $R$ of $A$ is a closed $*$-ideal of $A$. If $t \in R$ and $t = t^*$, then $C(t) \subseteq R$. Since $C(t)$ is $B^*$-equivalent and $t$ has zero spectrum, then $t = 0$. Therefore $R = 0$. This proves (1). (2) follows from (1) by Johnson's Theorem [3, Theorem 2, p. 539].

Given $t \in A$, $t = t^*$, then the spectrum of $t$ in $C(t)$ is real by [6, Lemma (4.8.1)(i), p. 240]. Then the spectrum of $t$ in $A$ is real. Therefore $*$ is a hermitian involution on $A$. By Shirali's Theorem [2], $*$ is symmetric on $A$. Therefore there exists a norm $|\cdot|$ on $A$ with the $B^*$-property, $|a*a| = |a|^2$ for all $a \in A$, by [6, Corollary (4.7.16), p. 237]. Assume that $|\cdot|_1$ is another norm on $A$ with the $B^*$-property. Given $a \in A$, $C(a*a)$ is a $B^*$-algebra in some norm. Then by [6, Corollary (4.8.6), p. 241], $|a*a| = |a|^2$. Therefore $|a| = |a|_1$. This completes the proof of the proposition.

In the next proposition we prove that commutative locally $B^*$-equivalent algebras are $B^*$-equivalent. Clearly any closed $*$-subalgebra of a locally $B^*$-equivalent algebra is locally $B^*$-equivalent. Therefore the proposition implies that every maximal commutative $*$-subalgebra of a locally $B^*$-equivalent algebra is $B^*$-equivalent.

**Proposition 2.2.** Assume that $A$ is a commutative Banach $*$-algebra which is locally $B^*$-equivalent. Then $A$ is $B^*$-equivalent.

**Proof.** We may assume that $A$ has an identity. Y. Katznelson has shown that when every continuous complex function operates on a commutative semisimple Banach algebra $B$, then $B$ is $B^*$-equivalent [4, Theorem 2]. When $B$ is a commutative semisimple Banach $*$-algebra with hermitian involution, then Katznelson's proof establishes that $B$ is $B^*$-equivalent if every continuous real function of a real variable operates on $B$. Now assume that $\phi$ is a continuous real function with real domain $\mathbb{D}$. If $t \in A$ and $t$ has range in $\mathbb{D}$, then $t = t^*$ by (1) and (3) of Proposition 2.1. Then $C(t)^*$ is a sup norm complete $*$-subalgebra of $A$. It follows that $\phi \circ t \in \tilde{A}$ by [6, Theorem (4.8.7), p. 241]. Therefore $A$ is $B^*$-equivalent by Katznelson's Theorem.
The next result shows that the property of local $B^*$-equivalence is preserved under continuous $*$-homomorphisms. If $B$ is any $*$-algebra, we denote by $B_t$ the set of selfadjoint elements of $B$.

**Proposition 2.3.** Assume that $A$ is locally $B^*$-equivalent and that $I$ is a closed $*$-ideal of $A$. Then $A/I$ is locally $B^*$-equivalent.

**Proof.** Assume that $R \in A/I$ and $R = R^*$. $R = s + i$ for some $s \in A$. Set $t = (s + s^*)/2$. Since $s - s^* \in I$, then $s - t = (s - s^*)/2 \in I$. Therefore $R = t + I$ where $t \in A_s$. Now set $B = C(t) + I$. Define a map $\phi: C(t)/(I \cap C(t)) \to B/I$ by

$$\phi(a + I \cap C(t)) = a + I$$

for $a \in C(t)$. $\phi$ is a $*$-isomorphism of $C(t)/I \cap C(t)$ onto $B/I$. It follows that $B/I$ is $B^*$-equivalent. Then $R \in B/I$ which is a closed $*$-subalgebra of $A/I$. Therefore $C(R) \subseteq B/I$, and since $B/I$ is $B^*$-equivalent, then $C(R)$ is $B^*$-equivalent. It follows that $A/I$ is locally $B^*$-equivalent.

Next we prove a sequence of three lemmas. These lemmas are the basic ingredients in the proof of our main result, Theorem 4.1.

**Lemma 2.4.** Assume that $A$ is locally $B^*$-equivalent. Assume that $B$ is a $*$-subalgebra of $A$, and that $I$ is a closed $*$-ideal of $A$ such that $I \subseteq B$. Then if $I$ is $B^*$-equivalent and $B/I$ is $B^*$-equivalent, we have $B$ is $B^*$-equivalent.

**Proof.** Let $| \cdot |$ be the unique norm on $A$ with the $B^*$-property (Proposition 2.1(4)). We prove that $| \cdot |$ is a complete norm on $B$. Let $B^c$ be the completion of $B$ with respect to $| \cdot |$. $I$ is complete in the norm $| \cdot |$ by hypothesis, so that $I$ is a closed $*$-ideal of $B^c$. Define the usual quotient norm $|a + I| = \inf_{b \in I} |a - b|$ on $B^c/I$. By [6, Theorem (4.9.2), p. 249], $| \cdot |'$ is a norm with the $B^*$-property on $B^c/I$. Since $B/I$ is a $B^*$-algebra in some norm, $B/I$ is complete in the norm $| \cdot |'$ by [6, Corollary (4.8.6), p. 241]. Assume now that $\{b_n\} \subseteq B$ and $|b_n - b_m| \to 0$. Then $|(b_n - b_m) + I|' \to 0$. Therefore there exists $b \in B$ such that $|(b_n - b) + I|' \to 0$. Then we can choose $\{a_n\} \subseteq I$ such that $|(b_n - b) - a_n| \to 0$. Then $|a_n - a_m| \to 0$, and since $| \cdot |$ is complete on $I$, there exists $a \in I$ such that $|a_n - a| \to 0$. Finally $|b_n - (b + a)| \to 0$, so that $| \cdot |$ is complete on $B$.

We will always denote the given norm in $A$ by $\| \cdot \|$, and $\nu(a)$ is the spectral radius of an element $a$ in $A$, $\nu(a) = \inf_n \|a^n\|^{1/n}$. When $E$ is a subset of $A$, $\bar{E}$ is the closure of $E$ in the norm $\| \cdot \|$.

**Lemma 2.5.** Assume that $A$ is locally $B^*$-equivalent. Assume that $D$ is a $*$-subalgebra of $A$, and that there exists $K > 0$ such that $K\nu(t) \geq \|t\|$ for all $t \in D_s$. Then $\bar{D}$ is $B^*$-equivalent.

**Proof.** $*$ is symmetric on $A$ by Proposition 2.1(3). It follows by [6, Lemma 4.7.10, p. 234] that $\nu(h + k) \leq \nu(h) + \nu(k)$ whenever $h, k \in A_s$. Given $h \in (\bar{D})_s$ and
$\epsilon > 0$, we can choose $k \in D_s$ such that $\|h - k\| < \epsilon$. Then
\[
\|h\| \leq \|h - k\| + \|k\| \leq \epsilon + K\nu(k) \\
\leq \epsilon + K(\nu(h) + \nu(k - h)) \leq (1 + K)\epsilon + K\nu(h).
\]
Since $\epsilon$ was an arbitrary positive number, then $\|h\| \leq K\nu(h)$.
It follows that $\bar{B}$ is $B^*$-equivalent by Theorem 2.4 and Lemma 2.6 of [7].

Given $a, b \in A$, let $a \circ b = a + b - ab$. When $a$ and $b$ are commuting selfadjoint idempotents, then $a \circ b$ is a selfadjoint idempotent which in some sense is the least upper bound of $a$ and $b$.

**Lemma 2.6.** Assume that $A$ is locally $B^*$-equivalent. Assume that $B$ is a $*$-subalgebra of $A$ with the properties:

1. For any $t \in B_s$, $C(t) \subset B$ and $C(t)$ is the closed linear span of the selfadjoint idempotents in $C(t)$.

2. When $f$ is a selfadjoint idempotent of $B$, then $(1 - f)B(1 - f) \neq 0$.

Then there exists a selfadjoint idempotent $e \in B$ such that $(1 - e)\bar{B}(1 - e)$ is $B^*$-equivalent.

**Proof.** Assume that the lemma is false. Then $\bar{B}$ is not $B^*$-equivalent. Therefore by Lemma 2.5 with $K = 2$, there exists $h_1 \in B_s$ such that $\nu(h_1) < \frac{1}{2}\|h_1\|$. Then by (1) there exists an element $s_1 \in C(h_1)$ such that $s_1 = \lambda_1 g_1 + \cdots + \lambda_n g_n$, where the $\lambda_k$ are real scalars and the $g_k$ are selfadjoint idempotents in $C(h_1)$, and such that $\nu(s_1) < \frac{1}{2}\|s_1\|$. Set $e_1 = g_1 \circ g_2 \circ \cdots \circ g_n$. Note that $s_1(1 - e_1) = s_1(1 - g_1)(1 - g_2) \cdots (1 - g_n) = 0$ and $(1 - e_1)s_1 = 0$. By assumption, $(1 - e_1)\bar{B}(1 - e_1)$ is not $B^*$-equivalent, and then, by Lemma 2.5 with $K = 4$, there exists $h_2 \in (1 - e_1)B \cdot (1 - e_1)$ such that $h_2 = h_2^*$ and $\nu(h_2) < \frac{1}{4}\|h_2\|$. As before we can choose $s_2 \in C(h_2)$ such that $s_2 = \mu_1 f_1 + \cdots + \mu_m f_m$, where the $\mu_k$ are real scalars and the $f_k$ are selfadjoint idempotents in $C(h_2)$, and such that $\nu(s_2) < \frac{1}{4}\|s_2\|$. Let $e_2 = e_1 \circ f_1 \circ f_2 \cdots \circ f_m$. Note that $s_2 s_2 = (s_1 e_1)(s_2) = 0$ and $s_2 s_1 = 0$. Continuing in this fashion, we can choose a sequence $\{s_k\}$ such that each $s_k$ is selfadjoint, $s_k s_j = s_j s_k = 0$ for $k \neq j$, and $\nu(s_k) < \frac{1}{4^k}\|s_k\|$ for all $k \geq 1$. Let $C$ be a maximal commutative $*$-subalgebra of $A$ which contains the sequence $\{s_k\}$. By Proposition 2.2, $C$ is $B^*$-equivalent. This contradicts the inequalities $\nu(s_k) < \frac{1}{4^k}\|s_k\|$, $k \geq 1$. Therefore the lemma holds.

3. **Algebras with dense socle which are locally $B^*$-equivalent.** The socle of a semisimple algebra $A$ is the sum of the minimal left ideals of $A$, or 0 if $A$ has no minimal left ideals. In this section we prove that a Banach $*$-algebra with dense socle which is locally $B^*$-equivalent is $B^*$-equivalent. We need several facts about minimal ideals and the socle. Assume that $A$ is a semisimple (complex) Banach algebra. An idempotent $e$ of $A$ is a minimal idempotent of $A$ if $eAe = \{\lambda e | \lambda \text{ complex}\}$. When $e$ is a minimal idempotent of $A$, then $Ae$ is a minimal left ideal of $A$. Conversely every minimal left ideal of $A$ is of the form $Ae$ for some minimal idempotent $e$ of $A$; see [6, pp. 45–46]. If in addition $A$ has an involution $*$ with the property that $a^* a = 0$ implies $a = 0$, then the idempotent $e$ above may be chosen selfadjoint.
by [6, Lemma (4.10.1), p. 261]. We denote the socle of \( A \) as \( S_A \). Although defined as the sum of the minimal left ideals of \( A \), \( S_A \) is also the sum of the minimal right ideals of \( A \). Therefore given \( s, t \in S_A \), there exist minimal idempotents of \( A \), \( e_k, f_j \), \( 1 \leq k \leq n, 1 \leq j \leq m \), such that \( s \in e_1 A + \cdots + e_n A \) and \( t \in A f_1 + \cdots + A f_m \). By [7, Lemma 5.1, p. 358], \( e_k A f_j \) is either one dimensional or 0 for all \( k \) and \( j \). Therefore, \( s A t = \sum e_k A f_j \) which is finite dimensional. Now assume that \( S_A = A \). In this case \( A \) has no primitive ideal which contains \( S_A \). Therefore, \( A/S_A \) is a radical algebra. Then given an idempotent \( g \) in \( A \), the residue class \( g + S_A \) is an idempotent in the radical algebra \( A/S_A \). Since radical algebras contain no nonzero idempotents, \( g \in S_A \). Thus when \( S_A = A \), every idempotent of \( A \) is in \( S_A \).

When \( X \) is a normed linear space, \( \mathcal{B}(X) \) denotes the algebra of bounded operators on \( X \) which have finite dimensional range, and \( \mathcal{C}(X) \) denotes the algebra of compact operators on \( X \).

**Theorem 3.1.** Assume that \( A \) is a primitive Banach \(*\)-algebra with dense socle and that \( A \) is locally \( B^\ast \)-equivalent. Then \( A \) is \(*\)-isomorphic to \( \mathcal{C}(H) \) for some Hilbert space \( H \).

**Proof.** Assume that \( M \) is a minimal left ideal of \( A \). Then there exists a selfadjoint minimal idempotent \( f \) of \( A \) such that \( M = Af \). We introduce an inner product \((\cdot, \cdot)\) on \( Af \) by the rule \((xf, yf)f = f y^* x f \). That \((\cdot, \cdot)\) is an inner product on \( Af \) is verified in [6, Theorem (4.10.3), p. 261]. When \( a \in A \), define \( T_a \) on \( Af \) by \( T_a(xf) = axf \). \( a \to T_a \) is a \(*\)-representation of \( A \) into the bounded operators on the inner-product space \( Af \), again by [6, Theorem (4.10.3)]. Set \( K = \{ a \in A \mid aAf = 0 \} \). \( K \) is the kernel of the representation \( a \to T_a \). Since \( KAf = 0 \) and 0 is a primitive ideal of \( A \) by hypothesis, then \( K = 0 \) by [6, Theorem (2.2.9)(iv), p. 54]. Therefore the representation \( a \to T_a \) is faithful.

If \( A \) is finite dimensional, then \( Af \) is a finite-dimensional Hilbert space, and \( a \to T_a \) is a faithful \(*\)-representation of \( A \) onto \( \mathcal{C}(Af) \). In what follows we assume that \( A \) is not finite dimensional. Now we verify that \( A \) satisfies (1) and (2) of Lemma 2.6. Suppose that \( g \) is a selfadjoint idempotent of \( A \) such that \( (1-g)A(1-g) = 0 \). Let \(| \cdot |\) be the unique norm on \( A \) with the \( B^\ast \)-property (Proposition 2.1(4)). Then \(|x(1-g)|^2 = |(1-g)x|^2 = |(1-g)x^* x(1-g)| = 0 \) for all \( x \in A \). It follows that \( A = gAg \) which is finite dimensional since \( g \in S_A \). This contradiction proves that, for any selfadjoint idempotent \( g \) of \( A \), \( (1-g)A(1-g) \neq 0 \). This is (2) of Lemma 2.6. Now assume that \( t \in A \). \( C(t) \) is \( B^\ast \)-equivalent, and therefore, from [1, Corollary, p. 517] and [8, Theorem 4.1, p. 42], \( C(t) \) has dense socle. It follows that \( C(t) \) is the closed linear span of its selfadjoint idempotents. This verifies (1) of Lemma 2.6. Therefore Lemma 2.6 implies that there exists a selfadjoint idempotent \( e \in A \) such that \( (1-e)A(1-e) \) is \( B^\ast \)-equivalent. \( (1-e)A(1-e) \) is complete in the norm \(| \cdot |\). \( (1-e)A(1-e) \) is a nonzero semisimple Banach \(*\)-algebra with dense socle. Choose \( f \) a selfadjoint minimal idempotent of \( (1-e)A(1-e) \). Then \( ef = fe = 0 \) so that, for any \( x \in A \), \( f x f = f(1-e)x(1-e)f = \lambda f \) for some scalar \( \lambda \). Therefore \( f \) is a minimal
idempotent of \(A\). Let \((\cdot, \cdot)\) be the inner-product introduced on \(Af\) as in the first paragraph of the proof. Let \(|\cdot|_2\) be the corresponding norm on \(Af\). Then for any \(x \in A\), \(|x|_2 = |f(x, x)| = |f| = |x|_2^2\). Therefore, \(|\cdot|_1\) and \(|\cdot|_2\) are identical on \(Af\). Also, \((1-e)A(1-e) = (1-e)Af\) is a minimal left ideal of \((1-e)A(1-e)\) by the choice of \(f\). Then \((1-e)Af\) is a closed, and hence complete, subspace of \((1-e)A(1-e)\) in the norm \(|\cdot|_1\). Therefore \(|\cdot|_2\) is a complete norm on \((1-e)Af\).

But \(eAf\) is finite dimensional, hence complete. Therefore \(Af = eAf \oplus (1-e)Af\) is a Hilbert space. We denote the Hilbert space \(Af\) by \(H\).

As in the first paragraph of the proof, \(a \mapsto Ta\) is a faithful \(*\)-representation of \(A\) into the bounded operators on \(H\). When \(t \in S_A\), then \((Af)\) is finite dimensional, so that \(T_t \in \mathcal{F}(H)\). Let \(B = \{T_a \mid a \in A\}\). When \(T\) is a bounded operator on \(H\), denote the operator norm of \(T\) by \(\|T\|_\text{op}\). By Proposition 2.1(4), \(|a| = \|T_a\|_\text{op}\) for all \(a \in A\).

The norm \(|\cdot|_1\) on \(A\) dominates the norm \(|\cdot|_2\) by [6, Corollary (4.1.16), p. 187]. \(A\) has dense socle in the norm \(|\cdot|_1\), and therefore \(A\) has dense socle in the norm \(|\cdot|_2\).

It follows that \(\{T_s \mid s \in S_A\}\) is dense in \(B\) in the norm \(|\cdot|_\text{op}\). Every operator \(T_s\) in \(B\) is the operator norm limit of operators with finite-dimensional range, so that \(B \subset \mathcal{C}(H)\).

It remains to be shown that \(B = \mathcal{C}(H)\). Given \(F\) an operator with 1-dimensional range on \(H\), then there exist \(\phi, \psi \in H\) such that \(F(\gamma) = (\gamma, \phi) \cdot \psi\) for all \(\gamma \in H\). \(\phi = uf\) and \(\psi = vf\) for some \(u, v \in A\). Then \(T_vuf(x)f = vfu^*xf = (xf, uf)vf = F(xf)\) for all \(xf \in Af\). Therefore \(F = T_vuf \in B\), and it follows that \(\mathcal{F}(H) \subset B\). Given \(T = T^* \in \mathcal{C}(H)\), then by the Spectral Theorem for compact operators, there exists a sequence of real scalars \(\{\lambda_k\}\), and a corresponding orthogonal sequence of selfadjoint projections with finite-dimensional range \(\{E_k\}\), such that \(T = \sum_{k=1}^{\infty} \lambda_k E_k\), convergence being in operator norm. We have shown that \(\{E_k\} \subset B\). Let \(C\) be a maximal commutative \(*\)-subalgebra of \(B\) containing \(\{E_k\}\). By Proposition 2.2, \(C\) is complete in the operator norm. Therefore \(T \in C \subset B\), and the proof is complete.

Now we consider the general case when \(A\) is a locally \(B^*\)-equivalent algebra with dense socle.

**Theorem 3.2.** Assume that \(A\) is a Banach \(*\)-algebra with dense socle, and that \(A\) is locally \(B^*\)-equivalent. Then \(A\) is \(B^*\)-equivalent.

**Proof.** Just as in the proof of Theorem 3.1, there exists a selfadjoint idempotent \(e \in A\) such that \((1-e)A(1-e)\) is \(B^*\)-equivalent. Also since \(A\) has dense socle, \(e \in S_A\). If \(f\) is a minimal idempotent of \(A\), then \((Af)\) is a minimal closed two-sided ideal of \(A\). It follows that every element of the socle of \(A\) is contained in a finite sum of minimal closed two-sided ideals of \(A\). Furthermore, if \(M\) and \(N\) are distinct minimal closed two-sided ideals of \(A\), then \(M \cdot N \subset M \cap N = 0\). Therefore there exist minimal closed two-sided ideals of \(A\), \(M_k\), \(1 \leq k \leq n\), such that \(e \in M_1 + M_2 + \cdots + M_n\) and \(M_kM_j = 0\) when \(k \neq j\). Also setting \(D = \{x \in A \mid xM_k = 0, 1 \leq k \leq n\}\) then \(S_A \subset M_1 \oplus M_2 \oplus \cdots \oplus M_n \oplus D\). Denote this sum \(A \oplus B\). Each \(M_k\) is a closed \(*\)-ideal of \(A\) with dense socle and a primitive \(*\)-bra. Therefore by Theorem 3.1,
$M_k$ is $B^*$-equivalent for $1 \leq k \leq n$. Since $e \in M_1 \oplus \cdots \oplus M_n$, then $D \subseteq (1 - e)A(1 - e)$. Therefore the closed $\ast$-algebra $D$ is $B^*$-equivalent. Then $B$ is $B^*$-equivalent, hence complete in the norm $\| \cdot \|$. But since $S_{\lambda} < B$, $B$ is dense in $A$ in the norm $\| \cdot \|$. Then $A = B$ which completes the proof.

4. The main result. In this section we assume that $A$ is locally $B^*$-equivalent, and that whenever $t \in A_n$, the spectrum of $t$ is at most countable. Then the spectrum of $t$ is totally disconnected, and it follows that $C(t)$ is the closed linear span of its selfadjoint idempotents; see [6, p. 293]. We prove now that $A$ is $B^*$-equivalent.

Theorem 4.1. Assume that $A$ is a Banach $\ast$-algebra which is locally $B^*$-equivalent and which has the property that whenever $t \in A_n$, then the spectrum of $t$ is at most countable. Then $A$ is $B^*$-equivalent.

Proof. We assume that $A \neq 0$. When $E$ is a subset of $A$, let $L[E] = \{a \in A \mid aE = 0\}$.

By [1, Theorem 2.3, p. 513], $L[S_{\lambda}] = 0$. Also $S_{\lambda}$ is $B^*$-equivalent by Theorem 3.2. Let $\mathcal{J}$ be the set of all closed $\ast$-ideals $I$ of $A$ which are $B^*$-equivalent and have the property that $L[I] = 0$. $\mathcal{J}$ is partially ordered by inclusion and nonempty since $S_{\lambda} \in \mathcal{J}$. Set $K = \bigcup_{J \in \mathcal{J}} J$. Suppose for some selfadjoint idempotent $f$ in $K$, $(1 - f)K(1 - f) = 0$. Then $(1 - f)x*(1 - f) = 0$ for all $x \in K$, and therefore $(1 - f)x = 0$ for all $x \in K$. Then $A(1 - f) \subseteq L[K] = 0$, so that $f$ is a right identity for $A$. But $f \in J$ for some $J$ in $\mathcal{J}$. In this case $J = A$, and we are done. Therefore we may assume that $(1 - f)K(1 - f) \neq 0$ for every selfadjoint idempotent $f \in K$. Assume now that $t \in K$. Then $t \in J$ for some $J \in \mathcal{J}$, and then $C(t) \subseteq J \subseteq K$, since $J$ is closed. Also $C(t)$ is the closed linear span of its selfadjoint idempotents by the remarks preceding the statement of the theorem. This verifies (1) and (2) of Lemma 2.6. Therefore there exists a selfadjoint idempotent $e \in K$ such that $(1 - e)\bar{K}(1 - e)$ is $B^*$-equivalent. $e \in J$ for some $J$ in $\mathcal{J}$ and $J$ is $B^*$-equivalent. Let $\| \cdot \|$ be the unique norm on $A$ with the $B^*$-property. $J$ and $(1 - e)\bar{K}(1 - e)$ are complete with respect to $\| \cdot \|$. Assume $\{x_n\} \subseteq \bar{K}$ and $\|x_n - x_m\| \to 0$. Set $y_n = (1 - e)x_n + e x_n$. Note that $\{y_n\} \subseteq J$. Then for all $n \geq 1$, $x_n = (1 - e)x_n(1 - e) + y_n$. Then $\{y_n\}$ is Cauchy in $J$ and $(1 - e)x_n \cdot (1 - e)$ is Cauchy in $(1 - e)\bar{K}(1 - e)$ in the norm $\| \cdot \|$. It follows that $\{x_n\}$ converges in $\bar{K}$ in the norm $\| \cdot \|$. Therefore $\bar{K}$ is a closed $B^*$-equivalent $\ast$-ideal of $A$ in $\mathcal{J}$, and $\bar{K}$ is an upper bound for $\mathcal{J}$. By Zorn’s Lemma $\mathcal{J}$ has a maximal element $I$. Suppose that $I \neq A$. $A/I$ is locally $B^*$-equivalent by Proposition 2.3. $A/I$ is semisimple, and every selfadjoint element in $A/I$ has at most countable spectrum. Let $T$ be the closure of the socle of $A/I$ in $A/I$. Note that $T \neq 0$ by [1, Theorem 2.3, p. 513]. Let $M = \{x \in A \mid x + I \in T\}$. $M$ is a closed $\ast$-ideal of $A$ which properly contains $I$. Also $M/I = T$. $I$ is $B^*$-equivalent, and by Theorem 3.2, $M/I$ is $B^*$-equivalent. Therefore by Lemma 2.4, $M$ is $B^*$-equivalent. This contradicts the maximality of $I$. Therefore $A = I$, so that $A$ is $B^*$-equivalent.

Corollary 4.2. Assume that $A$ is a Banach $\ast$-algebra which is locally $B^*$-equivalent and such that $A$ has a separable dual space. Then $A$ is $B^*$-equivalent.
Proof. Let \(|\cdot|\) be the unique norm on \(A\) with the \(B^*\)-property. Given \(a \in A_n\), there exists a number \(m > 0\) such that \(m|a| \geq \|a\|\) for all \(a \in C(t)\). Since the dual space of \(A\) is separable, then the dual space of \(C(t)\) is separable. If \(\phi\) and \(\psi\) are any two distinct nonzero multiplicative linear functionals on \(C(t)\), then

\[
\|\phi - \psi\| = \sup \{ |(\phi - \psi)(a)| \mid a \in C(t), \|a\| \leq 1 \} \\
\geq \frac{1}{m} \sup \{ |(\phi - \psi)(a)| \mid a \in C(t), |a| \leq 1 \} = \frac{2}{m}.
\]

It follows that the carrier space of \(C(t)\) is at most countable. Therefore the spectrum of \(t\) is at most countable. Then the corollary follows from Theorem 4.1.

**Corollary 4.3.** Assume that \(A\) is a Banach \(*\)-algebra with the property that, whenever \(t \in A_n\), then the spectrum of \(t\) is at most countable. Then if \(\sqrt{\cdot}\) operates on \(A\), we have \(A\) is \(B^*\)-equivalent.

The corollary follows from Theorem 4.1 and [5, Corollaire, p. 169].

**Corollary 4.4.** Assume that \(A\) is a Banach \(*\)-algebra with the property that, whenever \(t \in A_n\), then the spectrum of \(t\) is at most countable. Then if every orthogonal sequence of selfadjoint idempotents of \(A\) is bounded, we have \(A\) is \(B^*\)-equivalent.

The corollary follows from Theorem 4.1 and the results of Katznelson [5, pp. 167–169].

**References**