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MULTIPLIERS FOR SPHERICAL HARMONIC EXPANSIONS

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Abstract. Sufficient conditions are given for an operator on the sphere that commutes with rotations to be bounded in L^p. The conditions are analogous to those of Hörmander's well-known theorem on Fourier multipliers.

1. **Introduction.** Let $\{m_j\}$ be a bounded sequence of complex numbers and let T be the operator defined on a suitable class of functions on the $(n-1)$-sphere S^{n-1} by $Tf(x)=\sum_{j=0}^{\infty} m_j Y_j(x)$ if $f(x)=\sum_{j=0}^{\infty} Y_j(x)$. Here $\sum Y_j(x)$ is the expansion of f into spherical harmonics Y_j of degree j. Such operators commute with the action of the rotation group $SO(n)$ on S^{n-1} and are bounded on $L^2(S^{n-1})$ with norm equal to $\sup |m_j|$. We shall give sufficient conditions for T to be bounded on $L^p(S^{n-1})$.

The basic condition we consider is $\mathscr{H}(q, \alpha)$ that

$$\sum_{j=2^b}^{2^{b+1}} |\Delta^\alpha m_j|^q \leqq A 2^{b(1-q\alpha)}$$

for all nonnegative integers b. Here q is a real number in the interval $1 \leqq q < \infty$, α is a nonnegative real number, and the difference operator Δ^α is defined as follows:

(i) if α is an integer then Δ^α is just the usual difference given inductively by $\Delta^0 m_j = m_j$, $\Delta^{\alpha+1} m_j = \Delta^\alpha m_{j+1} - \Delta^\alpha m_j$;

(ii) if $\alpha = a+\beta$, a an integer and $0<\beta<1$ then

$$\Delta^\alpha m_j = \left(\sum_{k=2^b}^{2^{b+1}} \frac{|\Delta^a m_k - \Delta^a m_j|^q}{|k-j|^{1+\beta q}}\right)^{1/q}$$

if $2^b \leqq j < 2^{b+1}$.

We denote by $H(q, \alpha)$ the space of sequences satisfying $\mathscr{H}(q, \alpha)$ and $\mathscr{H}(q, a)$ for all integers $a < \alpha$.

THEOREM 1. *Assume $q \geqq 2$, $\alpha q > n-1$ and the sequence $\{m_j\}$ lies in $H(q, \alpha)$. Then T is bounded on $L^p(S^{n-1})$ provided $|1/p - \frac{1}{2}| < 1/q$.*

COROLLARY. *Let $m(x)$ be a function of a real variable satisfying*

$$|x^k m^{(k)}(x)| \leqq A \quad \textit{for } k = 0, \ldots, a.$$

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If $m_j = m(j)$ then T is bounded on $L^p(S^{n-1})$ for $|1/p-\frac{1}{2}| < a/(n-1)$, $p \neq 1, \infty$.

An immediate consequence is the following weak version of the Littlewood-Paley theorem:

THEOREM 2. *If $\theta(x) \in C^{[n/2]}(R^1)$, $\theta(x)=1$ for $1 \leqq x \leqq 2$ and θ has support in $\frac{1}{2} \leqq x \leqq 4$, then for $1<p<\infty$ we have*

$$A_p \left\| \sum_{k=0}^{\infty} Y_k(x) \right\|_p \leqq \left\| \left(|Y_0(x)|^2 + \sum_{j=0}^{\infty} \left| \sum_{k=2^j-1}^{2^{j+1}} \theta(2^{-j}k) Y_k(x) \right|^2 \right)^{1/2} \right\|_p$$
$$\leqq B_p \left\| \sum_{k=0}^{\infty} Y_k(x) \right\|_p .$$

Our theorem is a generalization of Hörmander's theorem on Fourier multipliers [7, Chapter 4] and its refinement in Littman [3] and Peetre [5]. Our method of proof closely parallels Stein's proof of Hörmander's theorem.

An example of Askey and Wainger [1] will indicate the sharpness of our result· They show that if $m_j = j^{-\mu} \exp((-1)^{1/2} j^{\beta})$ for $0<\beta<1$ and $\mu>0$ then T is bounded on $L^p(S^{n-1})$ for $|1/p-\frac{1}{2}| < \mu/(n-1)\beta$ and T fails to be bounded if $|1/p-\frac{1}{2}| > \mu/(n-1)\beta$ (recent results of Stein [8] indicate that T is probably bounded if $|1/p-\frac{1}{2}| = \mu/(n-1)\beta$). It is not hard to show that $\{m_j\}$ belongs to $H(q, \mu/\beta)$ for any q, so the boundedness for $|1/p-\frac{1}{2}| < \mu/(n-1)\beta$ follows from our result.

Weaker versions of our result are contained in more general results of Stein [6] and the author [9]. The approach of this paper has been taken independently by N. Weiss (see announcements in [12]) who solves the more general problem of multipliers for the Peter-Weyl expansion of an arbitrary compact Lie group. We believe that our work is still of interest for three reasons. First, our proof is simpler since it does not involve the general theory of compact Lie groups. Second, we have used interpolation methods that may also be applicable to Weiss' work. Third, our results are not completely contained in his, since the sphere is a homogeneous space of $SO(n)$. If one lifts functions on S^{n-1} to $SO(n)$ and applies Weiss' results one obtains a version of Theorem 1 with $n-1$ replaced by roughly the dimension of $SO(n)$, which is $n(n-1)/2$.

Finally we mention related results of Muckenhoupt and Stein [4]. They show that restricted to zonal functions T is bounded on L^p for $|1/p-\frac{1}{2}| < 1/2(n-1)$ provided $\{m_j\}$ is in $H(1, 1)$. The restriction to zonal harmonics seems to be essential here.

2. **Properties of g-functions.** We denote points of S^{n-1} by x' and points of R^n by x. We write $r=|x|$ and $x'=x/|x|$, so $x=rx'$. If $f(x')$ is a function on S^{n-1} we denote by $u(x)=u(x', r)$ the harmonic function in the ball $|x|<1$ which has $f(x')$ for boundary values in a suitable sense. If $f(x') = \sum Y_k(x')$ then $u(x', r) = \sum r^k Y_k(x')$. We also have the Poisson integral formula

$$u(x) = c_n \int_{S^{n-1}} \frac{1-|x|^2}{|x-y'|^n} f(y')\, dy'.$$

For each positive real α we define the radial derivative d_α of order α as follows: if α is an integer $d_\alpha f(x', r) = (r(\partial/\partial r))^\alpha f(x', r)$, and if $\alpha = k - \beta$ with k an integer and $0 < \beta < 1$ then

$$d_\alpha f(x', r) = \frac{1}{\Gamma(\beta)} \int_0^1 d_k f(x', rt) |\log t|^{\beta - 1} \frac{dt}{t}.$$

It is easy to verify the semigroup property $d_\alpha d_\beta = d_{\alpha+\beta}$. If $u(x', r) = \sum_1^\infty r^k Y_k(x')$ then $d_\alpha u(x', r) = \sum_1^\infty k^\alpha r^k Y_k(x')$ as may be seen by direct computation. These properties are also a consequence of the fact that under the change of variable $r = e^{-t}$ the d_α become the usual fractional derivatives.

In analogy with the Euclidean theory described in [7, Chapter 4] we introduce the following auxiliary functions:

$$g_\alpha(f, x')^2 = \int_0^1 |d_\alpha u(x', r)|^2 |\log r|^{2\alpha - 1} \frac{dr}{r},$$

$$S(f, x')^2 = \iint_{(1-r)^2 \geqq 1 - x' \cdot y'} |d_1 u(y', r)|^2 (1-r)^{2-n} \, dr \, dy'$$

$$g_\lambda^*(f, x')^2 = \int_0^1 \int_{S^{n-1}} \left(\frac{1-r}{1 - r + (1 - x' \cdot y')^{1/2}} \right)^{\lambda(n-1)} |d_1 u(y', r)|^2 (1-r)^{2-n} \, dy' \, dr.$$

We will usually require $\lambda > 1$. The following theorem summarizes the properties of these functions we will use. It is a rather straightforward elaboration of the techniques of Stein [6] and [7], but we include the proof for completeness.

THEOREM 3. *Let* $f \in L^2(S^{n-1}) \cap L^p(S^{n-1})$ *and assume* $\int f(x') \, dx' = 0$. *Then*

(a) $\|g_1(f)\|_p \leqq A_p \|f\|_p$, $1 < p < \infty$;

(b) $\|f\|_p \leqq A_p \|g_1(f)\|_p$, $1 < p < \infty$;

(c) $\|g_\lambda^*(f)\|_p \leqq A_p \|f\|_p$, $2 \leqq p < \infty$, $\lambda > 1$;

(d) $g_\alpha(f, x') \leqq A_{\alpha\beta} g_\beta(f, x')$, $\beta > \alpha$;

(e) $S(f, x') \leqq A_\lambda g_\lambda^*(f, x')$.

Proof. Parts (a) and (b) are special cases of a very general theorem of Stein [6]. We define the Poisson semigroup by $T^t f(x') = u(x', e^{-t})$. Using the explicit form of the Poisson integral it is easy to check that the conditions of [6] are satisfied. Theorem 10 and its corollary of [6] give (a) and (b).

We deduce (c) from (a) in exactly the same manner as in the Euclidean case [7, Theorem 4.2]. We prove

$$(*) \qquad \int g_\lambda^*(f, x')^2 \psi(x') \, dx' \leqq A_\lambda \int g_1(f, x')^2 M\psi(x') \, dx'$$

where $M\psi$ is the maximal function

$$M\psi(x') = \sup_{-1 < r < 1} |1 - r|^{1-n} \int_{(1-r)^2 \geqq 1 - x' \cdot y'} |\psi(y')| \, dy'.$$

This function is easily seen to have all the properties of the usual maximal function, in particular, $\|M\psi\|_p \leqq A_p\|\psi\|_p$ for $1<p\leqq\infty$. Once we have established (*) we obtain

$$\|g_\lambda^*(f)\|_p \leqq A_{\lambda,p}\|g_1(f)\|_p \quad \text{for } 2 \leqq p < \infty$$

by setting $\psi\equiv 1$ for $p=2$, or by taking the supremum over all ψ with $\|\psi\|_q=1$ where $1/q+2/p=1$.

To establish (*) we compute

$$\int g_\lambda^*(f, x')^2\psi(x')\,dx'$$
$$= \int_0^1\int_{S^{n-1}} (1-r)|d_1u(y', r)|^2$$
$$\cdot\left[\int_{S^{n-1}} \left(\frac{1-r}{1-r+(1-x'\cdot y')^{1/2}}\right)^{\lambda(n-1)} (1-r)^{1-n}\psi(x')\,dx'\right] dy'\,dr$$

so it suffices to show

$$\sup_{0<\varepsilon<1} \int_{S^{n-1}} \left(\frac{\varepsilon}{\varepsilon+(1-x'\cdot y')^{1/2}}\right)^{\lambda(n-1)} \varepsilon^{1-n}|\psi(x')|\,dx' \leqq cM\psi(y').$$

Now we may compute that

$$\varepsilon^{1-n}\left(\frac{\varepsilon}{\varepsilon+(1-x'\cdot y')^{1/2}}\right)^{\lambda(n-1)} = \varepsilon^{1-n}\left(\frac{\varepsilon}{\varepsilon+1}\right)^{\lambda(n-1)}$$
$$+\int_0^{1-(1-x'\cdot y')^{1/2}} (1-r)^{1-n}h_\varepsilon(r)\,dr \qquad \text{for } x'\cdot y \geqq 0$$

where

$$h_\varepsilon(r) = \varepsilon^{1-n}\lambda(n-1)\left(\frac{\varepsilon}{\varepsilon+1-r}\right)^{\lambda(n-1)}\frac{(1-r)^{n-1}}{(\varepsilon+1-r)} \geqq 0.$$

From this we obtain

$$\varepsilon^{1-n}\int_{x'\cdot y'\geqq 0} \left(\frac{\varepsilon}{\varepsilon+1-x'\cdot y'}\right)^{\lambda(n-1)} |\psi(x')|\,dx'$$
$$= \int_0^1 \left[(1-r)^{1-n}\int_{(1-r)^2\geqq 1-x'\cdot y'} |\psi(x')|\,dx'\right]h_\varepsilon(r)\,dr$$
$$+\varepsilon^{1-n}\left(\frac{\varepsilon}{\varepsilon+1}\right)^{\lambda(n-1)}\int_{x'\cdot y'\geqq 0} |\psi(x')|\,dx'.$$

If we set $\psi\equiv 1$ we obtain $\int_0^1 h_\varepsilon(r)\,dr\leqq c$ independent of ε (note $\varepsilon^{1-n}(\varepsilon/(\varepsilon+1))^{\lambda(n-1)}$ is bounded since $\lambda>1$). Since $h_\varepsilon(r)\geqq 0$ we thus obtain

$$\varepsilon^{1-n}\int_{x'\cdot y'\geqq 0} \left(\frac{\varepsilon}{\varepsilon+(1-x'\cdot y')^{1/2}}\right)^{\lambda(n-1)} |\psi(x')|\,dx' \leqq c'M\psi(y'),$$

and the integral over $x'\cdot y'\leqq 0$ is easily dominated by

$$c\int_{x'\cdot y'\leqq 0} |\psi(x')|\,dx' \leqq c'M\psi(y').$$

Thus (*) is established.

To establish (d) it suffices to show $g_\alpha(f, x')\leqq A_{\alpha\beta}g_{\alpha+\beta}(f, x')$ for $0<\beta<1$ and $\beta<2\alpha$. To do this we begin with

$$d\hat{\alpha}_j u(x', r) = \frac{1}{\Gamma(\beta)}\int_0^1 d_{\alpha+\beta}u(x', rt)|\log t|^{\beta-1}\frac{dt}{t}.$$

We apply Schwartz' inequality and make use of the fact that

$$\int_0^1 |\log(rt)|^{-2\alpha}|\log t|^{\beta-1}\frac{dt}{t} = c_{\alpha\beta}|\log r|^{\beta-2\alpha}$$

if $0<\beta<1$ and $\beta<2\alpha$ to obtain

$$|d_\alpha u(x', r)|^2 \leqq c|\log r|^{\beta-2\alpha}\int_0^1 |d_{\alpha+\beta}u(x', rt)|^2|\log(rt)|^{2\alpha}|\log t|^{\beta-1}\frac{dt}{t}.$$

Thus

$$\begin{aligned}
g_\alpha(f, x')^2 &= \int_0^1 |d_\alpha u(x', r)|^2\cdot|\log r|^{2\alpha-1}\frac{dr}{r} \\
&\leqq c\int_0^1\int_0^1 |d_{\alpha+\beta}u(x', rt)|^2|\log r|^{\beta-1}|\log(rt)|^{2\alpha}|\log t|^{\beta-1}\frac{dt}{t}\frac{dr}{r} \\
&= c\int_0^1\int_0^t |d_{\alpha+\beta}u(x', r)|^2\left|\log\frac{r}{t}\right|^{\beta-1}|\log r|^{2\alpha}|\log t|^{\beta-1}\frac{dr}{r}\frac{dt}{t}.
\end{aligned}$$

Interchanging the order of integration and using the fact that

$$\int_r^1 \left|\log\frac{r}{t}\right|^{\beta-1}|\log t|^{\beta-1}\frac{dt}{t} = c_\beta|\log r|^{2\beta-1}$$

for $0<\beta<1$ we obtain

$$g_\alpha(f, x')^2 \leqq A_{\alpha\beta}\int_0^1 |d_{\alpha+\beta}(x', r)|^2|\log r|^{2\alpha+2\beta-1}\frac{dr}{r} = A_{\alpha\beta}g_{\alpha+\beta}(f, x')^2.$$

Finally (e) is trivial because

$$\left(\frac{1-r}{1-r+(1-x'\cdot y')^{1/2}}\right)^{\lambda(n-1)} \geqq 2^{\lambda(1-n)}$$

on the set where $(1-r)^2\geqq 1-x'\cdot y'$.

3. **Proof of the main theorem.** We assume now for technical convenience that $m_0=m_1=0$. It is clear that the theorem holds in general if we can establish it with this assumption. We form the function

$$M(t, r) = \sum_{j=2}^{\infty} m_j r^j Z_j(t)$$

where $Z_j(t)$ is the zonal harmonic of degree j. Recall that

$$Z_j(x'\cdot y') = \sum_k Y_{jk}(x')Y_{jk}(y')$$

if $\{Y_{jk}\}$ is an orthonormal base for the spherical harmonics of degree j. The crux of the proof is the following lemma which translates the hypotheses on $\{m_j\}$ into properties of $M(t, r)$.

LEMMA 1. *Assume* $\{m_j\}$ *lies in* $H(2, \alpha)$. *Then*

(a) $|d_\alpha M(t, r)| \leqq A_\alpha r^2(1-r)^{1-n-\alpha}$ *if* $\alpha > (n-1)/2$;

(b) $\int_{-1}^{1} (1-t)^\alpha |d_\alpha M(t, r)|^2 (1-t^2)^{(n-2)/2}\, dt \leqq A_\alpha r^2(1-r)^{1-n}$.

Let us complete the proof of the theorem assuming the lemma. We first do the case $q=2$. We will show that $\mathscr{H}(2, \alpha)$ for $\alpha > (n-1)/2$ implies $g_{\alpha+1}(Tf) \leqq A g_\lambda^*(f)$ pointwise for $\lambda = 2\alpha/(n-1) > 1$. In view of Theorem 3 this proves that T is bounded on L^p for $2 \leqq p < \infty$, and the usual duality argument shows that T is also bounded on L^p for $1 < p \leqq 2$.

Let $U(x', r)$ be the Poisson integral of $Tf(x')$. Then we have

$$U(x', rs) = \int_{S^{n-1}} M(x'\cdot y', r) u(y', s)\, dy'.$$

This is an immediate consequence of

$$U(x', r) = \sum m_k r^k Y_k(x') \quad \text{and} \quad u(x', r) = \sum r^k Y_k(x'),$$

and the properties of zonal harmonics. If we apply d_α in the r-variable and d_1 in the s-variable to both sides we obtain

$$d_{\alpha+1} U(x', rs) = \int_{S^{n-1}} d_\alpha M(x'\cdot y', r) d_1 u(y', s)\, dy'$$

and setting $r=s$ we have

$$d_{\alpha+1} U(x', r^2) = \int_{S^{n-1}} d_\alpha M(x'\cdot y', r) d_1 u(y', r)\, dy'.$$

We break up the integral into two regions and apply the Schwartz inequality:

$$\begin{aligned} &|d_{\alpha+1} U(x', r^2)|^2 \\ &\leqq \int_{(1-r)^2 \geqq 1-x'\cdot y'} |d_1 u(y', r)|^2\, dy' \int_{(1-r)^2 \geqq 1-x'\cdot y'} |d_\alpha M(x'\cdot y', r)|^2\, dy' \\ &\quad + \int_{(1-r)^2 \leqq 1-x'\cdot y'} \frac{|d_1 u(y', r)|^2}{(1-x'\cdot y')^\alpha}\, dy' \int_{(1-r)^2 \leqq 1-x'\cdot y'} (1-x'\cdot y')^\alpha |d_\alpha M(x'\cdot y', r)|^2\, dy'. \end{aligned}$$

Now by (a) of Lemma 1 we have

$$\begin{aligned} &\int_{(1-r)^2 \geqq 1-x'\cdot y'} |d_\alpha M(x'\cdot y', r)|^2\, dy' \\ &\qquad \leqq A_\alpha r^2 (1-r)^{2(1-n-\alpha)} \int_{(1-r)^2 \geqq 1-x'\cdot y'} dy' \leqq c r^2 (1-r)^{1-n-2\alpha}, \end{aligned}$$

and by (b) we have

$$\int_{(1-r)^2 \leqq 1-x'\cdot y'} (1-x'\cdot y')^\alpha |d_\alpha M(x'\cdot y', r)|^2\, dy' \leqq \int_{-1}^{1} (1-t)^\alpha |d_\alpha M(t, r)|^2 (1-t^2)^{(n-2)/2}\, dt \leqq cr^2(1-r)^{1-n}.$$

Thus

$$|d_{\alpha+1} U(x', r^2)|^2 \leqq c(I_1(r)+I_2(r))$$

where

$$I_1(r) = r^2(1-r)^{1-n-2\alpha} \int_{(1-r)^2 \geqq 1-x'\cdot y'} |d_1 u(y', r)|^2\, dy',$$

$$I_2(r) = r^2(1-r)^{1-n} \int_{(1-r)^2 \leqq 1-x'\cdot y'} \frac{|d_1 u(y', r)|^2}{(1-x'\cdot y')^\alpha}\, dy'.$$

We may then estimate (substituting r^2 for r)

$$g_{\alpha+1}(Tf, x')^2 = 2^{2\alpha+2} \int_0^1 |d_{\alpha+1} U(x', r^2)|^2 |\log r|^{2\alpha+1} \frac{dr}{r} \leqq c\int_0^1 I_1(r)|\log r|^{2\alpha+1} \frac{dr}{r} + c\int_0^1 I_2(r)|\log r|^{2\alpha+1} \frac{dr}{r}.$$

Using $r|\log r|^{2\alpha+1} \leqq c(1-r)^{2\alpha+1}$ we have

$$\int_0^1 I_1(r)|\log r|^{2\alpha+1} \frac{dr}{r} \leqq cS(f, x')^2 \leqq cg_\lambda^*(f, x')^2$$

and

$$\int_0^1 I_2(r)|\log r|^{2\alpha+1} \frac{dr}{r} \leqq c \iint_{(1-r)^2 \leqq 1-x'\cdot y'} \frac{|d_1 u(y', r)|^2}{(1-x'\cdot y')^\alpha} (1-r)^{2-n+2\alpha}\, dy'\, dr$$
$$\leqq 4^\alpha c \int_0^1 \int_{S^{n-1}} |d_1 u(y', r)|^2 \left(\frac{1-r}{1-r+(1-x'\cdot y')^{1/2}} \right)^{2\alpha} (1-r)^{2-n}\, dy'\, dr = 4^\alpha c g_\lambda^*(f, x')^2$$

for $\lambda = 2\alpha/(n-1)$. Thus we have $g_{2\alpha+1}(Tf, x') \leqq cg_\lambda^*(f, x')$ as desired.

We complete the proof of the theorem for $q>2$ by interpolation. We use the complex method (see Calderón [2] for details).

LEMMA 2. *The intermediate space* $[H(q_0, \alpha_0), H(q_1, \alpha_1)]_\theta$ *may be identified with* $H(q, \alpha)$ *for* $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1$.

This is a simple extension of results in Taibleson [10, II] (cf. [3] and [5]). We omit the details.

Now we consider Tf as a bilinear operator on $H(q, \alpha)\times L^p$. Write $\alpha=(n-1)/q+\varepsilon$. By the $q=2$ case we know it is bounded from $H(2,(n-1)/2+\varepsilon)\times L^{p_0}$ to L^{p_0} for any p_0 in $1<p_0<\infty$. On the other hand Sobolev's inequality shows that all sequences in $H(q_1,(n-1)/q_1+\varepsilon)$ are bounded (see [10, I]) so Tf is bounded from $H(q_1,(n-1)/q_1+\varepsilon)\times L^2$ to L^2, $1<q_1<\infty$. It follows from the bilinear interpolation theorem [2] that Tf is bounded from $H(q,(n-1)/q+\varepsilon)\times L^p$ to L^p for $1/q=(1-\theta)/2+\theta/q_1$ and $1/p=(1-\theta)/p_0+\theta/2$. It remains to show that for fixed $q>2$, one can choose p_0 and q_1 so as to obtain all values of p in the interval $|1/p-\frac{1}{2}|<1/q$. But if $p>2$ we have $\frac{1}{2}-1/p=1/q-(\theta/q_1+(1-\theta)/p_0)$ and so we need only take p_0 and q_1 sufficiently large, and a similar argument works for $p<2$. This completes the proof of Theorem 1. We turn now to the proof of Lemma 1.

Proof of Lemma 1. First we observe that it suffices to prove (b) in the case when α is an even integer. For we may write

$$d_\alpha M(t,r) = \sum_{j=2}^{\infty} j^\alpha m_j r^j Z_j(t)$$

which makes sense for complex α. Furthermore, if $\alpha=a+ib$ then

$$d_\alpha M(t,r) = \sum j^a(j^{ib}m_j)r^j Z_j(t)$$

and the sequence $\{j^{ib}m_j\}$ belongs to $H(q,a)$ if $\{m_j\}$ does with a norm growing at most like a power of b. In view of Lemma 2 one may thus apply the Phragmen-Lindelöf principle to interpolate inequality (b).

The proof for the case when α is an even integer is based on the recursion relation

$$tZ_j(t) = \frac{j+n-2}{2j+n-2}Z_{j+1}(t)+\frac{j}{2j+n-2}Z_{j-1}(t)$$

which follows immediately from the recursion relation for Gegenbauer polynomials

$$tC_j^p(t) = \frac{j+1}{2j+2p}C_{j+1}^p(t)+\frac{j-1+2p}{2j+2p}C_{j-1}^p(t)$$

[11, p. 514] and the fact that

$$Z_j(t) = \frac{j!\,\Gamma(n-2)}{\Gamma(j+n-2)}C_j^{(n-2)/2}(t) \qquad [11, p. 461].$$

We will also use the well-known estimates $\|Z_j\|_2\leqq cj^{(n-2)/2}$ and $|Z_j(t)|\leqq cj^{n-2}$, and the elementary inequality $\sum_2^\infty j^{s-1}r^j\leqq cr^2(1-r)^{-s}$ for $s>0$ which may be deduced by estimating the sum by $\int_2^\infty x^{s-1}e^{x\log r}\,dx$.

Now (a) is a simple consequence of the boundedness of $\{m_j\}$:

$$|d_\alpha M(t,r)| = \left|\sum_2^\infty j^\alpha r^j m_j Z_j(t)\right| \leqq c\sup|m_j|\sum_2^\infty j^{\alpha+n-2}r^j \leqq A_\alpha r^2(1-r)^{1-n-\alpha}.$$

Next we turn to (b). For $\alpha=0$ we have

$$\begin{aligned}\int_{-1}^{1} &|M(t,r)|^2(1-t^2)^{(n-2)/2}\,dt \\ &= \left\|\sum_2^\infty m_j r^j Z_j(t)\right\|_2^2 = \sum_2^\infty |m_j|^2 r^{2j}\|Z_j\|_2^2 \\ &\leqq c\sum_2^\infty j^{n-2}|m_j|^2 r^{2j} \leqq c\sum_{k=1}^\infty 2^{k(n-2)}r^{2^{k+2}}\sum_{j=2}^{2^{k+1}}|m_j|^2 \\ &\leqq cA\sum_{k=1}^\infty 2^{k(n-2)}r^{2^{k+2}}2^k \leqq cA\sum_{j=2}^\infty j^{n-2}r^{2j} \leqq A'r^2(1-r)^{1-n}\end{aligned}$$

if $\{m_j\}\in H(2,0)$.

We consider next the case $\alpha=2$. We write

$$\varphi(j) = j^2 m_j r^j \quad\text{and}\quad \psi(j) = (2j+n-2)^{-1}\varphi(j).$$

Using the recursion relations we compute

$$\begin{aligned}(1-t)\,d_2M(t,r) = (1-t)\sum\varphi(j)Z_j(t) &= -\tfrac{1}{2}\sum(\varphi(j+1)-2\varphi(j)+\varphi(j-1))Z_j(t) \\ &\quad+\frac{n-2}{2}\sum(\psi(j+1)-\psi(j-1))Z_j(t).\end{aligned}$$

Now if $\{m_j\}\in H(2,2)$ we easily compute (using Leibnitz's formula for differences) that

$$\sum_{j=2^p}^{2^{p+1}}|\varphi(j+1)-2\varphi(j)+\varphi(j-1)|^2 \leqq A'2^p r^{2^{p+1}}(1+(1-r)2^p+(1-r)^2 2^{2p}),$$

and

$$\sum_{j=2^p}^{2^{p+1}}|\psi(j+1)-\psi(j-1)|^2 \leqq A'2^p r^{2^{p-1}}(1+(1-r)2^p).$$

Thus we have

$$\begin{aligned}\int_{-1}^{1} &|(1-t)d_2M(t,r)|^2(1-t^2)^{(n-2)/2}\,dt \\ &= \sum\left|-\tfrac{1}{2}(\varphi(j+1)-2\varphi(j)+\varphi(j-1))+((n-2)/2)(\psi(j+1)-\psi(j-1))\right|^2\|Z_j\|_2^2 \\ &\leqq \sum_{p=1}^\infty\sum_{j=2^p}^{2^{p+1}}(\text{same expression}) \\ &\leqq A''\sum_{p=1}^\infty 2^{p(n-1)}r^{2^{p+1}}(1+(1-r)2^p+(1-r)^2 2^{2p}) \\ &\leqq A'''\sum_{j=2}^\infty j^{n-2}r^{2j}(1+j(1-r)+j^2(1-r)^2) \leqq Br^2(1-r)^{1-n}.\end{aligned}$$

We establish (b) for $\alpha=2k$ in an analogous fashion, using the recursion relations to write $(1-t)^k d_{2k}M(t,r)$ as a zonal harmonic series with coefficients involving $\Delta^{2k}\varphi(j)$ and related terms. We omit the details.

The deduction of Theorem 2 from Theorem 1 is a routine argument using Rademacher functions as in [4, Chapter 4]. We use Theorem 1 to show that $\sum_j \varepsilon_j\theta(2^{-j}k)$ for $\varepsilon_j = \pm 1$ and $(\sum_j \theta(2^{-j}k))^{-1}$ are L^p multipliers.

References

1. R. Askey and S. Wainger, *On the behavior of special classes of ultraspherical expansions*. II, J. Analyse **15** (1965), 221–244. MR **33** #1510.

2. A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113–190. MR **29** #5097.

3. W. Littman, *Multipliers in L^p and interpolation*, Bull. Amer. Math. Soc. **71** (1965), 764–766. MR **31** #3792.

4. B. Muckenhoupt and E. M. Stein, *Classical expansions and their relation to conjugate harmonic functions*, Trans. Amer. Math. Soc. **118** (1965), 17–92. MR **33** #7779.

5. J. Peetre, *Applications de la théorie des espaces d'interpolation dans l'analyse harmonique*, Ricerche Mat. **15** (1966), 3–36. MR **36** #4266.

6. E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Ann. of Math. Studies, no. 63, Princeton Univ. Press, Princeton, N. J.; Univ. of Tokyo Press, Tokyo, 1970. MR **40** #6176.

7. ———, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970.

8. ———, *L^p boundedness of certain convolution operators*, Bull. Amer. Math. Soc. **77** (1971), 404–405.

9. R. Strichartz, *A functional calculus for elliptic pseudodifferential operators* (to appear).

10. M. H. Taibleson, a) *On the theory of Lipschitz spaces of distributions on Euclidean n-space. I. Principal properties*, J. Math. Mech. **13** (1964), 407–479. MR **29** #462.

b) II. *Translation invariant*, J. Math. Mech. **14** (1965), 821–839. MR **31** #5087.

11. N. Ja. Vilenkin, *Special functions and the theory of group representations*, "Nauka", Moscow, 1965; English transl., Transl. Math. Monographs, vol. 22, Amer. Math. Soc., Providence, R. I., 1968. MR **35** #420.

12. N. Weiss, *Multipliers on compact Lie groups*, Proc. Nat. Acad. Sci. U.S.A. **68** (1971), 930–931.

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