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CORRESPONDENCE BETWEEN LIE ALGEBRA INVARIANT SUBSPACES AND LIE GROUP INVARIANT SUBSPACES OF REPRESENTATIONS OF LIE GROUPS[1]

BY
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Abstract. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{B}=\mathfrak{u}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$; also let U be a representation of G on H, a Hilbert space, with dU the corresponding infinitesimal representation of $\mathfrak{g}$ and $\mathfrak{B}$. For G semisimple Harish-Chandra has proved a theorem which gives a one-one correspondence between $dU(\mathfrak{g})$ invariant subspaces and $U(G)$ invariant subspaces for certain representations U. This paper considers this theorem for more general Lie groups.

A lemma is proved giving such a correspondence without reference to some of the concepts peculiar to semisimple groups used by Harish-Chandra. In particular, the notion of compactly finitely transforming vectors is supplanted by the notion of Δ_f, the Δ finitely transforming vectors, for $\Delta \in \mathfrak{B}$. The lemma coupled with results of R. Goodman and others immediately yields a generalization to Lie groups with large compact subgroup.

The applicability of the lemma, which rests on the condition $\mathfrak{g}\Delta_f \subseteq \Delta_f$, is then studied for nilpotent groups. The condition is seen to hold for all quasisimple representations, that is representations possessing a central character, of nilpotent groups of class $\leqq 2$. However, this condition fails, under fairly general conditions, for $\mathfrak{g}=N_4$, the 4-dimensional class 3 Lie algebra. N_4 is shown to be a subalgebra of all class 3 $\mathfrak{g}$ and the condition is seen to fail for all $\mathfrak{g}$ which project onto an algebra where the condition fails. The result is then extended to cover all $\mathfrak{g}$ of class 3 with general dimension 1. Finally, it is conjectured that $\mathfrak{g}\Delta_f \subseteq \Delta_f$ for all quasisimple representations if and only if class $\mathfrak{g} \leqq 2$.

0. **Introduction.** Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{u}(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Δ will be the sum of the squares of a basis for $\mathfrak{g}$ in $\mathfrak{u}(\mathfrak{g})$, and π a representation of G on a Banach space H. The space of (infinitely) differentiable vectors for π, $H^\infty(\pi)$, is given by $\{v \in H \mid \text{the function } g \mapsto \pi(g)v \text{ is } C^\infty \text{ on } G\}$. $d\pi$ is defined for $X \in \mathfrak{g}$ and $v \in H^\infty(\pi)$ by

$$d\pi(X)v = \lim_{h\to 0} h^{-1}[\pi(\exp hX)v - v].$$

The analytic vectors, $H^\omega(\pi)$, are given by $H^\omega(\pi)=\{v \in H \mid \text{the map } g \mapsto \pi(g)v \text{ is analytic on } G\}$. The analytic vectors are of critical importance in integrating

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representations of $\mathfrak{g}$ to representations of G. For an operator A in H, $\mathfrak{a}(A)$, the set of analytic vectors for A, is given by

$$\mathfrak{a}(A) = \left\{ v \in \bigcap_{j=1}^{\infty} \mathrm{Dom}\,(A^j) \mid (\exists s > 0) \sum_{j=1}^{\infty} \frac{\|A^j v\|}{j!} s^j < \infty \right\}.$$

For G semisimple K will denote the analytic subgroup arising from a maximal compact subalgebra of $\mathfrak{g}$. A vector v is compactly finitely transforming, i.e. $v \in K_f$, if v is contained in a finite-dimensional space which is invariant under $\pi(K)$. Harish-Chandra [6] used K_f to show the density of analytic vectors for an important class of representations of semisimple Lie groups. Subsequently, Nelson [9] established that $\mathfrak{a}(d\pi(\Delta))$ is dense and contained in $H^{\omega}(\pi)$ for π any unitary representation of a Lie group.

For $\psi \in K_f$ let $U = d\pi(\mathfrak{u}(\mathfrak{g}))\psi$. Harish-Chandra [6] also showed that under certain conditions $U \subseteq H^{\omega}(\pi)$ and there is a bijective correspondence between $d\pi(\mathfrak{g})$ invariant subspaces of U and closed $\pi(G)$ invariant subspaces of Cl (U), the closure of U. In order to find a subspace on which such a correspondence holds, for general G it seems natural in the light of Nelson's work to consider subspaces defined in terms of a single operator arising from $\mathfrak{u}(\mathfrak{g})$ under a representation π. For the group of strictly upper triangular 3×3 matrices $d\pi(\Delta)_f$, where Δ is computed with respect to the usual basis of $\mathfrak{g}$, the sum of $d\pi(\Delta)$'s eigenspaces provides an example of a space on which such a correspondence holds. This paper considers the suitability of $d\pi(b)_f$ for $b \in \mathfrak{u}(\mathfrak{g})$ as a space where we can develop a correspondence for general Lie groups. This space, or more specifically $d\pi(\Delta)_f$, has also been of interest as a convenient subspace of analytic vectors on which to study the action of operators arising from $\mathfrak{u}(\mathfrak{g})$ for many Lie groups.

In §1 a criterion for an invariant subspace correspondence is proved. §2 investigates conditions under which this criterion is applicable. The algebra invariance of $d\pi(b)_f$ is seen to be critical to the criterion and this condition is considered in §3. §4 gives an example of a low-dimensional nilpotent group N_4, for which $d\pi(\Delta)_f$ is not algebra invariant for a large class of representations. §5 considers the invariance of $d\pi(\Delta)_f$ for arbitrary nilpotent groups.

Many details and much of the spirit of this paper and my thesis are due to my thesis adviser Professor R. J. Blattner. It is a pleasure to acknowledge my gratitude.

In what follows we will set $\mathfrak{B} = \mathfrak{u}(\mathfrak{g}^C)$ for $\mathfrak{g}$ real and $\mathfrak{B} = \mathfrak{u}(\mathfrak{g})$ for $\mathfrak{g}$ complex. $d\pi$ is understood to be the representation of $\mathfrak{g}$ or $\mathfrak{B}$ on $H^{\infty}(\pi)$. $\langle a, \ldots, z \rangle$ will denote the linear span of the elements $\{a, \ldots, z\}$ and $\sum A_j$ will denote the linear span of $\bigcup A_j$.

1. **A criterion for correspondence.** We begin by introducing some concepts we will need.

Let A be an operator in a Banach space H. $H_{A,\lambda}$ will denote the eigenspace of A corresponding to the eigenvalue λ. When speaking of a fixed $b \in \mathfrak{B}$ and a fixed representation $d\pi$ we write H_λ or $H_{b,\lambda}$ for $H_{d\pi(b),\lambda}$. A_f, the set of A finitely transforming vectors, is defined by $A_f=\{v\in H \mid \exists$ a finite dimensional subspace $V\leqq \operatorname{dom} A$ with $v\in V$ and $AV\subseteq V\}$. We will write b_f for $d\pi(b)_f$ unless we wish to specify the representation.

Now let A be a symmetric operator in a Hilbert space H. Recall that A's eigenvalues are real and that different eigenvalues give rise to orthogonal eigenspaces. It is easily shown that $A_f=\sum H_{A,\lambda}$, and that this sum is direct where λ runs through all eigenvalues of A. In particular any A invariant subspace V of A_f is generated by eigenvectors and $V=\sum V\cap H_{A,\lambda}$.

PROPOSITION 1.1. *For A a positive selfadjoint operator in a Hilbert space H we define fractional powers of A using the spectral resolution E_λ of A, so that $A^m=\int_{-\infty}^{\infty}\lambda^m\,dE_\lambda$ for m a positive rational number. Then we have*

$$A_f = ((a+bA)^m+c)_f$$

provided $a+bA$ is positive (a, b, c are constants with $b\neq 0$). If A has no kernel, then we can allow $m<0$, as well.

Combinatoric arguments lead to a proof that $\mathfrak{a}(B)=\mathfrak{a}(\alpha B+\beta)$ for α,β constants with $\alpha\neq 0$. If A is symmetric then $A_f\subseteq\mathfrak{a}(A)$. Also, if A and B are essentially selfadjoint and have commuting spectral resolutions, then $A\geqq B\geqq 0$ implies $A^j\geqq B^j\geqq 0$ which in turn implies $\mathfrak{a}(A)\subseteq\mathfrak{a}(B)$. In the case of a unitary representation U we have the following chain of spaces due to Nelson [9] and Goodman [4]: $A_f\subseteq\mathfrak{a}(A)\subseteq\mathfrak{a}(A^{1/2})=H^\omega(U)$ where $A=\mathrm{Cl}\,(1-dU(\Delta))$, the closure of $1-dU(\Delta)$.

The following theorem gives a general criterion for the desired sort of equivalence.

THEOREM 1.2. *Let π be a representation of the Lie group G on a Hilbert space H such that $\pi(G)^*\subseteq\pi(G)$ and let $b\in\mathfrak{B}$. Suppose*

(1) $d\pi(\mathfrak{g})b_f\subseteq b_f$,

(2) $b_f\subseteq H^\omega(\pi)$,

(3) $\dim H_{b,\lambda}<\infty$ *for all eigenvalues λ, and*

(4) $d\pi(b)$ *is a symmetric operator in H.*

Let M' be the set of $d\pi(\mathfrak{B})$ invariant subspaces of $(d\pi(b))_f$ and let M be the set of closed $\pi(G)$ invariant subspaces of $\mathrm{Cl}\,(d\pi(b)_f)$. Then there is a 1-1 correspondence φ from M' onto M given by $\varphi(V')=\mathrm{Cl}\,(V')$ and $\varphi^{-1}(V)=V\cap(d\pi(b))_f$ for $V'\in M'$ and $V\in M$.

Before giving a proof we establish a lemma.

LEMMA 1.3. *Let π be a representation of a Lie group G on a Hilbert space H such that $\pi(G)^*\subseteq\pi(G)$ and let $b\in\mathfrak{B}$. Suppose $d\pi(b)$ is symmetric, $d\pi(b)_f$ is dense in H and $d\pi(b)_f\subseteq H^\infty$. Let $\rho(g)=\pi(g)|_K$ for all $g\in G$ where K is a closed $\pi(G)$ invariant subspace of H. Then $d\rho(B)_f$ is dense in K and $d\rho(b)_f=d\pi(b)_f\cap K$.*

Proof. Since H is a Hilbert space $H=K+K^{\perp}$ (direct sum) where $K^{\perp}$ is the closed orthogonal complement to K in H. Let P and $P^{\perp}$ be the projections onto K and $K^{\perp}$ respectively.

K is G invariant and $K^{\perp}$ is G invariant too, since $\pi(G)^* \subseteq \pi(G)$. If $v \in H^{\infty}$, then since P and $P^{\perp}$ are continuous and linear, Pv and $P^{\perp}v$ are in H^{∞} also. Moreover, $K^{\infty}(\rho)=H^{\infty}(\pi) \cap K$ and $K^{\perp\infty}(\rho^{\perp})=H^{\infty}(\pi) \cap K^{\perp}$ where $\rho^{\perp}$ is the representation of $K^{\perp}$ induced by π.

We now show $d\pi(b)_f=d\rho(b)_f+d\rho^{\perp}(b)_f$. Let us consider $v \in H_{d\pi(b),\lambda}$. Then $\lambda Pv + \lambda P^{\perp}v=\lambda v=d\pi(b)v=d\pi(b)Pv+d\pi(b)P^{\perp}v$. Since K and $K^{\perp}$ are $\pi(G)$ invariant, $K^{\infty}(\rho)$ and $K^{\perp\infty}(\rho^{\perp})$ are $d\pi(\mathfrak{B})$ invariant. Hence $0=(d\pi(b)-\lambda)Pv+(d\pi(b)-\lambda)P^{\perp}v$ so that Pv and $P^{\perp}v$ are in $H_{d\pi(b),\lambda}$. Thus $H_{d\pi(b),\lambda}=K_{d\rho(b),\lambda}+K^{\perp}_{d\rho^{\perp}(b),\lambda}$ (direct sum) and we have shown $d\pi(b)_f=d\rho(b)_f+d\rho^{\perp}(b)_f$ (direct sum).

Suppose now that $\mathrm{Cl}\,(d\rho(b)_f)\neq K$. Then there is a nonzero $k \in K$ such that $k \perp d\rho(b)_f$. But then $k \perp d\rho^{\perp}(b)_f$ and so $k \perp \mathrm{Cl}\,(d\rho(b)_f+d\rho^{\perp}(b)_f)$ or $K \perp H$. Thus $k=0$ and so $\mathrm{Cl}\,(d\rho(b)_f)=K$. Q.E.D.

Proof of Theorem 1.2. We define the maps φ and $\tilde{\varphi}$ by $\varphi(V')=\mathrm{Cl}\,(V')$ for $V' \in M'$ and $\tilde{\varphi}(V)=V \cap b_f$ for $V \in M$. We show $(\alpha)\varphi$ is a map from M' into M, $(\beta)\tilde{\varphi}$ is a map from M into M', $(\gamma)\tilde{\varphi}\circ\varphi=I'$, the identity operator on M', and $(\delta)\varphi\circ\tilde{\varphi}=I$, the identity operator on M. This then shows φ is 1-1 and the theorem holds.

(α) *φ maps M' into M.* For $V' \in M'$, $\varphi(V')=\mathrm{Cl}\,(V')$. Harish-Chandra [6] shows that a $d\pi(\mathfrak{g})$ invariant subspace of $H^{\omega}(\pi)$ has $\pi(G)$ invariant closure. $V' \in M'$ and so V' is $d\pi(\mathfrak{g})$ invariant and $V' \leqq b_f$ which is contained in $H^{\omega}(\pi)$ by hypothesis (2). Thus $\varphi(V')=\mathrm{Cl}\,(V') \subseteq \mathrm{Cl}\,(b_f)$ and $\varphi(V')$ is $\pi(G)$ invariant so φ maps M' into M.

(β) *$\tilde{\varphi}$ maps M into M'.* For $V \in M$, $\tilde{\varphi}(V)=V \cap b_f$. We must show $V \cap b_f$ is a $d\pi(\mathfrak{g})$ invariant subspace of b_f. Now, for $v \in V \cap b_f$ and $X \in \mathfrak{g}$,

$$d\pi(X)v = \lim_{t\to\infty} \frac{\pi(\exp tX)v-v}{t}$$

which is a limit of objects in the $\pi(G)$ invariant closed subspace V so $d\pi(X)v \in V$. On the other hand, $v \in b_f$ and hypothesis (1) tell us that $d\pi(X)v \in b_f$. Thus $d\pi(\mathfrak{g})(V \cap b_f) \subseteq V \cap b_f$ and so $\tilde{\varphi}$ maps M into M'.

(γ) *$\tilde{\varphi}\circ\varphi=I'$, the identity operator on M'.* For $V' \in M'$, $\tilde{\varphi}\circ\varphi(V')=\mathrm{Cl}\,(V') \cap b_f$. We must show $V'=\mathrm{Cl}\,(V') \cap b_f$. Now $V' \leqq b_f$ and $V' \leqq \mathrm{Cl}\,(V')$ so we must now show $\mathrm{Cl}\,(V') \cap b_f \subseteq V'$. In light of hypothesis (4) and the comment just before Proposition 1.1, $V'=\sum V' \cap H_{\lambda}$ and $\mathrm{Cl}\,(V') \cap b_f=\sum \mathrm{Cl}\,(V') \cap H_{\lambda}$ (where λ runs through the eigenvalues for each of these sums). Also $V' \cap H_{\lambda} \leqq \mathrm{Cl}\,(V') \cap H_{\lambda} \leqq H_{\lambda}$ for all λ and each of these subspaces is of finite dimension by hypothesis (3). Suppose $V' \cap H_{\lambda_0} \neq \mathrm{Cl}\,(V') \cap H_{\lambda_0}$ for some λ_0. Then we select $v_0 \in \mathrm{Cl}\,(V') \cap H_{\lambda_0}$ such that $v_0 \perp V' \cap H_{\lambda_0}$. But then $0 \neq v_0 \in \mathrm{Cl}\,(V')$ while $v_0 \perp V' \cap H_{\lambda}$ for $\lambda \neq \lambda_0$ and so $v_0 \perp \sum V' \cap H_{\lambda}$ or $v_0 \perp V'$. This is a contradiction and so $V' \cap H_{\lambda} = \mathrm{Cl}\,(V') \cap H_{\lambda}$ for all λ. Hence $V'=\mathrm{Cl}\,(V') \cap b_f$ and $\tilde{\varphi}\circ\varphi=I'$.

(δ) $\varphi \circ \tilde{\varphi} = I$, *the identity on* M. We show $\mathrm{Cl}\,(V \cap b_f) = V$. V is closed and $V \cap b_f \subseteq V$ so $\mathrm{Cl}\,(V \cap b_f) \subseteq V$. The other inclusion follows from Lemma 1.3 with $V = K$ and U playing the role of H. Q.E.D.

The proof of this theorem was inspired by Harish-Chandra's proof of his theorem on correspondences.

The following example served as a model for this theorem. Let G_3 be the real Lie group

$$G_3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \middle| \, x, y, z \in \boldsymbol{R} \right\}.$$

The corresponding Lie algebra $N_3 = \langle X, Y, Z \rangle$ where

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Von Neumann has shown the following: The set of nontrivial unitary irreducible representations of G_3 are given (up to equivalence) by the representations U_λ on $L^2(\boldsymbol{R})$ where

$$\left[U_\lambda \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \right) f \right](t) = e^{i\lambda(z+ty)} f(t+x)$$

for any nonzero $\lambda \in \boldsymbol{R}$. $dU_\lambda(N_3)$ then acts on $H^\infty(U_\lambda)$ which is equal to $S(\boldsymbol{R})$, the Schwartz space of rapidly decreasing functions on $\boldsymbol{R}$, $dU_\lambda(X)$ is d/dt, $dU_\lambda(Y)$ gives multiplication by $it\lambda$ and $dU_\lambda(Z)$ is multiplication by $i\lambda$. Thus for $\Delta = X^2 + Y^2 + Z^2$ we have

$$[dU_\lambda(\Delta) f](t) = f''(t) - \lambda^2(t^2+1) f(t) \quad \text{for } f \in S(\boldsymbol{R}),$$

a "generalized" Hermite operator. In fact, for $\lambda = 1$, $dU_1(\Delta)$ is exactly the Hermite operator plus a constant and thus has the same eigenvectors as the Hermite operator; viz.

$$f_n(t) = (-1)^n \exp(t^2/2)(d^n/dt^n) \exp(-t^2).$$

The f_n's span $P[t] \exp(-t^2/2)$, where $P[t]$ is the set of polynomials in t, and form a complete orthonormal base, and each eigenvalue has multiplicity one. $P[t] \exp(-t^2/2)$ is $dU_1(N_3)$-invariant and is irreducible. More generally, for $\lambda \neq 0$, $\{t^n \exp(-|\lambda| t^2/2)\}_{n=0}^\infty$ provides an orthogonal basis composed of finite sums of eigenvectors. The span, $P[t] \exp(-|\lambda| t^2/2)$, is again irreducible and invariant.

In Theorem 1.2 the necessity for requiring that $\dim H_{b,\lambda} < \infty$ can be seen by considering the representation $\pi = (U_1)^\omega$, the direct sum of countably many copies of U_1.

2. **Conditions under which Theorem 1.2 may be applied.** We now give results which can be used in applying Theorem 1.2. We assume that U is a unitary representation of a Lie group G on a Hilbert space H.

For any formally symmetric $b \in \mathfrak{B}$, $dU(b)$ is symmetric, so hypothesis (4) is satisfied for such an element. Recall that b is formally symmetric if $b=b^+$ where $+$ is the conjugate linear anti-isomorphism which extends the map $X \mapsto -X$ from $\mathfrak{g}$ to $\mathfrak{B}$.

Suppose $b=\Delta$ for some choice of basis of $\mathfrak{g}$. Then hypothesis (2) is satisfied. In fact, $H_{\Delta,\gamma} \subseteq \mathfrak{a}(\Delta)$ and so $\Delta_f \subseteq \mathfrak{a}(\Delta)$ (Δ is formally selfadjoint and so $dU(\Delta)$ is symmetric). Also, Nelson has shown in his paper on analytic vectors [9] that $\mathfrak{a}(\Delta) \subseteq H^{\omega}$ and so hypothesis (2) holds.

Hypothesis (3) is satisfied for $b=\Delta$ and U a (unitary) irreducible representation of a CCR group. In fact, Nelson and Stinespring [10] have shown that for all $\lambda > 0$ there is a function $k_\lambda \in L^1(G)$ such that $[\lambda - \mathrm{Cl}\,(dU(\Delta))]^{-1} = U_{k_\lambda}$. (Recall that $U_f = \int_G f(g)U(g)\,dg$ as usual.) Thus if U is irreducible unitary and G is a CCR group then U_{k_λ} is a compact operator. $[1-\mathrm{Cl}\,(dU(\Delta))]^{-1}$, $[1-\mathrm{Cl}\,(dU(\Delta))]$ and $dU(\Delta)$ all have the same eigenvectors so that the eigenspaces have finite dimension and also Δ_f is dense. Nelson and Stinespring's work [10, see the proof of Theorem 3.1] along with Langlands' work leads to a consideration of the case where Δ is replaced by a strongly elliptic element $b \in \mathfrak{u}(\mathfrak{g})$ (which is the infinitesimal generator of a bounded semigroup).

We can compare the condition $\dim H_{\Delta,\lambda} < \infty$ with an analogous condition used by Harish-Chandra [6] for semisimple G. For $\mathfrak{k}$ a maximal compact subalgebra of $\mathfrak{g}$ let K be the corresponding analytic subgroup of G. For $\mathscr{D} \in \Omega$, the equivalence classes of finite-dimensional irreducible representations of K, we define $H_{\mathscr{D}} = \{v \in H \mid \exists$ a finite dimensional $V \leqq H \ni \pi(K)|_V$ is a sum of elements in $\mathscr{D}\}$. Harish-Chandra defines $K_f = \sum_{\mathscr{D} \in \Omega} H_{\mathscr{D}}$ and shows K_f is dense for a large class of representations. Harish-Chandra's correspondence theorem requires that $\dim H_{\mathscr{D}} < \infty$ for all $\mathscr{D} \in \Omega$. We now give some results which compare these two approaches.

For a compact subgroup K of an arbitrary Lie group G or K as in the preceding paragraph we say K is *large* if $\dim H_{\mathscr{D}} < \infty$ for all $\mathscr{D} \in \Omega$ and all unitary irreducible representations of G. Harish-Chandra [6] showed that the K arising in the previous paragraph is large. Nelson and Stinespring [10] showed that Lie groups with large compact subgroup are CCR.

The next result is from a preliminary version of a paper by Goodman [5].

THEOREM 2.1. *Let K be a large compact subgroup of the Lie group G and let π be a unitary irreducible representation of G in a Hilbert space H. Since K is compact there is an* $\mathrm{Ad}_G(K)$ *invariant inner product on* $\mathfrak{g}$. *Let $\{X_k\}_{k=1}^n$ be an orthonormal basis for $\mathfrak{g}$ with respect to this inner product and set $\Delta = \sum_{k=1}^n X_k^2$ and* $A = \mathrm{Cl}\,[1-d\pi(\Delta)]$. *Then $A_f = K_f$ and $d\pi(\mathfrak{g})A_f \subseteq A_f$.*

It is also clear that $\dim H_{\Delta,\lambda} < \infty$ *for each eigenspace of* Δ.

Proof. Nelson and Stinespring [10] have shown that G is CCR and that A^{-1} is a compact operator. Hence A has discrete spectrum $\{\lambda_n\}_{n=1}^{\infty}$, $\lambda_n \to \infty$ with finite multiplicity for each eigenvalue, and $A_f = \sum_{n=1}^{\infty} H_{A,\lambda_n}$.

Now $\mathrm{Ad}(K)\Delta = \Delta$ by the way we selected Δ and so $\mathrm{ad}\, T\Delta = 0$ for $T \in \mathfrak{k}$, the subalgebra of $\mathfrak{g}$ corresponding to K. Thus if $k \in K$, $\pi(k)d\pi(\Delta) = d\pi(\Delta)\pi(k)$. Since the eigenvectors for A lie in $H^{\infty}(\pi)$ [9] it follows that the H_{A,λ_n} are K invariant and so $A_f \subseteq K_f$. Conversely, if $v \in H^{\infty}(\pi)$, let $v_n = E_n v$ where E_n is the projection onto H_{A,λ_n}. E_n commutes with the action of K and so $E_n\colon H_{\mathscr{D}} \to H_{\mathscr{D}}$. Suppose $v \in H_{\mathscr{D}}$ and let $V = \pi(K)v$. $E_n V \subseteq H_{\mathscr{D}}$ for each n. Since $\dim H_{\mathscr{D}} < \infty$, only finitely many v_n's are nonzero. Thus $v \in A_f$ and $K_f \subseteq A_f$.

The $\mathfrak{g}$-invariance of K_f and thus of A_f is now due to a result of Godement [3, Lemma 26]. Q.E.D.

Now consider G a connected semisimple Lie group and let $\mathfrak{g} = \mathfrak{k} + p$ be a Cartan decomposition. Assume that K, the subgroup corresponding to $\mathfrak{k}$, is compact and form an orthogonal basis $X_1, \ldots, X_m$ of $\mathfrak{k}$ and an orthogonal basis $X_{m+1}, \ldots, X_n$ of p (with respect to the K invariant inner product). If we set $\Delta = \sum_{j=1}^{n} X_j^2$ then we have $U(K)_f = dU(\Delta)_f$ for representations which send $z(\mathfrak{B})$, the center of $\mathfrak{B}$, into scalars. In fact,

$$c = -\sum_{j=1}^{m} X_j^2 + \sum_{j=m+1}^{n} X_j^2$$

is the Casimir element and is known to be in $z(\mathfrak{B})$. The problem then reduces to showing $(\Delta_K)_f = K_f$ where $\Delta_K = (\Delta - c)/2$ (the Casimir element for K). One can now prove $K_f = \sum H_{\mathscr{D}}$ and $\sum H_{\mathscr{D}} = (\Delta_K)_f$ using the theory of representations of compact groups.

We can also see that in this case if U is unitary and $[1 - \mathrm{Cl}\,(dU(\Delta))]^{-1}$ is compact then $\dim H_{\Delta,\lambda} < \infty$. Since $\Delta - 2\Delta_K$ is a constant we have $\dim H_{\Delta_K,\lambda} < \infty$. Δ_K is equal to a constant μ on $H_{\mathscr{D}}$ so $H_{\mathscr{D}} \subseteq H_{\Delta_K,\mu}$. Thus $\dim H_{\mathscr{D}} < \infty$.

3. **The condition $\mathfrak{g}b_f \subseteq b_f$.** We have seen that application of Theorem 1.2 for irreducible unitary representations of CCR Lie groups rests on condition (1) $\mathfrak{g}b_f \subseteq b_f$. The remainder of this paper is mainly concerned with this condition. Harish-Chandra showed that K_f is $\mathfrak{g}$-invariant for a large class of representations of a semisimple Lie group G. Godement [3] enlarged this to cover groups with large compact subgroup. In the case of the representation U_1 of the group G_3 for $b = \Delta$ we find $\Delta_f = P[t]\exp(-t^2/2)$ and $dU_1(\mathfrak{g}) = \langle M_1, M_t, D_t\rangle$ where M_f is multiplication by f and D_t is differentiation with respect to t—clearly $\mathfrak{g}\Delta_f \subseteq \Delta_f$. Roe Goodman established [5] that condition (1) fails for unitary irreducible representations of the solvable "$ax+b$" group. It was hoped that the condition would, however, hold for unitary irreducible representations of nilpotent groups.

The following lemma indicates how the commutation relations of the group play a critical role in Δ_f's invariance.

LEMMA 3.1. *Let φ be a representation of* $\mathfrak{g}$, *let* $a, b \in \mathfrak{B}$ *and suppose* $\varphi(b)$ *is symmetric. Then* $\varphi(a)b_f \subseteq b_f$ *if and only if* $\varphi([b, a])b_f \subseteq b_f$. *Thus* $\varphi(\mathfrak{g})b_f \subseteq b_f$ *if and only if* $\varphi([b, \mathfrak{g}])b_f \subseteq b_f$.

Proof. Note that $\varphi(b)b_f \subseteq b_f$. Thus if $\varphi(a)b_f \subseteq b_f$, then $\varphi([b, a])b_f = \varphi(b)\varphi(a)b_f - \varphi(a)\varphi(b)b_f \subseteq b_f$. For the converse implication suppose $\varphi(b)v = \lambda v$ and let V_2 be a finite-dimensional b-invariant subspace containing $\varphi([b, a])v$. We set $V_0 = \langle \varphi(a)v \rangle + V_2$. Now $\varphi(b)[\varphi(a)v] = \varphi(ab)v + \varphi([b, a])v = \lambda\varphi(a)v + \varphi([b, a])v \in V_0$. V_0 is thus a finite-dimensional $\varphi(b)$ invariant subspace which contains $\varphi(a)v$ and so $\varphi(a)v \in b_f$.

The final statement is easily proved by applying the first statement to a basis for $\mathfrak{g}$. Q.E.D.

COROLLARY 3.2. *Let φ be a representation of an abelian Lie algebra and let* $b \in \mathfrak{B}$ *be such that* $\varphi(b)$ *is symmetric. Then* $\varphi(\mathfrak{g})b_f \subseteq b_f$.

The proof is immediate from the lemma so condition (1) does hold for abelian groups.

Recall that a Lie algebra $\mathfrak{g}$ is nilpotent if ad X is a nilpotent endomorphism of $\mathfrak{g}$ for each $X \in \mathfrak{g}$. The class of $\mathfrak{g}$ is one less than the length of the composition chain $\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq \cdots \supseteq [\mathfrak{g}, [\mathfrak{g}, \ldots [\mathfrak{g}, \mathfrak{g}] \ldots] = \{0\}$.

Standard results on nilpotent groups including Jordan-Hölder bases, determination of the representation spaces and the form of the irreducible unitary representations can be found in Pukánsky [11] and in Kirillov [7].

Given a representation φ of $\mathfrak{g}$ we will always consider it to be a representation of $\mathfrak{B}$. We say φ is quasisimple if φ maps $z(\mathfrak{B})$ into scalars. For π a representation of G we say π is quasisimple if $d\pi$ is. (This definition is a slight generalization of the one given by Harish-Chandra. It also includes irreducible unitary representation.) In the above case φ is a linear functional on $z(\mathfrak{g})$ and thus we can often restrict ourselves to the case where dim $z(\mathfrak{g}) = 1$. χ, the central character of φ, is given by $\varphi(b) = \chi(b)1$ for $b \in z(\mathfrak{B})$.

PROPOSITION 3.3. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra of class* 2 *or* 1 *and let* φ *be a quasisimple representation of* $\mathfrak{g}$. *Then* $\varphi(\mathfrak{g})\Delta_f \subseteq \Delta_f$ *for any second order element* Δ *such that* $\varphi(\Delta)$ *is symmetric.*

Proof. If class $\mathfrak{g} = 1$, then $\mathfrak{g}$ is abelian and we are done. If class $\mathfrak{g} = 2$, then $[\mathfrak{g}, \mathfrak{g}] \subseteq z(\mathfrak{g})$ so $\varphi([X, Y])\Delta_f \subseteq \Delta_f$ for X, Y and $\mathfrak{g}$. If $v_1 \in \Delta_f$ and V_1 is a finite-dimensional Δ-invariant subspace containing v_1, let $V = \varphi(\mathfrak{g})V_1 + V_1$. We claim V is finite dimensional and Δ-invariant. If $X \in \mathfrak{g}$ and $v \in V$, $\varphi(\Delta)\varphi(X)v = \varphi(X)\varphi(\Delta)v + \varphi([\Delta, X])V \subseteq \varphi(X)V_1 + \varphi([\Delta, X])V_1$. The Poincaré-Birkhoff-Witt (PBW) theorem and Leibniz's law along with φ's quasisimplicity imply that $\varphi([\Delta, X]) \subseteq \varphi(\mathfrak{g})$ and we are done.

4. N_4: **A counterexample.** We now introduce N_4, a class 3 nilpotent Lie algebra. We will show that $\varphi(N_4)\Delta_f \nsubseteq \Delta_f$ for any second order elliptic element $\Delta \in \mathfrak{u}(N_4)$ and φ any quasisimple representation of N_4 which is nontrivial on $z(N_4)$.

We define N_4 to be the 4-dimensional class 3 Lie algebra given by $\langle Z, Y, W, X\rangle$ with $[X, W]=Y$, $[X, Y]=Z$ and $[W, Y]=[Z, N_4]=0$. $z(\mathfrak{B}_4)$, the center of the universal enveloping algebra of N_4, is all polynomials in Z and Y^2-2ZW.

We are concerned with the condition $\varphi(\mathfrak{g})\Delta_f \subseteq \Delta_f$ where $\Delta_f=\sum H_{\Delta,\lambda}$. If this condition holds, then for $v \in H_{\Delta,\lambda}$ we must have $\varphi(\mathfrak{g})v$ contained in a $\varphi(\Delta)$ invariant subspace V of H^∞ which is finite dimensional. The following theorem is crucial in violating the condition of finite dimensionality.

THEOREM 4.1. *Let φ be a quasisimple representation of N_4 on H^∞ such that $\varphi(Z) \neq 0$ and let b be a second order elliptic element of $\mathfrak{B}_4=\mathfrak{u}(N_4)$. If* (1) $\varphi(b)v=\lambda v$, (2) $v \in V \subseteq H^\infty$, (3) $\{\varphi(N_4)v\}\subseteq V$ *and* (4) $\varphi(b)V\subseteq V$, *then* $\{\varphi(\mathfrak{B}_4)v\}\subseteq V$.

Proof. We shall suppress the writing of φ. The proof is in the following three parts.

(1) clarify the form of b,

(2) show $\{\langle Y^j, Y^jX\rangle_{j=0}^{\infty}v\}=\{\mathfrak{B}_4 v\}$, and

(3) show for all j, $\{Y^jv, Y^jXv\}\subseteq V$.

The last step is by induction on j after we have computed bY^kv and bY^kXv.

For convenience we shall use the notation $X_1=Z$, $X_2=Y$, $X_3=W$ and $X_4=X$.

PROPOSITION 4.2. *If b is a 2nd order elliptic element of $\mathfrak{u}(N_4)$ then*

$$b = \pm \sum_{i,j=1}^{4} b_{ij}X_iX_j+\sum_{j=1}^{4} b_jX_j+b_0I$$

where $B=(b_{ij})$ is symmetric, positive definite and therefore has positive diagonal entries and eigenvalues and any principal minor has positive determinant.

Note. This proposition holds for arbitrary Lie algebras $\mathfrak{g}$ with 4 replaced by $\dim \mathfrak{g}$ in the proof.

Proof. We have for some $c_1 \in N_4+\boldsymbol{R}I$,

$$\begin{aligned} b &= \sum_{i,j=1}^{4} c_{ij}X_iX_j+c_1 \\ &= \sum_{i,j=1}^{4}\left(\frac{c_{ij}+c_{ji}}{2}\right)X_iX_j+\sum_{i,j=1}^{4}\frac{c_{ij}}{2}[X_i, X_j]+c_1. \end{aligned}$$

Thus $b_{ij}=(c_{ij}+c_{ji})/2$ yields a symmetric matrix, B. Thus, there exists an invertible orthogonal matrix $P={}^tP^{-1}=(p_{ij})$ such that $PBP^{-1}=D=(d_{ij})$, a diagonal matrix. Thus $B=({}^tP)DP$.

If we set $Y_i=\sum_{j=1}^4 p_{ij}X_j$, we find that

$$\sum_{i,j=1}^{4} b_{ij}X_iX_j = \sum_{j,i=1}^{4} d_{ij}Y_iY_j.$$

The ellipticity of the left-hand side leads to the conclusion that the d_{jj} all have the same sign and are not zero. The proposition now follows.

LEMMA 4.3. *Let $\{X_1, \ldots, X_n\}$ be a Jordan-Hölder base for a nilpotent Lie algebra $\mathfrak{g}$. Then, for $X_s \neq X_n$,*

$$X_n^t X_s \in \sum_{j=0}^{t} \mathfrak{g}_{n-1} X_n^j$$

where $\mathfrak{g}_{n-1} = \langle X_1, \ldots, X_{n-1}\rangle$.

Proof. The proof proceeds by induction on t. In fact,

$$X_n^{t-1}(X_n X_s) = X_n^{t-1} X_s X_n + X_n^{t-1} \sum_{r=1}^{s-1} c_r X_r$$

where $[X_n, X_s] = \sum_{r=1}^{s-1} c_r X_r$. Application of the induction hypothesis now finishes the proof.

PROPOSITION 4.4. *Assume the hypotheses of Theorem* 4.1. *Then $\{\mathfrak{B}_4 v\} = \{\mathfrak{B}_3 v\} + \{\mathfrak{B}_3 X v\}$ where $\mathfrak{B}_3 = \mathfrak{u}(\langle Z, Y, W\rangle)$.*

Proof. By the PBW theorem $\mathfrak{B}_4 = \sum_{j=0}^{\infty} \mathfrak{B}_3 X^j$. We prove by induction on j that for all j, $\{\mathfrak{B}_3 X^j v\} \subseteq \{\mathfrak{B}_3 v\} + \{\mathfrak{B}_3 X v\}$. Obviously this holds for $j=0$ or 1 and we assume it holds for $0 < s \leqq j$. We consider $aX^{j+1}v$ where $a \in \mathfrak{B}_3$ and let $b' = a/b_{44}$.

$$\begin{aligned} aX^{j+1}v &= b'X^{j-1}b_{44}X^2 v + b'X^{j-1}(\lambda - b)v \\ &= b'X^{j-1}\left(\lambda - \sum_{(i,s)\neq(4,4)} b_{is}X_i X_s - \sum_{s=1}^{4} b_s X_s - b_0\right)v. \end{aligned}$$

Lemma 4.3 and the induction hypothesis lead to the desired conclusion.

PROPOSITION 4.5. *Assume the hypotheses of Theorem* 4.1 *and also suppose $a_1, a_2 \in \mathfrak{B}$ and $\chi(Z) \neq 0$ where χ is the central character of φ. Then*

(a) *$a_1 Z a_2 v = \chi(Z) a_1 a_2 v$,*

(b) *$a_1 W a_2 v = (a_1/2\chi(Z)) Y^2 a_2 v - (\chi(Y^2 - 2ZW)/2\chi(Z)) a_1 a_2 v$.*

Proof. (a) $a_1 Z a_2 v = a_1 a_2 Z v = a_1 a_2 \chi(Z) v = \chi(Z) a_1 a_2 v$.

(b)
$$\begin{aligned} a_1 W a_2 v &= a_1\left[W - \frac{2Z}{2\chi(Z)}\, W + \frac{Y^2}{2\chi(Z)} - \frac{\chi(Y^2 - 2ZW)}{2\chi(Z)}\right] a_2 v \\ &= \frac{a_1}{2\chi(Z)}\, Y^2 a_2 v - \frac{\chi(Y^2 - 2ZW)}{2\chi(Z)}\, a_1 a_2 v. \qquad \text{Q.E.D.} \end{aligned}$$

We now conclude that

$$\{\mathfrak{B}_3 v\} = \{\mathfrak{u}(\langle Z, Y, W\rangle) v\} = \{\mathfrak{u}(\langle Y\rangle) v\}$$

and

$$\{\mathfrak{B}_3 X v\} = \{\mathfrak{u}(\langle Y\rangle) X v\}.$$

Proposition 4.4 now tells us that

$$\{\mathfrak{B}_4 v\} = \{\langle Y^j, Y^j X\rangle_{j=0}^{\infty} v\}. \tag{4.6}$$

We have now completed the first two parts of the proof of Theorem 4.1. The proof will be complete if we show the following.

Claim 4.7. For all $s \geqq 1$, $Y^s v$, $Y^{s-1}Xv \in V$.

Proof. We induce on s. We know from condition (3) of Theorem 4.1 that Yv, Xv and Wv are in V. By Proposition 4.5(b), $Y^2 v \in V$. Assume the claim holds for all s such that $1 \leqq s \leqq j$ and we will show the claim holds for $j+1$. We now compute $bY^t v$ and $bY^t Xv$ where b is as in Proposition 4.2. In fact, $bY^t v = Y^t bv + [b, Y^t]v = \lambda Y^t v + [b, Y^t]v$ and so we shall calculate $[b, Y^t]v$ (and $[b, Y^t X]v$ in turn). Using Proposition 4.2 we have

$$\begin{aligned}
[b, Y^t]v &= \sum_{i,j=1}^{4} b_{ij}[X_i X_j, Y^t]v + \sum_{j=1}^{4} b_j [X_j, Y^t]v + 0 \\
&= \sum_{i,j=1}^{4} b_{ij}\{X_i[X_j, Y^t] + [X_i, Y^t]X_j\}v + b_4[X_4, Y^t]v \\
&= \left\{\sum_{i=1}^{4} b_{i4}X_i[X, Y^t] + \sum_{j=1}^{4} b_{4j}[X, Y^t]X_j + b_4[X, Y^t]\right\}v.
\end{aligned}$$

We have used the fact that ad Y is 0 on $\langle X_1, X_2, X_3\rangle$. We notice that $[X, Y^t]v = t\chi(Z)Y^{t-1}v$. Thus

$$[b, Y^t]v = \left\{\sum_{j=1}^{4} b_{j4}(X_j t\chi(Z)Y^{t-1} + t\chi(Z)Y^{t-1}X_j) + b_4 t\chi(Z)Y^{t-1}\right\}v$$

so that

$$[b, Y^t]v \in \left\{\sum_{j=0}^{t} CY^j + 2b_{34}t\chi(Z)Y^{t-1}W + b_{44}t\chi(Z)(Y^{t-1}X + XY^{t-1})\right\}v.$$

But, $XY^{t-1}v = Y^{t-1}Xv + (t-1)\chi(Z)Y^{t-2}v$. This fact and Proposition 4.5 yield

$$bY^t v \in \left\{\sum_{j=0}^{t} CY^j\right\}v + b_{34}tY^{t+1}v + 2b_{44}t\chi(Z)Y^{t-1}Xv. \tag{4.8}$$

We list some identities which will be of use.

(4.9) $[X, Y^m] = mZY^{m-1}$.

(4.10) $[X^2, Y^m] = mZXY^{m-1} + mZY^{m-1}X = 2mZY^{m-1}X + m(m-1)Z^2Y^{m-2}$.

(4.11) $AB = BA + [A, B]$ for any $A, B \in \mathfrak{B}_4$.

We now calculate the effect of left multiplication by b acting on $Y^t Xv$ in terms of the basis suggested by (4.6). It is sufficient to calculate $[b, Y^t X]v$. By using Leibniz's rule and the specific form for b we have

$$\begin{aligned}
[b, Y^t X]v = &\sum_{i,j=1}^{4} b_{ij}X_i[X_j, Y^t]Xv \\
&+ \sum_{i,j=1}^{4} b_{ij}[X_i, Y^t]X_j Xv + \sum_{i,j=1}^{4} b_{ij}Y^t X_i[X_j, X]v \\
&+ \sum_{i,j=1}^{4} b_{ij}Y^t[X_i, X]X_j v + \sum_{j=1}^{4} b_j[X_j, Y^t X]v.
\end{aligned}$$

We note again that ad Y kills $\langle Z, Y, W\rangle$. Also we combine the second pair of sums and use (4.11),

$$\begin{aligned}[b, Y^tX]v &= \sum_{j=1}^{4} b_{i4}X_i[X, Y^t]Xv+\sum_{j=1}^{4} b_{4j}[X, Y^t]X_jXv \\ &\quad+\sum_{i,j=1}^{4} b_{ij}Y^t\{2[X_j, X]X_i+[X_i, [X_j, X]]\}v+\sum_{j=1}^{4} b_j[X_j, Y^tX]v \\ &= \sum_{s=1}^{4} b_{s4}t\chi(Z)\{X_sY^{t-1}+Y^{t-1}X_s\}Xv \\ &\quad+\sum_{i,j=1}^{4} b_{ij}Y^t\{2[X_j, X]X_i+[X_i, [X_j, X]]\}v+\sum_{j=1}^{4} b_j[X_j, Y^tX]v.\end{aligned}$$

We have used (4.2) and (4.9). We now use (4.11) and separate the second sum.

$$\begin{aligned}[b, Y^tX]v &= \sum_{s=1}^{4} b_{s4}t\chi(Z)\{2Y^{t-1}X_s+[X_s, Y^{t-1}]\}Xv \\ &\quad+\sum_{i,j=1}^{4} 2b_{ij}Y^t[X_j, X]X_iv \\ &\quad+\sum_{i,j=1}^{4} b_{ij}Y^t[X_i, [X_j, X]]v+\sum_{j=1}^{4} b_j[X_j, Y^tX]v.\end{aligned}$$

Again we need only consider the higher order coefficients. Thus,

$$\begin{aligned}bY^tXv \in &\sum_{l=0}^{t+2} CY^lv+\sum_{l=0}^{t} CY^lXv \\ &+b_{34}t\chi(Z)2Y^{t-1}WXv+b_{44}t\chi(Z)2Y^{t-1}X^2v \\ &+2b_{33}Y^t[W, X]Wv+2b_{43}Y^t[W, X]Xv.\end{aligned}$$

Finally, rearranging and using (4.5(b)) and (4.2) we have

$$\begin{aligned}bY^tXv \in &\sum_{l=0}^{t+2} CY^lv+\sum_{l=0}^{t} CY^lXv \\ &+b_{34}(t-2)Y^{t+1}Xv-(b_{33}/\chi(Z))Y^{t+3}v+b_{44}2t\chi(Z)Y^{t-1}X^2v.\end{aligned} \tag{4.12}$$

We define

$$V_r = \sum_{l=0}^{r+2} CY^lv+\sum_{l=0}^{r} CY^lXv.$$

We now consider the term $b_{44}Y^sX^2v = Y^s(\lambda-(b-b_{44}X^2))v$. By (4.2),

$$b_{44}Y^sX^2v = \left\{\lambda Y^s-\sum_{i,j=1;(i,j)\neq(4,4)}^{4} b_{ij}Y^sX_iX_j-\sum_{j=1}^{4} b_jY^sX_j-b_0Y^s\right\}v.$$

Close inspection of this yields the following

$$b_{44}Y^sX^2v \in V_{s+1}-(2b_{34}/2\chi(Z))Y^{s+2}Xv-(b_{33}/4\chi(Z)^2)Y^{s+4}v.$$

Incorporating this into (4.12) gives the desired calculation.

$$bY^tXv \in V_t - (b_{33}/2\chi(Z))(t+2)Y^{t+3}v - b_{34}(t+2)Y^{t+1}Xv. \tag{4.13}$$

Now we are prepared to finish the proof of Claim 4.7. First we show $Y^{j+1}v \in V$. By inductive hypothesis $Y^{j-2}Xv \in V$ and so $bY^{j-2}Xv \in V$. (4.13) coupled with the fact that (b_{ij}) is positive definite immediately implies $Y^{j+1}v \in V$.

Showing that $Y^{j+1}Xv \in V$ is more complicated. First note that since $Y^{j+1}v \in V$ we must have $bY^{j+1}v \in V$. According to (4.8) (and using the fact that V is a vector space)

$$b_{34}Y^{j+2}v + 2\chi(Z)b_{44}Y^jXv \in V. \tag{4.14}$$

In similar fashion we observe that $Y^{j-1}v \in V$, $bY^{j-1}Xv \in V$ and (4.13) leads to

$$b_{33}Y^{j+2}v + 2\chi(Z)b_{34}Y^jXv \in V. \tag{4.15}$$

An obvious linear combination of (4.14) and (4.15) yields the fact that $(b_{34}b_{43} - b_{33}b_{44})Y^jXv \in V$. But $(b_{34}b_{43} - b_{33}b_{44})$ is minus the determinant of a principal minor of the positive definite matrix (b_{ij}) and so is nonzero.

We have completed the proof of Claim 4.7 and Theorem 4.1.

If we have $\chi(Z)=0$, then $\varphi(Y^2-2ZW)=\varphi(Y^2)=\varphi(Y)^2$, a constant. $\varphi(Y)$ may change the sign of some things and leave others constant, but $\varphi(Y)$ is then the sum of two constant operators. In this case the representation acts on $\langle Y, W, X\rangle$ like the sum of two quasisimple representations. We thus have the following theorem.

THEOREM 4.16. *Let φ be a quasisimple representation of N_4 and set $\chi=\varphi|z(\mathfrak{B}_4)$. Then*

(a) *if $\chi(Z)=0$, then for all second order elliptic $b \in \mathfrak{B}_4$ with $\varphi(b)$ symmetric we have $\mathfrak{g}b_f \subseteq b_f$.*

(b) *If $\chi(Z)\neq 0$ and φ is infinite dimensional, then for all second order elliptic $b \in \mathfrak{B}_4$ with $\varphi(b)$ symmetric, $\mathfrak{g}b_f \nsubseteq b_f$.*

5. **Reductions in the nilpotent case.** We now consider the question of which nilpotent Lie algebras, $\mathfrak{g}$, have second order elliptic $b \in \mathfrak{B}$ such that $\mathfrak{g}b_f \nsubseteq b_f$ for some representation.

Consider P, a homomorphism from a Lie algebra $\mathfrak{g}$ onto N_4, and $b \in \mathfrak{u}(\mathfrak{g})$ such that $P(b)$ is a second order elliptic element of N_4. Then there exist irreducible unitary representations U of G (the simply connected group corresponding to $\mathfrak{g}$) with $dU(\mathfrak{g})b_f \nsubseteq b_f$. In fact, exponentiate P to a homomorphism π of G onto G_4, the analytic group corresponding to N_4. Composition of π with an appropriate representation of G_4 will do. Any of the representations $U_{\lambda,\mu}$ with $\lambda\neq 0$ given by Dixmier [2] are unitary, irreducible and have nontrivial action on Z.

It is obvious that N_4 can in fact be replaced by any Lie algebra for which $\mathfrak{g}b_f \nsubseteq b_f$ for all second order elliptic b.

We have seen that it is sufficient to consider nilpotent Lie algebras for which $\dim z(\mathfrak{g})=1$ as long as we are interested in quasisimple representations. By dividing a nilpotent group by its center or other subgroups which arise from the composition series we can "lower the class (length of composition chain)" of the group, i.e. we can study representations of a lower class homomorphic image of the group. It is thus critical to examine the class 3 nilpotent Lie algebras.

THEOREM 5.1. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra with class* $\mathfrak{g}=3$ *and with* $z(\mathfrak{g})=\langle Z\rangle$ $(\dim z(\mathfrak{g})=1)$. *Then*

(a) *there exist* X *and* Y *in* $\mathfrak{g}$ *such that* $\langle Z, Y, X\rangle\cong N_3$ *and* Y *is a pure commutator, that is, there exist* M *and* N *in* $\mathfrak{g}$ *such that* $Y=[M, N]$,

(b) *there exists* $l\in\mathfrak{g}$ *such that* $[l, [l, \mathfrak{g}]]=\langle Z\rangle$,

(c) $\mathfrak{g}$ *contains a subalgebra isomorphic to* $N_4=\langle A, B, C, D\rangle$ *which is characterized by* $[D, C]=B$, $[D, B]=A$, $[C, B]=0=[A, N_4]$.

Proof. (a) Since class $\mathfrak{g}=3$ and $\dim z(\mathfrak{g})=1$ we have $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]=\langle Z\rangle$. Thus there exist $L, M, N\in\mathfrak{g}$ with $[L, [M, N]]=Z$. Set $X=L$ and $Y=[M, N]$.

(b) Now $[M, [M, N]]\in z(\mathfrak{g})$ so $[M, Y]=0$ or else we are done. Similarly $[N, Y]=0$ and $[X, [X, N]]=0$. We also must have $\mathrm{ad}^2\,(X+M)(N)=0+Z+[M, [X, N]]+0=0$. So $[M, [N, X]]=Z$ and by the Jacobi identity $[N, [X, M]]=-2Z$. A direct computation now reveals that $\mathrm{ad}^2\,(M+N)(X)=3Z$ so that we may take $l=M+N$.

(c) Consider now $l\in\mathfrak{g}$ given by part (b). There exists $W\in\mathfrak{g}$ such that $[l, [l, W]]=Z$. Let a be the constant such that $[W, [l, W]]=aZ$. Then $N_4=\langle A, B, C, D\rangle$ is given by $\langle Z, [l, W], -al+W, l\rangle$. Q.E.D.

Unfortunately, Theorem 5.1 does not settle the question just yet, since N_4 is not a homomorphic image of every class 3 nilpotent Lie algebra.

It is our conjecture that for a nilpotent Lie algebra $\mathfrak{g}$, class $\mathfrak{g}\leqq 2$ if and only if for all second order elliptic elements $\Delta\in\mathfrak{u}(\mathfrak{g})$, $\mathfrak{g}\Delta_f\subseteq\Delta_f$ for all unitary irreducible representations of the Lie group corresponding to $\mathfrak{g}$. While we are unable to prove this we offer the following theorem.

THEOREM 5.2. *Let* G *be a real nilpotent Lie group with Lie algebra* $\mathfrak{g}$ *such that class* $\mathfrak{g}=3$, $\dim z(\mathfrak{g})=1$ *and* general dimension $(\mathfrak{g})=p$. *Then there is an element* $\Delta\in\mathfrak{u}(\mathfrak{g})$ *equal to the sum of squares of a basis with the following properties.*

(a) *For all unitary irreducible representations* U *of* G *with* $dU(Z)\neq 0$, U *is realizable on* $L^2(\boldsymbol{R}^p)$ *and*

$$dU(\Delta) = D^2_{x_p}-\sum_{j=0}^{4} M^j_{x_p}a_j$$

(*acting on* $S(\boldsymbol{R}^p)$, *the Schwartz space of* $\boldsymbol{R}^p$) *with* $a_4\in\boldsymbol{R}$, $a_4>0$ *and* $a_j\in\boldsymbol{P}(S(\boldsymbol{R}^{p-1}))$, *the polynomial differential operators on* $S(\boldsymbol{R}^{p-1})$.

(b) *If* $p=1$, *then* (*letting* $x_1=t$)

$$dU(\mathfrak{g})\Delta_f \supseteq \langle M_1, M_{x_t}, D_t\rangle\Delta_f \nsubseteq \Delta_f.$$

$D_j=D_{x_j}$ *is the operation of differentiation with respect to* x_j. (*The careful reader will note the domain of these operators.*)

Proof. The proof is conducted by applying an explicit formula for dU due to Dixmier [1, Lemmas 30 and 31] to a carefully chosen basis. It is a matter of general theory [11] that we can form a direct decomposition $\mathfrak{g}=\langle Z\rangle+\langle Y\rangle+\bar{\mathfrak{g}}+\langle X\rangle$ with $\mathfrak{g}_1=(\mathrm{ad}\ Y)^{-1}(0)=\langle Z\rangle+\langle Y\rangle+\bar{\mathfrak{g}}$ and $\langle Z, Y, X\rangle\cong N_3$. Also U is induced from a unitary irreducible representation, T, of G_1, the subgroup corresponding to $\mathfrak{g}_1$ and U is realizeable on $L^2_{\mathbf{C}}(\boldsymbol{R}^p)$ with $H^\infty=S(\boldsymbol{R}^p)$ and $dU(\mathfrak{B})=\boldsymbol{P}(S(\boldsymbol{R}^p))$. In fact, we have from Dixmier that

$$
\begin{aligned}
&dU(X) = D_p,\\
(5.3)\qquad &dU(l) = \sum_{j=0}^{n} (M_{x_p})^j/j!\ dT(\mathrm{ad}^j\ X(l)) \quad \text{for } l\in\mathfrak{g}_1.
\end{aligned}
$$

(We interpret dT as a representation on $S(\boldsymbol{R}^{p-1})$ which is "contained" in $S(\boldsymbol{R}^p)$.)

(a) $\mathfrak{g}$ contains N_4 as a subalgebra. In fact, there is a base of $\mathfrak{g}$ such that $\langle X_1, X_2, \ldots, X_w, \ldots, X_n\rangle=\mathfrak{g}$ with $\langle X_1, \ldots, X_{n-1}\rangle=\mathrm{ad}^{-1}\ X_2(0)$ and

$$N_4 \cong \langle X_1, X_2, X_w, X_n\rangle.$$

Applying 5.3 and noting that class $\mathfrak{g}=3$ we have

$$dU(l) = \sum_{k=0}^{2} M^k_{x_p}\ dT[\mathrm{ad}^k\,(X_n)(l)]/k!.$$

We can even choose T so that $dT(X_2)=0$ and so the action of dU is given by $X_1\mapsto M_i$, $X_2\mapsto i\lambda M_{x_p}$, $X_w\mapsto dT(X_w)+0+M^2_{x_p}(i\lambda/2)$, and $X_n\mapsto D_p$ for some $\lambda\in\boldsymbol{R}$. If we take Δ to be the sum of squares of the basis $\{X_1, X_2, \ldots, X_w, \ldots, X_n\}$ we see that Δ is sent into the desired form. We now show $a_4>0$. $dU(X_w)^2$ gives a positive real contribution to a_4. The only other contributions will be of the form $dT([X_n, [X_n, X_j]]/2)^2$. But class $\mathfrak{g}=3$ and so $[X_n, [X_n, X_j]]\in z(\mathfrak{g})$ which is sent to some multiple of $i\lambda$. Thus $a_4\in\boldsymbol{R}$ and $a_4>0$.

(b) Clearly $dU(\mathfrak{g})\Delta_f\supseteq\langle D_p, M_{x_p}, M_1\rangle\Delta_f$ for all p. If we take $p=1$, we have $\Delta=D^2-\sum_{j=0}^4 a_jt^j$ where $D=D_p$ and we write t for M_{x_p}.

The proof is now entirely analogous to the proof of Theorems 4.1 and 4.16. We consider that $\Delta v=\lambda v\in\langle D, t^2, t, 1\rangle V\leqq V$ and $\Delta V\subseteq V$ for a subspace $V\leqq H^\infty=S(\boldsymbol{R})$—and we show dim $V\not<\infty$ since $\{t^jv\}_{j=1}^\infty\subseteq dU(\mathfrak{B})V\subseteq V$. In fact, $\boldsymbol{P}(S(\boldsymbol{R}))v=P[t]v+P[t]Dv$ where $P[t]$ is the polynomials in t. $t^jv, t^{j-1}Dv\in V$ are proven in just the same way as before. Q.E.D.

Bibliography

1. J. Dixmier, *Sur les représentations unitaires des groupes de Lie nilpotents*. II, Bull. Soc. Math. France **85** (1957), 325–388. MR **20** #1928.

2. ———, *Sur les représentations unitaires des groupes de Lie nilpotents*. III, Canad. J. Math. **10** (1958), 321–348. MR **20** #1929.

3. R. Godement, *A theory of spherical functions*. I, Trans. Amer. Math. Soc. **73** (1952), 496–556. MR **14**, 620.

4. R. Goodman, *Analytic domination by fractional powers of a positive operator*, J. Functional Analysis **3** (1969), 246–264. MR **39** #801.

5. R. Goodman, *Spaces of analytic vectors for representations of Lie groups* (preliminary version).

6. Harish-Chandra, *Representations of a semisimple Lie group on a Banach space*. I, Trans. Amer. Math. Soc. **75** (1953), 185–243. MR **15**, 100.

7. A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk **17** (1962), no. 4 (106), 57–110. (Russian) MR **25** #5396.

8. R. P. Langlands, *On Lie semi-groups*, Canad. J. Math. **12** (1960), 686–693. MR **22** #12401.

9. E. Nelson, *Analytic vectors*, Ann. of Math. (2) **70** (1959), 572–615. MR **21** #5901.

10. E. Nelson and W. F. Stinespring, *Representation of elliptic operators in an enveloping algebra*, Amer. J. Math. **81** (1959), 547–560. MR **22** #907.

11. L. Pukánsky, *Leçons sur les représentations des groupes*, Monographies de la Société Mathématique de France, no. 2, Dunod, Paris, 1967. MR **36** #311.

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