ON THE EQUIVALENCE OF MULTIPLICITY
AND THE GENERALIZED TOPOLOGICAL DEGREE

BY

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Abstract. In this paper we first extend the definition of the multiplicity (as defined by J. Cronin-Scanlon) of operators of the form $I+C+T$ to operators of the form $H+C+T$. We then show that the generalized topological degree (as defined by F. E. Browder and W. V. Petryshyn) of operators of the form $H+C+T$ is also defined. Finally, we show that when both the multiplicity and generalized topological degree of $H+C+T$ are defined, they are equal.

1. **Introduction.** In [3], [4], [5], [6] and [7] J. Cronin-Scanlon studied operators on a Banach space of the form $(I+C+T)(x)=y$, where C is a compact linear operator and T satisfies a contraction condition near the origin. The Leray-Schauder degree of $I+C+T$, in general, is not defined. Cronin-Scanlon's method for investigating the existence of solutions of the equation $(I+C+T)(x)=y$ was to define the multiplicity of $I+C+T$, denoted by $m(I+C+T)$, which had properties similar to the topological degree of an operator. Using these properties she was then able to examine such questions as existence of solutions and the number of solutions. In [5] she actually defined the degree of a mapping in the form $I+C+T$ (along with some additional conditions) defined on a Hilbert space in terms of $m(I+C+T)$.

In [1] and [2] F. E. Browder and W. V. Petryshyn extended the definition of topological degree to A-proper mappings. This class of mappings is much larger than the class of mappings for which the Leray-Schauder degree is defined. In this paper we shall show that the mappings of the form $I+C+T$ can be investigated through the generalized degree of Browder and Petryshyn. We shall show that not only is the generalized topological degree of $I+C+T$ defined with respect to some neighborhood of θ, but it is, in fact, equal (up to a multiplicative ± 1) to $m(I+C+T)$. Thus we see that the theory of generalized topological degree can then be applied to the integral equations that were studied by Schmidt [14].

In §2 instead of stating the results which we need from [3], we shall extend the definition of multiplicity of an operator to include mappings of the form $H+C+T$, where H is a linear homeomorphism. In §3 we shall show that not only is the map $H+C+T$ (defined on a Banach space with a certain projectional structure) A-proper with respect to some neighborhood of θ, but also that it satisfies a theorem

of Browder and Petryshyn which will give the generalized degree of $H+C+T$ some especially nice properties. In §4 we shall then show that if the Banach space on which $H+C+T$ is defined has the projectional structure that allows the generalized topological degree of $H+C+T$ to be defined, then this degree is, in fact, ± 1 times the multiplicity of $H+C+T$. Finally in §5 we shall give an example to which the above work can be applied.

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2. Multiplicity. Let X be a Banach space, H a linear homeomorphism mapping X into X and C a compact linear map from X into X. Let $X_1 = N(H+C) = $ null space of $H+C$ and $X_1^* = N(H^*+C^*)$. In [13] F. Riesz obtained results for $N(I+C)$ where C is compact. Analogously, and hence presented without proof, we obtain the following results: (1) X_1 is finite dimensional, (2) X_1^* is finite dimensional and dimension $X_1 = $ dimension X_1^*, and (3) there exists a projection from X onto X_1, E_1, such that $X = X_1 + X^1$ where $X^1 = E^1 X = (I - E_1)X$.

Now let $x_1, \ldots, x_n$ and $f_1, \ldots, f_n$ be bases for X_1 and X_1^* respectively. Then there exist $g_1, \ldots, g_n \in X^*$ and $y_1, \ldots, y_n \in X$ such that

(1) for $x \in X$, $E_1(x) = \sum_{i=1}^n g_i(x)x_i$ $(g_i(x_j) = \delta_{ij})$, and

(2) $f_i(y_j) = \delta_{ij}$.

Denote by C_0 the transformation

$$Cx + \sum_{i=1}^n g_i(x)y_i.$$

C_0 is then compact and $N(H+C_0) = \{\theta\}$. Thus $H+C_0$ has a linear inverse $R = H_1 + C_1$ where H_1 is a linear homeomorphism $(H_1 = H^{-1})$ and C_1 is linear and compact. Then, analogous to the result of Cronin-Scanlon [7], we can show that, for all $x \in X$, $R(H+C)(x) = (I - E_1)x = E^1 x$.

Along with the above facts we shall need the Implicit Function Theorem which is due to Hildebrandt and Graves [9].

THEOREM 1 (IMPLICIT FUNCTION THEOREM). *Let X, Y, Z be Banach spaces and let U, V, W be open sets in X, Y, Z, respectively. Let L be a function with domain $U \times V \times W$ and range a subset of X. Assume that the following conditions are satisfied:*

(1) *there is a point $(x_0, y_0, z_0) \in U \times V \times W$ such that $x_0 = L(x_0, y_0, z_0)$,*

(2) *there exists a positive number $c < 1$ such that $\|L(x_1, y, z) - L(x_2, y, z)\| \leq c\|x_1 - x_2\|$ for every $(x_1, y, z), (x_2, y, z) \in U \times V \times W$,*

(3) *L is uniformly continuous on $U \times V \times W$.*
Then the following conditions hold:

(1) *for each $(y, z) \in V \times W$ there is at most one point $(x, y, z) \in U \times V \times W$ which satisfies the equation*

(1) $$x = L(x, y, z),$$

(2) *there exist open sets V_1 and W_1, spherical neighborhoods of y_0 and z_0, respectively, and a function F with domain $V_1 \times W_1$ and range a subset of X such that the point $(F(y, z), y, z)$ is an element of $U \times V \times W$ and $x = F(y, z)$ is a solution of equation (1) for every $(y, z) \in V_1 \times W_1$,*

(3) *the solution $F(y, z)$ of equation (1) is uniformly continuous on $V_1 \times W_1$.*

We are now ready to consider the equation

$$(2) \qquad (H + C + T)(x) = y.$$

We suppose that T satisfies the following properties:

(1) T is defined on some ball B in X centered at θ,

(2) $T(\theta) = \theta$, and

(3) if $x, y \in B$, then $\|Tx - Ty\| \leqq M(x, y)\|x - y\|$ where $M(x, y)$ is a positive-valued function such that

$$\lim_{(x,y) \to (\theta,\theta)} M(x, y) = 0.$$

We note that when $(H + C)^{-1}$ exists, equation (2) can be multiplied by $(H + C)^{-1}$ and solved by the Implicit Function Theorem. If $H + C$ does not have an inverse, we multiply both sides of equation (2) by $E_1 R$ and $E^1 R$, and hence reduce equation (2) to the equations

$$(3) \qquad E_1 R T(x_1 + x^1) = E_1 R y,$$

$$(4) \qquad x^1 + E^1 R T(x_1 + x^1) = E^1 R y,$$

where $x_1 = E_1 x$ and $x^1 = E^1 x$.

If we then apply the Implicit Function Theorem to equation (4), we have the following theorem.

THEOREM 2 (CRONIN-SCANLON [3]). *There exist open sets, V_1 and W_1, spherical neighborhoods of $x_1 = \theta$ and $y = \theta$, respectively, and a uniformly continuous function F with domain $V_1 \times W_1$ and range in X, such that $x^1 = F(x_1, y)$ is a solution to equation (4).*

We then substitute $x^1 = F(x_1, y)$ into equation (3) to obtain

$$(5) \qquad E_1 R T(x_1 + F(x_1, y)) - E_1 R y = \theta.$$

Solving equation (5) is obviously equivalent to solving equation (2). Thus we make the following definition.

DEFINITION 1. *The multiplicity of solutions of the equation*

$$(H + C + T)(x) = y_0,$$

denoted by $m(H + C + T)$, is defined as

$$m(H + C + T) = +1, \quad \text{when } (H + C)^{-1} \text{ exists,}$$

$$= d(E_1 R T(x_1 + F(x_1, y_0)) - E_1 R(y_0)_{x_1}, B \cap X_1, \theta), \quad \text{otherwise,}$$

where B is a sufficiently small ball in X centered at θ and the degree above is the usual Brouwer degree.

Cronin-Scanlon, in [3], using the definition of multiplicity similar to Definition 1, places an additional condition on T and then derives methods for calculating $m(I+C+T)$. These methods would also be applicable to calculating $m(H+C+T)$.

3. **The generalized degree of** $H+C+T$. In [1] and [2], F. E. Browder and W. V. Petryshyn defined a generalized topological degree. We shall be interested in a slight variation of some of their definitions and results. Definitions 2–4 below are the same as those given in [2] except that we shall replace their mapping P_n, which maps X_n into X, by the inclusion map. The inclusion map will surely satisfy the properties of the mapping P_n. We then obtain the following definitions.

DEFINITION 2 [2, DEFINITION 3.1]. *Let X be a real Banach space. By a projective approximation scheme for mappings from X into X, denoted by Γ, we mean a sequence of oriented finite-dimensional subspaces $\{X_n\}$ and a sequence of linear projections from X onto X_n, $\{Q_n\}$, such that $Q_n x \to x$ for each $x \in X$.*

It can be shown [2] under the conditions of Definition 2 that we also get (1) $\{Q_n\}$ is uniformly bounded and (2) if F is a given finite-dimensional subspace of X, then there exists an integer $n_F \geq 1$ such that Q_n is injective on F for each $n \geq n_F$.

DEFINITION 3 [2, DEFINITION 1.2]. *Let G be an open subset of X and W a mapping of $\overline{G}$ (the closure of G) into X. W is said to be A-proper on $\overline{G}$ with respect to a given approximation scheme $\Gamma = (\{X_n\}, \{Q_n\})$ if and only if, for any sequence $\{n_j\}$ of positive integers with $n_j \to \infty$ and a corresponding sequence $\{x_{n_j} \mid x_{n_j} \in X_{n_j}\}$ with $x_{n_j} \in \overline{G}$ such that $\|Q_{n_j} W x_{n_j} - Q_{n_j} y\| \to 0$ for some $y \in X$, there exists an infinite subsequence $\{x_{n_{j(k)}}\}$ and an element $x \in X$ such that $x_{n_{j(k)}} \to x$ and $Wx = y$.*

Let us next introduce the notation $W_n = Q_n W$ and proceed to define the generalized topological degree as follows.

DEFINITION 4 [2, DEFINITION 1.3]. *Let W be an A-proper continuous mapping from $\overline{G}$ into X with respect to Γ. Let $G_n = X_n \cap G$ be bounded for all n and suppose that $a \notin W(\partial G)$.*

We define $D(W, G, a)$, the degree of W on G over a with respect to Γ, as follows: Let Z' be the set of all integers together with $\pm\infty$. Then $D(W, G, a)$ is the subset of Z' given by $D(W, G, a) = \{\gamma \mid \gamma \in Z'$, there exists an infinite sequence $\{n_j\}$ of positive integers with $n_j \to \infty$ such that $d(W_{n_j}, G_{n_j}, Q_{n_j} a) \to \gamma\}$.

To keep the concepts of generalized degree separate from those of the more classical degree, we shall use a capital D to denote the generalized degree and a lower case d to denote the Brouwer degree (when in R^n) and the Leray-Schauder degree.

The results we obtain in this paper do not use the most general form of the generalized degree as defined above. Whenever we are confronted with the generalized degree of a function we shall be able to apply the following theorem. This theorem is Theorem 2 in [2] and for that reason we do not include the proof.

THEOREM 3. *Let h and K be two continuous mappings of $\overline{G}$ into X such that if we assume $G_n = X_n \cap G$ are bounded and set $h_n = Q_n h|_{\overline{G}_n}$, the following hypotheses are satisfied:*

(1) *h is an A-proper homeomorphism of G onto an open subset $h(G)$ of X with h mapping $\overline{G}$ homeomorphically onto $\overline{h(G)}$.*

(2) *For each n, h_n is a homeomorphism of G_n onto the open subset $h_n(G_n)$ of X_n and h_n maps $\overline{G}_n$ homeomorphically onto $\overline{h(G)}$.*

(3) *There exists a continuous function $\alpha: [0, \infty) \to [0, \infty)$ such that $r_i \to 0$ whenever $\alpha(r_i) \to 0$, for which $\|h_n u - h_n v\| \geqq \alpha(\|u - v\|)$ for all u and v in G_n and all $n \geqq 1$.*

(4) *K is compact.*

Consider now the mapping $W = h + K: \overline{G} \to X$ and suppose that $a \notin W(\partial G)$. Then the following are true:

(1) *W is A-proper.*

(2) *There exists an integer $n_0 \geqq 1$ such that $d(W_n, G_n, Q_n a)$ is defined for each $n \geqq n_0$.*

(3) *There exists an integer $n_1 \geqq 1$ such that, for all $n \geqq n_1$, $|d(W_n, G_n, Q_n a)| = |d(I + K h^{-1}, h(G), a)|$ and is independent of the choice of n.*

(4) *$D(W, G, a) = \{\pm d(I + K h^{-1}, h(G), a)\}$.*

In [2], condition (2) required that h_n be an orientation preserving homeomorphism. If this condition is assumed, then result (3) becomes

$$d(W_n, G_n, Q_n a) = d(I + K h^{-1}, h(G), a)$$

and condition (4) becomes

$$D(W, G, a) = \{d(I + K h^{-1}, h(G), a)\}.$$

Since our result will be unchanged by a multiplicative ± 1, it is not necessary to assume that h_n be orientation preserving. For convenience, however, we shall assume that h_n is orientation preserving in our lemmas and theorems. This will eliminate many cumbersome ± 1's.

In this paper we wish to apply the generalized topological degree to the mapping $H + C + T$. This mapping is not, in general, A-proper with respect to any open set G. For this reason we say that a map f is *locally A-proper at* θ if there exists an open neighborhood of θ, say N, such that f is A-proper with respect to N. We are then able to prove the following lemma.

LEMMA 1. *$I + T$ is locally A-proper at θ.*

Proof. Let K_1 be the uniform bound on $\|Q_n\|$. Using condition (3) in the definition of T we choose r so that $M(x, y) \leqq k < \text{minimum } \{\tfrac{1}{2}, 1/K_1, K_1\}$ in $B_r \times B_r$ and $r_2 = r/4K_1$. We shall prove that $I + T$ is A-proper on B_{r_2}.

Let $G = B_{r_2}$ and denote $I + T$ by W. Let $\{n_j\}$ be a subsequence of the integers, $x_{n_j} \in \overline{G}_{n_j}$ and suppose there exists a $y \in X$ such that $\|W_{n_j} x_{n_j} - Q_{n_j} y\| \to 0$.

We note that since

$$\|W_{n_j}x_{n_j}-y\| \leqq \|W_{n_j}x_{n_j}-Q_{n_j}y\|+\|Q_{n_j}y-y\|,$$

the above assumption and Definition 2 imply that $\|W_{n_j}x_{n_j}-y\| \to 0$. We also note that since

$$\|W_{n_j}x_{n_j}\| = \|Q_{n_j}(I+T)x_{n_j}\| \leqq K_1(1+k)\|x_{n_j}\| < 2K_1r_2 = r_1,$$

then $y \in B_{r_1}$ where $r_1 = 2K_1r_2$.

Claim. There exists a $z \in \bar{B}_r$ such that $Wz = y$.

Define f_y by $f_y(x) = y - Tx$, let $t_0 \in B_{r_2}$ and define $t_{n+1} = f_y(t_n)$. In an approach similar to that used in the proof of the Banach Fixed Point Theorem we can show that $t_n \in B_r$ for all n and $\{t_n\}$ is a Cauchy sequence. Let $t_n \to z$. Then $Wz = y$. We then have that

$$\begin{aligned}
\|W_{n_j}x_{n_j}-y\| &= \|Q_{n_j}Wx_{n_j}-Wz\| = \|(x_{n_j}-z)-(Tz-Q_{n_j}Tx_{n_j})\| \\
&\geqq \|x_{n_j}-z\| - \|(Tz-Q_{n_j}Tz)+(Q_{n_j}Tz-Q_{n_j}Tx_{n_j})\| \\
&\geqq \|x_{n_j}-z\| - \|Q_{n_j}Tz-Q_{n_j}Tx_{n_j}\| - \|Tz-Q_{n_j}Tz\| \\
&\geqq \|x_{n_j}-z\|(1-K_1k) - \|Tz-Q_{n_j}Tz\|.
\end{aligned}$$

Since $\|W_{n_j}x_{n_j}-y\| \to 0$, $\|Tz-Q_{n_j}Tz\| \to 0$ and $(1-K_1k)>0$, we have that $x_{n_j} \to z$.

We now use Lemma 1 to prove that $H+T$ is also locally A-proper at θ.

THEOREM 4. *$H+T$ is locally A-proper at θ.*

Proof. We shall show that $H+T$ is A-proper on some ball, $B_r = G$, such that $B_r \subset H^{-1}(B_{r_2})$, where r_2 is as in Lemma 1.

Let $\{n_j\}$ be a subsequence of integers. Suppose that $x_{n_j} \in \bar{G}_{n_j}$ and that there exists a $y \in X$ such that $\|Q_{n_j}(H+T)x_{n_j} - Q_{n_j}y\| \to 0$. But this is the same as

$$\|Q_{n_j}(I+TH^{-1})(Hx_{n_j})-Q_{n_j}y\| \to 0.$$

Furthermore, it is easy to show that TH^{-1} will also satisfy the conditions of Lemma 1 (with $\|H^{-1}\|M(x,y)$ replacing $M(x,y)$ in condition (3) on T). Therefore by Lemma 1, $Hx_{n_j} \to z$ and $(I+TH^{-1})z = y$. But then $x_{n_j} \to H^{-1}(z)$ and

$$(H+T)(H^{-1}z) = y.$$

Thus $H+T$ is locally A-proper at θ.

Thus we have that $H+T$ is A-proper on some B_r. It is not hard, then, to show that $(H+T)+C$ is A-proper on B_r. However, we are able to get more than $(H+T)+C$ being A-proper. We shall, in the following lemmas, show that $(H+T)+C$ is such that it will satisfy Theorem 3.

LEMMA 2. *$h=H+T$ is a homeomorphism from some ball B_r onto $h(B_r)$. $h(B_r)$ is open and $h(\bar{U})=\overline{h(U)}$ for any open set U such that $\bar{U} \subset B_r$.*

Proof. The fact that h is a homeomorphism follows from

$$\|x-y\|[m-k] \leqq \|Hx-Hy\| - \|Ty-Tx\| \leqq \|hx-hy\|$$
$$\leqq \|Hx-Hy\| + \|Tx-Ty\| \leqq (\|H\|+k)\|x-y\|$$

where $m=\|H^{-1}\|^{-1}$ and B_r is chosen so that $M(x, y)<k$ for $x, y \in B_r$ and $k<m$. The proof that $h(B_r)$ is open and $h(\overline{U})=\overline{h(U)}$ is similar to that for the case when $H=I$ and T is a strict contraction.

LEMMA 3. *h_n is a homeomorphism from some ball $X_n \cap B_r$ onto $h_n(X_n \cap B_r)$. $h_n(X_n \cap B_r)$ is open and $h_n(\overline{U})=\overline{h(U)}$ for any open set U such that $\overline{U} \subset B_r$ $(h_n = Q_n h)$.*

Proof. We let $k_n = I_n + Q_n T H^{-1}$ where I_n is the identity map on X_n. Then since

$$(1 - kK_1\|H^{-1}\|)\|x-y\| \leqq \|k_n x - k_n y\| \leqq (1 + kK_1\|H^{-1}\|)\|x-y\|,$$

we see that if we choose r such that $m(x, y)<k<1/K_1\|H^{-1}\|$ for $x, y \in B_r$, then k_n is a homeomorphism on B_r. Then $h_n = k_n H$ will also be a homeomorphism. The last part of the lemma is similar to the corresponding part of Lemma 2.

LEMMA 4. *h_n satisfies condition (3) of Theorem 3.*

Proof. We note that

$$\|h_n u - h_n v\| = \|k_n H u - k_n H v\| \geqq \|Hu - Hv\|(1 - kK_1\|H^{-1}\|)$$
$$\geqq \frac{1}{\|H^{-1}\|}\|u-v\|(1 - kK_1\|H^{-1}\|)$$

where k_n is as in Lemma 3. Thus we let

$$\alpha(r) = \|H^{-1}\|^{-1}(1 - kK_1\|H^{-1}\|)r$$

and condition (3) of Theorem 3 is satisfied.

Since C is compact, the mapping $H+C+T$ satisfies the hypotheses of Theorem 3. We then have the following theorem.

THEOREM 5. *There exists an r such that $D(H+C+T, B_r, \theta)$ is defined and such that $D(H+C+T, B_r, \theta)=\{d(I+Ch^{-1}, h(B_r), \theta)\}$ where $h=H+T$.*

We can now use the properties of generalized degree to investigate the existence of solutions of $(H+C+T)(x)=\theta$. In [5], Cronin-Scanlon proved the invariance under homotopy of the special degree defined in that paper. We see that since $D(H+C+T, B_r, \theta)$ is defined for some r, the invariance under homotopy follows from that corresponding property in the paper by Browder and Petryshyn [2].

We should note that the definition of multiplicity was for the equation

$$(H+C+T)(x) = y$$

for y in a sufficiently small neighborhood of θ. We can, however, use the property

that if $D(H+C+T, B_r, \theta)$ is defined then there exists a neighborhood of θ, N, such that for any $y \in N$, $D(H+C+T, B_r, y)$ is defined and is equal to

$$D(H+C+T, B_r, \theta).$$

4. **The equality of the generalized degree and multiplicity.** In this section we shall show that the multiplicity is, in fact, a special case of the generalized degree. We shall show that

(6) $$|m(H+C+T)| = |d(I+Ch^{-1}, h(B_r), \theta)|.$$

When $H+C$ is nonsingular, then $m(H+C+T)=1$. It is not hard to show that when $H+C$ is injective so also is $H+C+T$. It is then clear that $d(I+Ch^{-1}, h(G), \theta) = \pm 1$ and that (6) is satisfied.

When $H+C$ is singular, we proceed much as Cronin-Scanlon did in [3]. We define Φ and Ψ as follows:

$$\Phi(x) = x+E^1RT(x),$$

$$\Psi(x) = E^1x+E_1RT[E_1(x)+F(E_1x, E^1x)],$$

where $R=(H+C_0)^{-1}=H_1+C_1$ and F is as in (5). We note that

(7) $$R(H+C+T) = R(h+C) = \Psi\Phi.$$

This can be rewritten in the form

(8) $$(I+C_1H_1^{-1})H_1(I+Ch^{-1}) = \Psi\Phi h^{-1}.$$

Before stating Theorem 7 we state the product theorem which can be found in [11].

THEOREM 6. *Let $f=I+C_1$ and $g=I+C_2$ be two mappings of subsets of the Banach space X into X such that*

(1) *C_1 and C_2 are compact,*

(2) *G is a bounded open set such that C_1 is defined on $\bar{G}$ and C_2 is defined on the closure of A, a bounded open set containing $(I+C_1)(\bar{G})$,*

(3) *$p \notin gf(\partial G) \cup g[\partial f(G)]$.*
Then

$$d(gf, G, p) = \sum_{(i)} d(f, g, d_i)d(g, D_i, p)$$

where $\sum_{(i)}$ means that the sum over the bounded components of $X-f(\partial G)$, D_i, and d_i is an arbitrary element of D_i. Only a finite number of the terms on the right will be nonzero.

Using the above product theorem we prove a slightly more general product theorem that will help us investigate equation (8).

THEOREM 7. *Let $f=H+C_1$ and $g=I+C_2$ be two mappings of subsets of the Banach space X into X such that*

(1) *H is a homeomorphism which satisfies the hypotheses of Theorem 3 and C_1 and C_2 are compact.*

(2) *G is a bounded open set such that H and C_1 are defined on $\bar{G}$ and C_2 is defined on D where D is a bounded open set containing $f(\bar{G})$.*

(3) $p \notin gf(\partial G) \cup g[\partial f(G)]$.

Then

$$D(gf, G, p) = \sum_{(i)} d(g, D_i, p)D(f, G, d_i)$$

where the sum is taken over all of the components of $X - f(\partial G)$, D_i, and d_i is an arbitrary element of D_i. Only a finite number of the terms on the right will be nonzero.

It should be remarked that since f is such that each of the sets $D(f, G, d_i)$ is the singleton set, by $\sum_{(i)} d(g, D_i, p)D(f, G, d_i)$ we merely mean the sum and product of the integers involved.

Proof. Since H is such that it satisfies the hypotheses of Theorem 3, then $gf = (I + C_2)(H + C_1) = H + C_1 + C_2(H + C_1)$ also satisfies Theorem 3 and

$$D(gf, G, p) = \{d(I + C_1 H^{-1} + C_2(H + C_1)H^{-1}, H(G), p)\}.$$

Since $I + C_1 H^{-1} + C_2(H + C_1)H^{-1} = (I + C_2)(I + C_1 H^{-1}) = gf_1$, $(f_1 = I + C_1 H^{-1})$ we see that

$$d(I + C_1 H^{-1} + C_2(H + C_1)H^{-1}, H(G), p)$$
$$= d((I + C_2)(I + C_1 H^{-1}, H(G), p) = d(gf_1, H(G), p).$$

We now use Theorem 6 to get

$$d(gf_1, H(G), p) = \sum_{(i)} d(I + C_1 H^{-1}, H(G), e_i)d(I + C_2, E_i, p),$$

where E_i is a component of $X - f_1(\partial H(G))$ and $e_i \in E_i$.

It is not hard to show that $p \notin g[f(\partial G)] \cup g[\partial f(G)]$ implies that

$$p \notin g[f_1(\partial H(G))] \cup g[\partial f_1(H(G))]$$

(so that Theorem 6 can be applied).

Since $f_1(H(G))$ clearly equals $f(G)$, we see that $E_i = D_i$ and $e_i = d_i$.

Therefore, since

$$D(f, G, d_i) = D(H + C_1, G, d_i) = \{d(I + C_1 H^{-1}, H(G), d_i)\},$$

we see that

$$D(gf, G, p) = \sum_{(i)} d(g, D_i, p)D(f, G, d_i).$$

COROLLARY 1. *If $f(\theta) = \theta$ is an isolated θ-point of g, then there exists an r such that, when $G = B_r$,*

$$d(g, D_i, p)D(f, G, d_i) \neq \{0\}$$

for only one i.

Corollary 1 follows from Theorem 7 in much the same way as Lemma 5.2 follows from Lemma 5.1 in [3].

We can now apply Corollary 1 to equation (8) and obtain

$$(9) \quad d(I + C_1 H_1^{-1}, D_i, \theta) D(H_1(I + Ch^{-1}), G, d_i) = d(\Psi, D_i, \theta) D(\Phi h^{-1}, G, d_i).$$

We can then make use of the following lemmas.

LEMMA 5. $d(I + C_1 H_1^{-1}, D_i, \theta) = \pm 1$.

Proof. Once we note that $I + C_1 H_1^{-1}$ is injective, the lemma follows from Leray-Schauder [10, pp. 56–59].

LEMMA 6. Φh^{-1} *is injective on some neighborhood of θ and satisfies the hypotheses of Theorem 3.*

Proof. Suppose that $\Phi(h^{-1}x) = \Phi(h^{-1}y)$, i.e.

$$h^{-1}x + E^1 R T^{-1}x = h^{-1}y + E^1 R Th^{-1}y.$$

But then we also have $x + hE^1 RTh^{-1}x = y + hE^1 RTh^{-1}y$. We note that

$$\|(x + hE^1 RTh^{-1}x) - (y + hE^1 RTh^{-1}y)\| \geqq \|x - y\| - \|hE^1 RTh^{-1}y - hE^1 RTh^{-1}x\|$$
$$\geqq \|x - y\|[1 - \|hE^1 R\| M(h^{-1}y, h^{-1}x) \|h^{-1}\|].$$

Choose r small enough so that $x, y \in B_r$ implies that

$$1 - \|hE^1 R\| \, \|h^{-1}\| M(h^{-1}y, h^{-1}x) > 0.$$

Then if $x \neq y$, we have a contradiction.

Φh^{-1} satisfies the hypotheses of Theorem 3 since it is very similar to the mapping $H + T + C$ (with $C \equiv 0$).

LEMMA 7. $D(\Phi h^{-1}, G, d_i) = \{\pm 1\}$ (+1 *or* −1 *but not both*).

Proof. Using the fact that in G, Φh^{-1} is injective, this lemma follows in the same manner as Lemma 5.

LEMMA 8. *There exists a B_r such that $d(\Psi, D_i, \theta) = d(\Psi, B_r, \theta)$.*

Proof. Note that $\Psi(\theta) = \theta$ and $\theta \in D_i$. Choose B_r such that $B_r \subset D_i$. Then since θ is an isolated θ-point of Ψ, we get the above result.

The following lemma is due to Cronin-Scanlon [3, Lemma 5.4].

LEMMA 9. $d(\Psi, B_r, \theta) = d(\Psi|_{X_1}, X_1 \cap B_r, \theta)$.

The above lemmas and equation (9) then yield

$$(10) \qquad\qquad D(H_1(I + Ch^{-1}), G, d_i) = \{\pm m(H + C + T)\}.$$

It then remains to show that

$$(11) \qquad\qquad D(H_1(I + Ch^{-1}), G, d_i) = \pm D(H + C + T, G, \theta).$$

We should note that since $\theta \in D_i$, we can choose $d_i = \theta$.

To prove (11) we denote Ch^{-1} by K and proceed with the following. Suppose that $\{y_1, \ldots, y_m\}$ is an $\varepsilon/2$-net of $\overline{K(G)}$ (exists since $\overline{K(G)}$ is compact). Since, for any $x \in X$, $Q_n(x) \to x$, we see that for any $\varepsilon/2 > 0$ there exists an N_i such that $n \geq N_i$ implies that $\|Q_n(y_i) - y_i\| < \varepsilon/2$. Let $N = \max\{N_1, \ldots, N_m\}$ and consider X_n where $n \geq N$. Then for any $y \in \overline{K(G)}$ there exists a y_i such that $\|y - y_i\| < \varepsilon/2$. We then have that

$$\|Q_n(y_i) - y\| \leq \|Q_n(y_i) - y_i\| + \|y_i - y\| < \varepsilon.$$

Thus $\{Q_n(y_1), \ldots, Q_n(y_m)\}$ is an ε-net of $\overline{K(G)}$.

We next define F_ε by

$$F_\varepsilon(x) = \sum_{i=1}^{m} m_i(x) Q_n(y_i) \Big/ \sum_{i=1}^{m} m_i(x),$$

where

$$m_i(x) = \varepsilon - \|x - Q_n(y_i)\| \quad \text{if } \|x - Q_n(y_i)\| \leq \varepsilon,$$
$$= 0 \quad\quad\quad\quad\quad\quad\; \text{if } \|x - Q_n(y_i)\| > \varepsilon,$$

and choose ε sufficiently small so that

(12) $$d(I + F_\varepsilon K, X_n \cap G, \theta) = d(I + K, G, \theta).$$

Let us now choose N_1 such that, for $n \geq N_1$,

$$d(Q_n H_1(I + K), G \cap X_n, \theta) = d(I + H_1 K H_1^{-1}, H_1(G), \theta),$$

and suppose from this point on that n is always at least as large as N and N_1. Now consider the following lemmas.

LEMMA 10 [2, PROPOSITION 1].

$$d(Q_n H_1(I + K), G \cap X_n, \theta) = d(Q_n H_1(I + F_\varepsilon K), G \cap X_n, \theta).$$

LEMMA 11. *For n sufficiently large we have $d(Q_n H_1, D_i, \theta) = d(I, H_1(D_i), \theta) = 1$ if $\theta \in H_1(D_i)$.*

Proof. This follows from Theorem 3.

We now state the following theorem which can be found in [8].

THEOREM 8. *Suppose (1) the mapping f is continuous in the closure of a bounded open set $\Omega \subset R^n$, (2) the mapping g is continuous on $f(\overline{\Omega})$, and (3) $g(y) \neq u_0$ for $y \in f(\partial\Omega)$, where u_0 is fixed in R^n. Then*

$$d(gf, \Omega, u_0) = \sum_{(i)} d(f, \Omega, d_i) d(g, D_i, u_0)$$

where we sum over all of the components of $R^n - f(\partial\Omega)$, D_i, that are contained in $f(\Omega)$ and where d_i is an arbitrary point in D_i.

Theorem 8 and the preceding lemmas allow us to prove the following product theorem.

THEOREM 9. *Let H_1 and $f = I + K$ be two mappings of subsets of the Banach space X into X such that*

(1) *H_1 is a homeomorphism which satisfies the hypotheses of Theorem 3 and K is compact.*

(2) *G is a bounded open subset of X such that K is defined on $\bar{G}$ and H_1 is defined on the closure A of some bounded open set containing $f(\bar{G})$.*

(3) *$p \notin f(\partial G)$.*

Then

$$D(H_1 f, G, p) = \sum_{(i)} d(f, G, d_i) D(H_1, D_i, p)$$

where the sum is taken over all of the components, D_i, of $X - f(\partial G)$ that are contained in $f(\bar{G})$ and d_i is an arbitrary point in D_i.

Proof. For sufficiently large n, Lemma 10 and Theorem 8 allow us to write

$$d(Q_n H_1 f, X_n \cap G, p) = d(Q_n H_1(I + F_\varepsilon K), X_n \cap G, p)$$
$$= \sum_{(i)} d(Q_n H_1, X_n \cap D_i, p) d(I + F_\varepsilon K, X_n \cap G, d_i).$$

Our choice of ε allows us to state that $d(I + F_\varepsilon K, X_n \cap G, d_i) = d(I + K, G, d_i)$ and our choice of n allows us to write $\{d(Q_n H_1, X_n \cap D_i, p)\} = D(H_1, D_i, p)$. Thus we have the desired result.

We then have the following corollary.

COROLLARY 2. *If $f(\theta) = \theta$ and θ is an isolated θ-point of H_1, then*

$$D(H_1 f, G, \theta) = d(f, G, \theta) D(H_1, D_i, \theta)$$

for one component of $X - f(\partial G)$, D_i.

With the above corollary and Lemma 11 we have

$$D(H_1(I + K), G, \theta) = D(H_1(I + Ch^{-1}), G, \theta) = \{d(I + Ch^{-1}, G, \theta)\}.$$

This result along with equation (10) yields

$$D(H + C + T, B_r, \theta) = \{\pm m(H + C + T)\}.$$

Thus the value obtained by using the generalized degree and the multiplicity are the same up to a multiplicative constant ± 1.

5. **An example.** In this section we shall merely restate the example due to Cronin-Scanlon in [3]. Consider $X = C[0, 1]$ (the continuous functions in $[0, 1]$) with the sup-norm. It is shown in [12] that if we let

$$x_0(t) = \chi_{[0,1]}(t) \quad \text{(the characteristic function of the set } [0, 1]),$$
$$x_1(t) = t\chi_{[0,1]}(t),$$
$$x_2(t) = x_1(2t) + \chi_{[0,1]}(2t - 1) - x_1(2t - 1),$$
$$x_{2^n + i}(t) = x_2(2^n t - i + 1), \quad i = 1, \ldots, \quad n = 1, 2, \ldots,$$

then $\{x_i\}$ is a basis for $C[0, 1]$. Since $\{x_i\}$ is a basis for $C[0, 1]$, each $x \in C[0, 1]$ can be expressed uniquely as $x = \sum_{i=0}^{\infty} a_i x_i$ (where the a_i are real constants).

Let $X_n = \text{span} \{x_0, \ldots, x_n\}$ and $Q_n(x) = Q_n(\sum_{i=0}^{\infty} a_i x_i) = \sum_{i=0}^{n} a_i x_i$. This sequence of subspaces and projections will be a projective approximation scheme for mappings from $C[0, 1]$ into $C[0, 1]$.

We then define C by $C(x(t)) = y(t)$ where $y(t) = x(1) \cdot t$ and define T by $T(x(t)) = y(t)$ where $y(t) = [x(t)]^2$. Then $I + C + T$ will satisfy the conditions set forth in this paper. This is then an example of a mapping for which the generalized degree is defined on some ball about the origin (and hence the multiplicity) while neither the Leray-Schauder degree nor the extended degree due to Cronin-Scanlon [5] is defined.

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