MAPPINGS FROM 3-MANIFOLDS ONTO 3-MANIFOLDS (*)

BY

ALDEN WRIGHT

Abstract. Let $f$ be a compact, boundary preserving mapping from the 3-manifold $M^3$ onto the 3-manifold $N^3$. Let $Z_p$ denote the integers mod a prime $p$, or, if $p = 0$, the integers. (1) If each point inverse of $f$ is connected and strongly 1-acyclic over $Z_p$, and if $M^3$ is orientable for $p > 2$, then all but a locally finite collection of point inverses of $f$ are cellular. (2) If the image of the singular set of $f$ is contained in a compact set each component of which is strongly acyclic over $Z_p$, and if $M^3$ is orientable for $p 
eq 2$, then $N^3$ can be obtained from $M^3$ by cutting out of $\text{Int } M^3$ a compact 3-manifold with 2-sphere boundary, and replacing it by a $Z_p$-homology 3-cell. (3) If the singular set of $f$ is contained in a 0-dimensional set, then all but a locally finite collection of point inverses of $f$ are cellular.

I. Introduction. We suppose throughout the introduction that $f: M^3 \rightarrow N^3$ is a compact, boundary preserving mapping from the 3-manifold $M^3$ onto the 3-manifold $N^3$ (where $M^3$ and $N^3$ may or may not have boundary). Let $Z_p$ denote the integers modulo a prime $p$, or, if $p = 0$, the integers.

If $f^{-1}(x)$ is connected and strongly 1-acyclic over $Z_p$ for all $x \in N^3$, and if $M^3$ is orientable for $p > 2$, then in Corollary 1 it is shown that all but a locally finite collection of point inverses are cellular. This implies that $N^3$ can be obtained from $M^3$ by cutting out of $\text{Int } M^3$ a locally finite collection of compact 2-manifolds, each bounded by a 2-sphere, and replacing them by a 3-cell (see Corollary 3). Thus, if $M^3$ is compact, $N^3$ is a factor in a connected sum decomposition of $M^3$.

Now suppose that the image of the singular set of $f$ is contained in a compact set $X$ each component of which is strongly acyclic over $Z_p$. If $M^3$ is orientable for $p 
eq 2$, then $N^3$ can be obtained from $M^3$ by cutting out of $M^3$ a finite number of compact 3-manifolds, each bounded by a 2-sphere, and replacing each by a $Z_p$-homology 3-cell. In particular, if $X$ has a neighborhood which is an irreducible 3-manifold with boundary (or if $N^3$ is irreducible), then $N^3$ is a factor in a connected sum decomposition of $M^3$. This extends Theorem 1 of Lambert in [9]. In the special case where the image of the singular set is contained in a Cantor set,
we can say in addition that all but a finite number of point inverses are cellular. This was previously proved by the author using other techniques.

Lemma 5 restates one of Armentrout’s results on approximating cellular maps with homeomorphisms. Using this lemma, we combine the results of Theorems 1 and 3 in Theorem 5. Thus if $M^3$ is compact and orientable for $p \neq 2$, and if the image of the point inverses of $f$ which are not connected and strongly 1-acyclic over $Z_p$ is contained in a compact set $X$ each component of which is strongly acyclic over $Z_p$, then $N^3$ can be obtained from $M^3$ by cutting out of Int $M^3$ a finite number of 3-manifolds each bounded by a 2-sphere, and replacing each by a $Z_p$-homology 3-cell. Theorem 6 combines Theorems 1 and 4 in a similar fashion.

In Theorem 7, we extend a result of McMillan [13] to show that if the image of the singular set of $f$ is contained in a (nonclosed) 0-dimensional set, then all but a locally finite collection of point inverses are cellular.

Let $G$ be a nontrivial abelian group. A compact set $X \subset M$ is strongly $k$-acyclic over $G$ if for each open set $U \subset M$ containing $X$, there is an open set $V$ such that $X \subset V \subset U$ and such that the inclusion induced homomorphism $i_* : H_k(V; G) \to H_k(U; G)$ is zero. (If $X$ is connected and strongly $k$-acyclic over $G$ for $1 \leq k \leq n$, then $X \subset M$ has property $w^k(G)$ in the sense of [8].) The compact set $X \subset M$ is strongly acyclic over $G$ if it is connected and strongly $k$-acyclic over $G$ for all $k \geq 1$.

We refer the reader to [13 (especially Lemma 1)] for further facts about strong acyclicity. In particular, for any positive integer $k$, a compact set $X$ in the interior of a 3-manifold $M^3$ is strongly $k$-acyclic over $G$ if and only if each component of $X$ is strongly $k$-acyclic over $G$. Also $X$ is strongly acyclic over $Z$ if and only if $X$ is connected and $H^*(X; Z) = 0$ (see [7]).

The compact set $X \subset M$ has property $UV^n$ if for each open set $U \subset M$ containing $X$, there is an open set $V$ such that $X \subset V \subset U$ and such that $V$ is contractable in $U$. A set $X$ in a 3-manifold $M^3$ is cellular in $M^3$ if $X = \bigcap_{i=1}^n F_i$ where each $F_i$ is a 3-cell, and $F_{i+1} \subset Int F_i$ for all $i$.

If $\sigma$ is a loop in a space $M$, we will denote its homology class in $H_1(M; G)$ by $[\sigma]$. The symbol $Z_p$ for $p>0$ will denote the finite cyclic group of order $p$. The symbol $Z_0$ will denote the integers.

A manifold will be assumed to be connected and to have no boundary unless otherwise specified. We assume that all manifolds have a piecewise-linear structure. A 3-manifold is irreducible if every polyhedral 2-sphere in it bounds a polyhedral 3-cell. If $M^3$ and $N^3$ are 3-manifolds, possibly with boundary, the connected sum $M^3 \# N^3$ of $M^3$ and $N^3$ is obtained by removing the interior of a 3-cell from the interior of each, and then sewing the two manifolds together along the resulting boundary components, using an orientation reversing homeomorphism if $M^3$ and $N^3$ are oriented.

A map or mapping is a continuous function. A monotone map is a map all of whose point inverses are connected. A map $f : M \to N$ is compact (proper) if, for any compact set $K$ in $N$, $f^{-1}(K)$ is compact. If $f : M \to N$ is a compact monotone
map, then the point inverses of $M$ form a monotone upper semicontinuous decomposition of $M$ whose associated decomposition space is homeomorphic to $N$. Conversely, if $G$ is a monotone upper semicontinuous decomposition of $M$, the projection map $p: M \to M/G$ is a compact monotone map.

Let $\{X_a\}_{a \in A}$ be a collection of compact subsets of a space $M$. Then $\{X_a\}_{a \in A}$ is a **locally finite collection** if for $y \in M$, $y$ has a neighborhood $U$ which intersects only a finite number of elements of the collection.

### 11. Maps all of whose point inverses are strongly acyclic.

**Lemma 1.** If $X$ is a compact connected subset of a space $M$ and if $X$ is strongly $k$-acyclic over $\mathbb{Z}$ in $M$ for $1 \leq k \leq n$, then $X$ is strongly $k$-acyclic over $\mathbb{Z}_p$ in $M$ for $1 \leq k \leq n$ and for any prime $p > 1$.

**Proof.** Let $W$ and $V$ be chosen so that $X \subset W \subset V \subset U$ and so that the inclusion induced homomorphisms $i_*: H_k(V; \mathbb{Z}) \to H_k(U; \mathbb{Z})$ and $j_*: H_k(W; \mathbb{Z}) \to H_k(V; \mathbb{Z})$ are zero for $1 \leq k \leq n$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
0 & \longrightarrow & H_k(W; \mathbb{Z}) \otimes \mathbb{Z}_p \\
\downarrow i_* \otimes \text{id} & & \downarrow i_* \\
0 & \longrightarrow & H_k(V; \mathbb{Z}) \otimes \mathbb{Z}_p \\
\downarrow j_* \otimes \text{id} & & \downarrow j_*' \\
0 & \longrightarrow & H_k(U; \mathbb{Z}) \otimes \mathbb{Z}_p \\
\end{array}
$$

The horizontal rows, which are exact, are from the universal coefficient theorem. By our choice of $W$ and $V$, the outer vertical maps are zero. Using a diagram chasing argument, we see that $j_* i_*$ is the zero homomorphism.

**Lemma 2.** Let $M^3$ and $N^3$ be 3-manifolds, and let $f: M^3 \to N^3$ be a compact, monotone, onto map. Let $p$ be 0 or a prime, and suppose $M^3$ is orientable if $p \neq 2$. If $f^{-1}(y)$ is strongly 1-acyclic over $\mathbb{Z}_p$ for every $y \in N^3$, then each $f^{-1}(y)$ is strongly acyclic over $\mathbb{Z}_p$ in $M^3$.

**Proof.** By Alexander duality and Theorem 3 of [8] we see that $H^k(f^{-1}(y); \mathbb{Z}_p) = 0$ for $k \geq 2$. Then the continuity of $H^*$ and the universal coefficient theorem for cohomology show that $f^{-1}(y)$ is strongly acyclic over $\mathbb{Z}_p$ for all $y \in N^3$. (For more details, see Theorems 4.4 and 3.2 of [7].)

**Lemma 3.** Let $M^3$ and $N^3$ be 3-manifolds, and $f: M^3 \to N^3$ be a compact, monotone, onto map such that $f^{-1}(y)$ is strongly 1-acyclic over $\mathbb{Z}_p$ for each $y \in N^3$. If $H_1(N^3; G) = 0$, then $H_1(M^3; G) = 0$.

The proof of Lemma 3 is similar to the proof of Theorem 2.1 of [15].
If $M^n$ and $N^n$ are $n$-manifolds with boundary, a map $f: M^n \to N^n$ is said to be boundary preserving if $f|_{\text{Bd } M^n}$ is a homeomorphism of $\text{Bd } M^n$ onto $\text{Bd } N^n$, and if $f^{-1}(\text{Bd } N^n) = \text{Bd } M^n$. A 2-manifold with boundary $S$ is properly embedded in a 3-manifold with boundary $M^3$ if $S \cap \text{Bd } M^3 = \text{Bd } S$.

A $Z_p$-homology (homotopy) 3-cell is a compact $Z_p$-acyclic (contractible) 3-manifold with boundary. A cube-with-handles is obtained by adding orientable 1-handles to a 3-cell. We define a $Z_p$-homology (homotopy) cube-with-handles similarly. We will say that a set $X$ is the intersection of a decreasing sequence of $(Z_p$-homology, homotopy) cubes-with-handles if $X = \bigcap_{i=1}^\infty K_i^3$ where each $K_i^3$ is a $(Z_p$-homology, homotopy) cube-with-handles and $K_i^{3+1} \subset \text{Int } K_i^3$.

**Theorem 1.** Let $p$ denote 0 or a prime, and let $M^3$ and $N^3$ be compact 3-manifolds, possibly with boundary, where $M^3$ is orientable if $p > 2$. Let $f: M^3 \to N^3$ be a monotone, onto, boundary preserving map. Let $U$ be an open subset of $N^3$. If $f^{-1}(x)$ is strongly $1$-acyclic over $Z_p$ for all $x \in U$, then $\{x \in U : f^{-1}(x)$ is not cellular$\}$ is a finite set.

**Remark.** This theorem was first proved for $p = 0, 2$ in [16]. It has since been generalized by D. R. McMillan in [13].

**Proof.** The case where $p = 0$ reduces to the case where $p = 2$ by Lemma 1. By the proofs of Theorems 1 and 2 of [11] and by Kneser’s Theorem [6] it is sufficient to prove that $\{x \in U : f^{-1}(x)$ is not $UV^\infty$\} is finite.

We can apply Lemma 2 to see that $f^{-1}(x)$ is strongly acyclic over $Z_p$ for each $x \in U$. By Theorem 2 of [12], $f^{-1}(x)$ is the intersection of a decreasing sequence of $Z_p$-homology cubes-with-handles.

Let $q$ be the rank (i.e. the minimum number of generators) of $\pi_1(M^3)$. By a corollary to the Grushko-Neumann Theorem (p. 192 of [10]), there are at most $q$ disjoint $Z_p$-homology 3-cells in $M^3$ which are not homotopy 3-cells. Thus there are at most $q$ points in $U$ whose inverse images are not the intersection of a decreasing sequence of homotopy cubes-with-handles.

Let $x \in U$, where $f^{-1}(x)$ is the intersection of a decreasing sequence of homotopy cubes-with-handles. We will complete the proof by showing that $f^{-1}(x)$ is $UV^\infty$. Let $U'$ be an open set in $M^3$ containing $f^{-1}(x)$. There is a homotopy cube-with-handles $H^3$ such that

$$f^{-1}(x) \subset \text{Int } H^3 \subset H^3 \subset U' \cap f^{-1}(U).$$

Let $W$ be an open 3-cell in $U$ such that $x \in W$ and $f^{-1}(W) \subset \text{Int } H^3$. Define inductively $G_0, G_1, G_2, \ldots$ by letting $G_0 = \pi_1(f^{-1}(W))$, and by letting

$$G_i = G_{i-1}(X_1 X_2 X_1^{-1} X_2^{-1} X_3^{\infty}).$$

(See p. 74 of [10] for notation.) In other words, $G_i$ is the subgroup of $G_{i-1}$ generated by all elements of the form $w u v^{-1} u^{-1} \tau^p$ where $u, v, \tau \in G_{i-1}$. Let $F_0, F_1, F_2, \ldots$ be the corresponding subgroups of $\pi_1(H^3)$.
The subgroup $G_1$ certainly contains the commutator subgroup of $G_0$. The image of $G_1$ in $H_1(f^{-1}(W); Z)$ is $p \cdot H_1(f^{-1}(W); Z)$. Thus

$$\pi_1(f^{-1}(W)) / G_1 \cong H_1(f^{-1}(W); Z) / p \cdot H_1(f^{-1}(W); Z) \cong H_1(f^{-1}(W); Z_p).$$

Let $\delta \in \pi_1(f^{-1}(W))$. Since $H_1(f^{-1}(W); Z_p) = 0$ (by Lemma 3), $\delta \in G_1$. Thus $\delta$ is a product of elements of the form $u v u^{-1} v^{-1} \tau \rho$ where $u, v, \tau \in G_0$. By applying the same argument to $u, v, and \tau$, we see that $u, v, \tau \in G_1$. Thus $\delta \in G_2$. By repeating this argument, $\delta \in \bigcap_{n=0}^\infty G_1$. By Corollary 2.12 on p. 109 of [10], $\bigcap_{n=1}^\infty F_i = 1$. Thus $\delta = 1$ in $\pi_1(H^3)$, and $f^{-1}(x)$ is $UV^\infty$.

**Corollary 1.** Let $M^3$ and $N^3$ be 3-manifolds, possibly with boundary, and let $f: M^3 \to N^3$ be a compact, monotone, boundary preserving, onto map. Let $p$ denote 0 or a prime, and suppose that $M^3$ is orientable if $p > 2$. If $f^{-1}(x)$ is strongly 1-acyclic over $Z_p$ in $M^3$ for all $x \in U$, then $\{x \in U : f^{-1}(x)$ is not cellular$\}$ is a locally finite set in $N^3$.

**III. Maps where the image of the singular set lies in a strongly acyclic set.** We state below a slightly strengthened version of Theorem 2 of [13]: here we assume that $M^3$ is orientable only if $p > 2$, and thus the 1-handles which are attached to $\text{Bd} \ Q_i$ to obtain $H_i$ may be attached in a nonorientable fashion. (See the statement of Theorem 2 for the definition of $Q_i$ and $H_i$.) The only additional difficulty in the proof is when we have $S_i \subset \text{Bd} \ Z_p^*$ and $S_k \subset \text{Bd} \ Z_k^*$ topologically parallel. (See p. 133 of [12].) As before, each loop in $S_i Z_p$-bounds in $Z_p^*$, and the same argument shows that $S_i$ is a 2-sphere if $S_i$ is not homeomorphic to a projective plane. But if $S_i$ is a projective plane, it must contain an orientation-reversing simple closed curve since $S_i$ is two-sided. This contradicts the fact that every simple closed curve in $S_i Z_p$-bounds in $Z_p^*$, since $p = 0, 2$.

**Theorem 2.** Let $p$ denote 0 or a prime. Let $X$ be a compact, proper subset of $\text{Int} \ M^3$, where $M^3$ is a 3-manifold, possibly with boundary. Suppose $M^3$ is orientable if $p > 2$, and suppose that $X$ has the following property relative to $M^3$ and $p$. For each open set $U \subset M^3$ with $X \subset U$, there is an open set $V, X \subset V \subset U$, such that, under inclusion, $H_1(V - X; Z_p) \to H_1(U; Z_p)$ is zero. Then $X = \bigcap_{i=1}^\infty H_i$, where $H_i$ is a compact polyhedron in $M^3$, each component of $H_i$ is a 3-manifold with nonempty boundary, $H_{i+1} \subset \text{Int} \ H_i$ and each $H_i$ has the following structure: it is obtained from a compact polyhedron $Q_i$, each component of which is a 3-manifold whose boundary consists entirely of 2-spheres, by adding to $\text{Bd} \ Q_i$, a finite number of (solid, possibly nonorientable) 1-handles.

Let $f: M \to N$ be a map. Then let $S_f = \{x \in M : f^{-1}(x)$ is nondegenerate$\}$.

**Theorem 3.** Let $p$ denote 0 or a prime. Let $M^3$ and $N^3$ be piecewise-linear 3-manifolds, possibly with boundary, where $M^3$ is orientable if $p \neq 2$. Let $X$ be a compact subset of $\text{Int} \ N^3$ such that each component of $X$ is strongly acyclic over $Z_p$. Let $f: M^3 \to N^3$ be a compact, boundary preserving map with $f(S_f) \subset X$. Then $N^3$ can
be obtained from $M^3$ by cutting out of Int $M^3$ a finite number of polyhedral 3-manifolds which are each bounded by a 2-sphere, and replacing each by a polyhedral $Z_p$-homology 3-cell.

**Proof.** By Theorem 2 of [12], $X$ is the intersection of a decreasing sequence of $Z_p$-homology cubes-with-handles. Thus we can assume that $N^3$ is a $Z_p$-homology cube-with-handles, and that each two-sided surface in Int $N^3$ separates $N^3$.

The first half of the proof will be to show that $f^{-1}(X)$ has the following property in Int $M^3$: for each open set $U \subseteq$ Int $M^3$ with $f^{-1}(X) \subseteq U$, there is an open set $V$, with $f^{-1}(X) \subseteq V \subseteq U$, such that, under inclusion, $H_1(V - f^{-1}(X); Z_p) \rightarrow H_1(U; Z_p)$ is zero.

Let $U$ be an open set in Int $M^3$ with $f^{-1}(X) \subseteq U$. Since $Cl(M) \subseteq U$, $f(U)$ is open. Let $Z^3$ be a compact polyhedron in $f(U)$ such that each component of $Z^3$ is a 3-manifold with boundary, and such that $X \subseteq$ Int $Z^3$. Since $X$ is strongly 1-acyclic over $Z_p$, there is an open set $W$ containing $X$ such that, under inclusion

$$H_1(W - X; Z_p) \rightarrow H_1(Z^3; Z_p)$$

is zero.

Let $V = f^{-1}(W)$, and let $[\sigma] \in H_1(V - f^{-1}(X); Z_p)$ where we can assume that $\sigma$ is a finite, pairwise disjoint collection of (oriented, if $p \neq 2$) simple closed curves such that $f(\sigma)$ is polyhedral in $Z^3$. Let $F^3$ be a regular neighborhood of $f(\sigma)$ in (Int $Z^3$) $- X$. We can triangulate $Z^3$ so that $F^3$ and $f(\sigma)$ are subcomplexes of the triangulation. Then the homeomorphism $f^{-1}((Bd Z^3 \cup F^3)$ induces a triangulation of $f^{-1}(Bd Z^3 \cup F^3)$. Since each of the finite number of components of $f^{-1}(Z^3)$ is a 3-manifold with boundary, by Theorem 5 of [2] there is a triangulation of $f^{-1}(Z^3)$ which is compatible with the above triangulation of $f^{-1}(Bd Z^3 \cup F^3)$. Using the relative simplicial approximation theorem, there is a piecewise-linear, nondegenerate map $g$ from $f^{-1}(Z^3)$ onto $Z^3$ such that

$$g \mid f^{-1}(Bd Z^3 \cup F^3) = f \mid f^{-1}(Bd Z^3 \cup F^3),$$
$$g^{-1}(Bd Z^3 \cup F^3) = f^{-1}(Bd Z^3 \cup F^3).$$

By subdividing we can assume that $g$ is simplicial.

At this point we divide the remainder of the first half of the proof into three cases: Case 1 ($p = 0$), Case 2 ($p = 2$), and Case 3 ($p > 2$).

**Case 1 ($p = 0$).** Since $f(\sigma) \subseteq W - X$, $[f(\sigma)] = 0$ in $H_1(Z^3; Z)$. Thus $f(\sigma)$ must bound a 2-complex $L^2$ in $Z^3$ where each component of $L^2$ is an orientable, two-sided 2-manifold with boundary. We can adjust $L^2$ slightly so that it is in general position mod $f(\sigma)$ with respect to our last triangulation of $Z^3$. Then $g^{-1}(L^2)$ will be a 2-complex in $f^{-1}(Z^3) \subseteq U$, where each component of $g^{-1}(L^2)$ is a two-sided 2-manifold with boundary. Thus, since $M^3$ is orientable, each component of $g^{-1}(L^2)$ is orientable. Since $\sigma$ bounds $g^{-1}(L^2)$, $[\sigma] = 0$ in $H_1(U; Z)$, and the inclusion-induced homomorphism $H_1(V - f^{-1}(X); Z) \rightarrow H_1(U; Z)$ is trivial.
Case 2 \((p=2)\). The proof is essentially the same as Case 1, except that \(L^2\) and \(g^{-1}(L^2)\) may not be orientable.

Case 3 \((p > 2)\). Note that

\[
H_1(Z^3; Z) / G \simeq H_1(Z^3; Z) \otimes Z_p \simeq H_1(Z^3; Z_p)
\]

where \(G\) is the subgroup of \(H_1(Z^3; Z)\) generated by elements of the form \(p[y]\) where \([y] \in H_1(Z^3; Z)\). Since \([f(\sigma)] = 0\) in \(H_1(Z^3; Z_p)\), there is a 1-cycle \([\tau] \in H_1(Z^3; Z)\) so that \([f(\sigma)] = p[\tau]\) in \(H_1(Z^3; Z)\). We can assume that \(\tau\) is a finite, pairwise disjoint collection of polyhedral, oriented, simple closed curves which are in general position with respect to our last triangulation of \(Z^3\). Then \(g^{-1}(\tau)\) is a finite, pairwise disjoint collection of simple closed curves in \(f^{-1}(Z^3)\). We can find a regular neighborhood \(T^3\) of \(\tau\) so close to \(\tau\) that \(g^{-1}(T^3)\) is a regular neighborhood of \(g^{-1}(\tau)\). We can find a 1-cycle \([\delta] \in H_1(Bd T^3; Z)\) so that \([f(\sigma)] = [\delta]\) in \(H_1(Z^3 - Int T^3; Z)\). We can assume that \(\delta\) is a finite collection of mutually exclusive, oriented, simple closed curves on \(Bd T^3\). Then there is a 2-complex \(L^2 \subset Z^3 - Int T^3\) where each component of \(L^2\) is a two-sided, orientable, 2-manifold, and where \(Bd L^2 = f(\sigma) \cup \delta\) (homologically \(f(\sigma) - \delta\)). We can assume that \(L^2\) is in general position mod \(f(\sigma)\) with respect to our last triangulation of \(Z^3\). Then \(g^{-1}(L^2)\) will be a 2-complex where each component of \(g^{-1}(L^2)\) is a two-sided 2-manifold with boundary. Thus \(g^{-1}(L^2)\) is orientable.

Since \(L^2\) is two-sided in \(Z^3\), \(\delta\) is two-sided in \(Bd T^3\). Thus \(g^{-1}(\delta)\) is two-sided in \(g^{-1}(Bd T^3)\), and using this two-sidedness, we can induce an orientation of \(g^{-1}(\delta)\) which is consistent with that on \(g^{-1}(L^2)\). Thus \([g^{-1}(\delta)] = [\sigma]\) in \(H_1(f^{-1}(Z^3); Z)\).

Let \(\alpha\) be a meridional curve on \(Bd T^3\) which is in general position with respect to \(\delta\). Then \(\alpha\) will intersect \(\delta\) algebraically \(\pm p\) times. Since the two-sidedness of \(\delta\) is preserved by \(g^{-1}\), each component of \(g^{-1}(\alpha)\) which is a meridional curve must intersect \(g^{-1}(\delta)\) algebraically \(\pm p\) times. Thus, \([g^{-1}(\delta)] = p[g^{-1}(\tau)]\) in \(H_1(T^3; Z)\).

Therefore, \([\sigma] = p[g^{-1}(\tau)]\) in \(H_1(Z^3; Z)\), and the inclusion-induced homomorphism \(H_1(V - X; Z_p) \to H_1(U; Z_p)\) is trivial. This completes Case 3.

By Theorem 2, we can find a compact polyhedron \(H_0^3\), where each component of \(H_0^3\) is a 3-manifold with nonempty boundary, and where \(H_0^3\) has the following structure: it is obtained from a compact polyhedron \(Q_0^3\), each component of which is a 3-manifold whose boundary consists entirely of 2-spheres, by adding to \(Bd Q_0^3\) a finite number of (solid, possibly nonorientable) 1-handles.

We can also assume that each 1-handle is attached to only one boundary component of \(Bd Q_0^3\) since we can add 1-handles to \(Bd Q_0^3\) which join different components of \(Bd Q_0^3\) without destroying the property that \(Bd Q_0^3\) consists entirely of 2-spheres.

We claim that each component of \(Bd Q_0^3\) separates \(M^3\). For suppose that \(S_0\) is a component of \(Bd Q_0^3\) that does not separate \(M^3\). Then there is a polyhedral simple closed curve \(J\) which intersects \(S_0\) at exactly one point which is a piercing point.

It is easy to see that we can choose \(J\) so that it does not intersect any of the 1-handles.
which are added to \(Q^3_0\) to obtain \(H^3_0\). Let \(S_1\) be the component of \(\text{Bd} \ H^3_0\) which is obtained from \(S_0\) by adding handles. Then \(J\) intersects \(S_1\) only in the same piercing point. Since \(f^{-1}\|f(\text{Bd} \ H^3_0)\) is a homeomorphism, \(f(J)\) is a loop in \(N^3\) which intersects \(f(S_1)\) in exactly one piercing point. Thus \(f(S_1)\) does not separate \(N^3\). But \(f(S_1)\) is a 2-sided surface in \(N^3\), so \(f(S_1)\) must separate \(N^3\). This is a contradiction, so \(S_0\) does separate \(M^3\).

Let \(Q^3\) be the closure of the “inside” complementary domains of the “outermost” boundary components of \(Q^3_0\). (Here, “inside” and “outermost” are relative to \(\text{Bd} \ M^3\), which is connected.) Thus we have “filled in the holes” in \(Q^3_0\) to obtain \(Q^3\), and each component of \(Q^3\) has connected boundary. We define \(H^3\) to be \(Q^3\) union the 1-handles of \(H^3_0 - Q^3_0\) which are not already contained in \(Q^3\).

There are properly embedded polyhedral disks \(B^2_1, \ldots, B^2_t\) in \(H^3\) such that the 1-handles which are added to \(Q^3\) to obtain \(H^3\) are regular neighborhoods of \(B^2_1, \ldots, B^2_t\) in \(H^3\). Let these 1-handles be \(N(B^2_1), \ldots, N(B^2_t)\). Each \(B^2_i\) is mapped properly into \(f(H^3)\) by \(f\), and furthermore, \(f(B^2_i)\) has no singularities near \(\text{Bd} \ B^2_i\). So by Dehn’s Lemma, there exist nonsingular properly embedded polyhedral disks \(D^2_1, \ldots, D^2_t\) in \(f(H^3)\) with \(\text{Bd} \ D^2_i = f(\text{Bd} B^2_i)\). By a cutting and pasting argument, we can choose \(D^2_1, \ldots, D^2_t\) to be disjoint. We can also find disjoint regular neighborhoods \(N(D^2_1), \ldots, N(D^2_t)\) of \(D^2_1, \ldots, D^2_t\) in \(f(H^3)\) so that

\[
f(N(B^2_i) \cap \text{Bd} \ H^3) = N(D^2_i) \cap \text{Bd} f(H^3).
\]

For each \(i\), there is a homeomorphism \(h_i: f(\text{Bd} B^2_i) \to N(D^2_i)\) such that

\[
h_i| (\text{Bd} H^3 \cap N(B^2_i)) = f| (\text{Bd} H^3 \cap N(B^2_i)).
\]

We define a homeomorphism

\[
h: M^3 - \text{Int} \ Q^3 \to (N^3 - \text{Int} f(H^3)) \cup \left( \bigcup_{i=1}^r N(D^2_i) \right)
\]

by \(h| (M^3 - \text{Int} \ H^3) = f| (M^3 - \text{Int} \ H^3)\), and by \(h| N(B^2_i) = h_i\) for each \(i = 1, \ldots, r\).

Then \(h(\text{Bd} Q^3)\) is a finite disjoint collection of 2-spheres in \(N^3\) each of which bounds a \(Z_p\)-homology 3-cell. Furthermore, these homology 3-cells are disjoint since each component of \(h(\text{Bd} Q^3)\) is outermost in the sense that it can be joined to \(\text{Bd} N^3\) with an arc which misses \(h(\text{Bd} Q^3)\) except at one end point.

Let \(K^3_1, \ldots, K^3_k\) be these homology 3-cells, and let \(Q^3_1, \ldots, Q^3_m\) be the corresponding components of \(Q^3\) so that \(h^{-1}(\text{Bd} K^3_i) = \text{Bd} Q^3_i\). Each \(Q^3_i\) is a 3-manifold with 2-sphere boundary. Then \(h\) is a homeomorphism from \(M^3 - (\bigcup_{i=1}^k Q^3_i)\) onto \(N^3 - (\bigcup_{i=1}^m K^3_i)\). Thus we obtain \(N^3\) from \(M^3\) by cutting out the \(Q^3_i\)'s and replacing each with the corresponding \(K^3_i\).

**Remark.** If we define \(*Q^3_i\) to be the closed 3-manifold obtained from \(Q^3_i\) by sewing a 3-cell onto \(\text{Bd} Q^3_i\), and if we define \(*K^3_i\) to be the closed 3-manifold obtained from \(K^3_i\) in the same way, then

\[
M^3 \# *K^3_1 \# \cdots \# *K^3_m \cong N^3 \# *Q^3_1 \# \cdots \# *Q^3_m.
\]
We should also note that we have shown that for any open set $U$ in $M^3$ which contains $X$, then $f^{-1}(X)$ has a polyhedral neighborhood $H^3 \subset U$ where each component of $H^3$ is formed by adding 1-handles to a 3-manifold with 2-sphere boundary. Furthermore, we have shown that these 1-handles are attached in an orientable fashion to the 2-sphere boundary.

**Corollary 2.** Let $M^3$ and $N^3$ be compact 3-manifolds, possibly with boundary. Let $X$ be a compact proper set in $\text{Int } N^3$ with the following property: For each open set $U \subset \text{Int } N^3$ with $X \subset U$, there is an open set $V, X \subset V \subset U$, such that under inclusion $H_1(V - X; \mathbb{Z}_p) \rightarrow H_1(V; \mathbb{Z}_p)$ is zero. Suppose also that $X$ has a polyhedral neighborhood each component of which is an orientable, irreducible 3-manifold with boundary. If there is a boundary preserving map $f$ from $M^3$ onto $N^3$ such that $f(S_1) \subset X$, then $M^3$ can be obtained from $N^3$ by removing the interiors of a finite number of 3-manifolds each of which is bounded by a 2-sphere, and by replacing each by a 3-cell.

**Proof.** By using Theorem 2 and the fact that $X$ has a polyhedral neighborhood each component of which is an irreducible 3-manifold with boundary, we see that $X$ has a polyhedral neighborhood each component of which is a cube-with-handles. Thus we can assume that $N^3$ is a cube-with-handles. The remainder of the proof of Theorem 3 now goes through with the weaker hypothesis on $X$.

**Theorem 4.** Let $M^3$ and $N^3$ be 3-manifolds, possibly with boundary, and let $f: M^3 \rightarrow N^3$ be an onto, compact, boundary preserving mapping from $M^3$ onto $N^3$ such that $f(S_1) \subset X$ where $X$ is a closed 0-dimensional set in $N^3$. Then $f$ is monotone, and $\{x \in N^3 : f^{-1}(x) \text{ is not cellular in } M^3 \}$ is a locally finite subset of $N^3$.

**Proof.** Let $x \in X$, and let $U$ be an arbitrarily small open 3-cell containing $x$. Then there is a polyhedral 3-manifold with boundary $K^3$ so that $x \in \text{Int } K^3 \subset K^3 \subset U$ and so that $\text{Bd } K^3 \cap X = \emptyset$. In fact, using Theorem 2 of [12] and the fact that $U$ is irreducible, we can see that $K^3$ can be chosen to be a cube-with-handles. Then $f^{-1}(K^3)$ is a connected neighborhood of $f^{-1}(x)$ which can be chosen “arbitrarily close” to $f^{-1}(x)$. Thus $f$ is monotone.

We can cover $X$ with the interiors of a locally finite collection of mutually exclusive collection of cubes-with-handles. Thus, in order to prove the theorem, it suffices to consider the case where $N^3$ is a cube-with-handles, and where $M^3$ is a compact 3-manifold with connected boundary. In this case, we will prove that all but a finite number of point inverses of $f$ are cellular.

The set $X$ is strongly 1-acyclic over $Z_2$ in $N^3$, and thus by the remark following the proof of Theorem 3, we have $f^{-1}(X) = \bigcap_{i=1}^\infty H^3_i$, where $H^3_i$ is a 3-manifold with connected boundary, and where $H^3_i \subset \text{Int } H^3_{i-1}$. We can assume that $H^3_i$ is obtained from a compact polyhedron $Q^3_i$ where each component of $Q^3_i$ is a 3-manifold with 2-sphere boundary, by adding to $\text{Bd } Q^3_i$ a finite number of (orientable, solid) 1-handles. We also have that each 1-cycle in $\text{Bd } H^3_i$ bounds in $\text{Int } H^3_{i-1}$. We have assumed that $M^3$ is compact and that $H_1(M^3; Z_2)$ is finitely generated;
so it is easy to show that there is an integer \( N \) so that there are not more than \( N \) disjoint 3-manifolds with 2-sphere boundary and nontrivial \( \mathbb{Z}_2 \)-homology in \( \text{Int} M^3 \). Therefore, all but at most \( N \) components of \( f^{-1}(X) \) are the intersection of a decreasing sequence of \( \mathbb{Z}_2 \)-homology cubes-with-handles.

If \( Z_2^p \) is a \( \mathbb{Z}_2 \)-homology cube-with-handles, the inclusion-induced homomorphism \( H_1(\text{Bd} Z_2^p; \mathbb{Z}_2) \to H_1(Z_2^p; \mathbb{Z}_2) \) is onto. Thus, if \( Z_2^p \subset \text{Int} Z_2^{p-1} \) where \( Z_2^{p-1} \) is another \( \mathbb{Z}_2 \)-homology cube-with-handles, and if each 1-cycle in \( \text{Bd} Z_2^p \) \( \mathbb{Z}_2 \)-bounds in \( \text{Int} Z_2^{p-1} \), then the inclusion-induced homomorphism \( H_1(Z_2^p; \mathbb{Z}_2) \to H_1(Z_2^{p-1}; \mathbb{Z}_2) \) is trivial. Therefore, each component of \( f^{-1}(X) \) which is the intersection of \( \mathbb{Z}_2 \)-homology cubes-with-handles must be strongly 1-acyclic over \( \mathbb{Z}_2 \). This shows that at most a finite number of point inverses of \( f \) are not strongly 1-acyclic over \( \mathbb{Z}_2 \).

We can now apply Theorem 1 which implies that only a finite number of the strongly 1-acyclic over \( \mathbb{Z}_2 \) point inverses of \( f \) are not cellular.

**IV. Maps almost all of whose point inverses are strongly 1-acyclic over \( \mathbb{Z}_p \).**

**Lemma 4.** Let \( f: M \to N \) be a compact map from a metric space \( M \) onto a metric space \( N \). Let \( X \) be a closed set in \( N \). Let \( G \) be a decomposition of \( M \) defined by
\[
G = \{ f^{-1}(y) : y \in X \} \cup \{ x \in M : f(x) \notin X \}.
\]
Let \( Q = M/G \) and let \( \pi: M \to Q = M/G \) be the projection map for the decomposition \( G \). Let \( p: Q \to N \) be defined so as to make the following diagram commute:
\[
\begin{array}{ccc}
M & \xrightarrow{\pi} & Q \\
\downarrow f & & \downarrow p \\
N & & \\
\end{array}
\]
Then
1. \( G \) is upper semicontinuous and hence \( \pi \) is continuous and compact.
2. The decomposition \( \{ p^{-1}(y) : y \in N \} \) is upper semicontinuous and hence \( p \) is continuous and compact.

**Proof.** Lemma 4 follows from the fact that \( \{ f^{-1}(y) : y \in N \} \) is an upper semicontinuous decomposition of \( M \).

**Lemma 5.** Let \( p: Q \to N^3 \) be a compact, monotone map from a metric space \( Q \) onto a 3-manifold \( N^3 \), possibly with boundary. Let \( X \) be a closed set in \( N^3 \) containing \( \text{Bd} N^3 \). Suppose that \( p|_{p^{-1}(X)} \) is a homeomorphism, and that \( W = Q - p^{-1}(X) \) is an open 3-manifold. If \( p^{-1}(x) \) is cellular for all \( x \in N^3 - X \), then there is a homeomorphism \( h: N^3 \to Q \) such that \( h|X = p^{-1}|X \).

The proof of Lemma 9 is the same as the proof of Theorem 1 of [1].

Suppose \( f: M^3 \to N^3 \) is a mapping. We let \( A^p_f = \{ x \in M^3 : f^{-1}f(x) \) is either not connected or is not strongly 1-acyclic over \( \mathbb{Z}_p \) \).
Theorem 5. Let \( p \) denote 0 or a prime, and let \( M^3 \) and \( N^3 \) be compact 3-manifolds, possibly with boundary, where \( M^3 \) is orientable if \( p \neq 2 \). Let \( Y \) be a compact set in \( \text{Int } N^3 \) each component of which is strongly acyclic over \( \mathbb{Z}_p \). Let \( f: M^3 \to N^3 \) be an onto, boundary preserving map such that \( f(A^3) \subset Y \). Then \( N^3 \) can be obtained from \( M^3 \) by cutting out of \( M^3 \) a finite number of polyhedral 3-manifolds, each bounded by a 2-sphere, and replacing each by a \( \mathbb{Z}_p \)-homology 3-cell.

Proof. By Theorem 1 there are only a finite number of points \( x_1, x_2, \ldots, x_n \) in \( N^3 - Y \) whose inverses under \( f \) are not cellular in \( M^3 \). Let
\[
X = Y \cup \{ x_1, x_2, \ldots, x_n \} \cup \text{Bd } N^3.
\]
We use this \( X \) to define \( Q, \pi: M^3 \to Q, \) and \( p: Q \to N^3 \) as in Lemma 4. Since \( \pi(M^3 - f^{-1}(X)) \) is a homeomorphism from \( M^3 - f^{-1}(X) \) onto \( W = Q - p^{-1}(X) \), \( W \) is an open 3-manifold. And since \( p|p^{-1}(X) \) is one-to-one and continuous, \( p|p^{-1}(X) \) is a homeomorphism. Therefore, by Lemma 5, there is a homeomorphism \( h: N^3 \to Q \). In particular, \( Q \) is a 3-manifold \( Q^3 \). Let
\[
X' = Y \cup \{ x_1, \ldots, x_n \}.
\]
Then \( \pi(S_a) \cap p^{-1}(X') = h(X') \), and \( X' \) is strongly acyclic over \( \mathbb{Z}_p \), so the map \( \pi \) satisfies the hypotheses of Theorem 3.

Theorem 6. Let \( p \) denote 0 or a prime, and let \( M^3 \) and \( N^3 \) be 3-manifolds, possibly with boundary, where \( M^3 \) is orientable if \( p > 2 \). Let \( Y \) be a closed 0-dimensional set in \( \text{Int } N^3 \), and let \( f: M^3 \to N^3 \) be an onto, compact, boundary preserving map such that \( f(A^3) \subset Y \). Then \( \{ x \in N^3 : f^{-1}(x) \text{ is not cellular in } M^3 \} \) is a locally finite subset of \( N^3 \).

Proof. By Corollary 1, the set \( \{ x \in N^3 - Y : f^{-1}(x) \text{ is not cellular in } M^3 \} \) is a locally finite subset of \( N^3 \).

Let
\[
X = Y \cup \text{Bd } N^3 \cup \{ x \in N^3 - Y : f^{-1}(x) \text{ is not cellular} \}.
\]
Let \( Q, \pi: M^3 \to Q, p: Q \to N^3, \) and \( h: N^3 \to Q \) be defined as in Lemmas 4 and 5. Let
\[
X' = Y \cup \{ x \in N^3 - Y : f^{-1}(x) \text{ is not cellular} \}.
\]
Then \( \pi(S_a) \cap p^{-1}(X') = h(X') \), and thus \( \pi(S_a) \) is contained in a closed 0-dimensional set in \( Q \). Theorem 4 can be applied to the map \( \pi: M^3 \to Q^3 \) to say that
\[
\{ y \in Q^3 : \pi^{-1}(y) \text{ is not cellular in } M^3 \}
\]
is a locally finite subset of \( Q^3 \). The image under \( p \) (or \( h^{-1} \)) of this set is
\[
\{ x \in N^3 : f^{-1}(x) \text{ is not cellular in } M^3 \}
\]
which must then be a locally finite subset of \( N^3 \).
V. Further applications. The following lemma is a slight generalization of Lemma 5 of [13]. While the proof of Lemma 5 of [13] suffices to prove our Lemma 6, a proof is included here for completeness and since part of the proof will be needed to prove Theorem 7.

**Lemma 6.** Let $M^3$ and $N^3$ be 3-manifolds. Let $f: M^3 \to N^3$ be a compact, monotone mapping so that $f(S_t)$ is 0-dimensional. Let $x \in N^3$. If there is an open set $U$ containing $f^{-1}(x)$ so that the inclusion-induced homomorphism from $H_1(U; Z)$ into $H_1(M^3; Z)$ is trivial, then $f^{-1}(x)$ is strongly 1-acyclic over $Z$.

**Proof.** Let $B^3$ be an open 3-cell in $N^3$ with compact closure so that $x \in B^3$ and $W=f^{-1}(B^3)$ is contained in $U$. Let $K_1, K_2, K_3, \ldots$ be a locally finite collection of compact sets in $W$ so that $\bigcup_{i=1}^{\infty} K_i = W$ and each $K_i$ is contained in an open 3-cell $B_i \subset W$. Let

$$e_i = \inf \{ \rho(x, y) : x \in K_i \text{ and } y \in W - B^3 \}$$

where $\rho$ is a metric on $M^3$. Let

$$C_i = \{ x \in N^3 : \text{diam} (f^{-1}(x)) \geq e_i \text{ and } f^{-1}(x) \cap K_i \neq \emptyset \}.$$ 

It is easy to see that each $C_i$ is a closed set. Let $C = \bigcup_{i=1}^{\infty} C_i$.

We will show that $\{f(K_i)\}$ is a locally finite collection in $B^3$. Let $x_0 \in B^3$ and let $V$ be a neighborhood of $x_0$ in $B^3$ with compact closure. Since $f$ is a compact map, $f^{-1}(V)$ has compact closure. Since $\{K_i\}$ is a locally finite collection in $W$, $f^{-1}(V)$ intersects only a finite number of the $K_i$'s, and thus $V$ intersects only a finite number of the $f(K_i)$'s. Using the fact the $\{f(K_i)\}$ is a locally finite collection, we see that $C$ is a closed 0-dimensional subset of $B^3$.

Consider the following commutative diagram where the horizontal maps are induced by inclusion, and the vertical maps are induced by $f$.

$$
\begin{array}{ccc}
H_1(W - f^{-1}(C); Z) & \xrightarrow{\alpha} & H_1(W; Z) \\
| & & | \\
H_1(B^3 - C; Z) & \longrightarrow & H_1(B^3; Z)
\end{array}
$$

First, we claim that $\alpha$ is an epimorphism. Let $[\delta] \in H_1(W; Z)$ where $\delta$ is a simple closed curve. Let $O$ be an open set in $B^3$ so that $f(\delta) \subset O$ and $(\text{Bd } O) \cap C = \emptyset$. By applying Lemma 2 of [13], we see that $\delta$ is homologous in $f^{-1}(O)$ to a 1-cycle in $f^{-1}(O) - f^{-1}(O \cap C) \subset W - f^{-1}(C)$.

Finally, we claim that $\alpha$ is the zero homomorphism. Let $[\tau] \in H_1(W - f^{-1}(C); Z)$ where $\tau$ is a simple closed curve. We can also suppose that $f(\tau)$ is a simple closed curve, and that $f(\tau)$ bounds an orientable surface $S$ in $B^3 - C$. By our choice of the $\delta_i$'s, for each $y \in B^3 - C$, there is an open set $V_y$ so that $f^{-1}(V_y)$ is contractible in $W$. Let $\mathcal{V} = \{ V_y : y \in B^3 - C \}$. We can find a triangulation $T$ of $S$ which is so fine that
for each 2-simplex $\sigma \in T$, there is a $V_\sigma \in V$ so that $\sigma \subseteq V_\sigma$. Using the fact that $f$ is monotone, we can find a map $h$ from the 1-skeleton of $T$ into $W - f^{-1}(C)$ so that, if $\sigma$ is a 2-simplex of $T$, $h(\partial \sigma) \subseteq f^{-1}(V_\sigma)$. (See the proof of Theorem 2.1 of [15] for details.) We can also suppose that $hf|_\tau = \text{id}$. Since each $V_\sigma$ is contractible in $W$, $h$ can be extended to a map $H$ which takes the surface $S$ into $W$ and which takes $\partial S$ onto $\tau$. Thus, $a[\tau] = 0$ in $H_1(W; \mathbb{Z})$.

**Theorem 7.** Let $M^3$ and $N^3$ be 3-manifolds, possibly with boundary. Let $f$ be a compact, monotone, boundary preserving mapping from $M^3$ onto $N^3$ such that $f(S_f)$ is 0-dimensional. Then $\{x \in N^3 : f^{-1}(x) \text{ is not cellular}\}$ is a locally finite subset of $N^3$.

**Proof.** By a procedure similar to the first part of the proof of Lemma 6, we can find a closed set $C \subseteq f(S_f) \subseteq N^3$ so that, if $x \notin C$, then there is an open set $U_x$ where $f^{-1}(x) \subseteq U_x$ and $U_x$ is contractible in $M^3$. By Lemma 6, if $x \in N^3 - C$, then $f^{-1}(x)$ is strongly 1-acyclic over $\mathbb{Z}$. Thus $f(A^0) \subseteq C$, and $C$ is a closed 0-dimensional set. Theorem 7 now follows from Theorem 6.

Let $f: M^3 \to N^3$ be an onto, compact, boundary preserving map as before. Many of our earlier results have shown that $\{x \in N^3 : f^{-1}(x) \text{ is not cellular in } M^3\}$ is a locally finite subset of $N^3$. The following three corollaries concern mappings of this type.

**Corollary 3.** Let $M^3$ and $N^3$ be 3-manifolds, possibly with boundary. Let $f: M^3 \to N^3$ be a compact, monotone, boundary preserving map such that $\{x \in N^3 : f^{-1}(x) \text{ is not cellular}\}$ is a locally finite subset of $N^3$. Then

(i) For each $x \in N^3$ and each open set $U$ containing $f^{-1}(x)$, there is an open set $V$ with $f^{-1}(x) \subseteq V \subseteq U$, such that $V - f^{-1}(x)$ is homeomorphic to $S^2 \times (0, 1)$.

(ii) $N^3$ can be obtained from $M^3$ by cutting out of $M^3$ a locally finite collection of mutually exclusive, polyhedral 3-manifolds, each with 2-sphere boundary, and replacing each by a 3-cell.

**Proof.** (i) If $f^{-1}(x)$ is cellular, this follows from Theorem 1 of [3].

Let $x_1, x_2, x_3, \ldots$ be the points in $N^3$ such that $f^{-1}(x_i)$ is not cellular for $i = 1, 2, 3, \ldots$. Let $X = \{x_1, x_2, x_3, \ldots\} \cup \text{Bd } N^3$. Let the 3-manifold $Q^3$, the maps $\pi: M^3 \to Q^3$, $p: Q^3 \to N^3$, and the homeomorphism $h: N^3 \to Q^3$ be defined as in Lemmas 4 and 5. It will be sufficient to show that $f^{-1}(x_i)$ has the required neighborhood. We are given an open set $U \supseteq f^{-1}(x_1)$. Let $U'$ be an open set in $M^3$ so that $f^{-1}(x_i) \subseteq U' \subseteq U$ and $U' \cap f^{-1}(x_i) = \emptyset$ for $i \geq 2$. Then $h^{-1}(p(U'))$ is an open set containing $x_1$ in $N^3$. Let $W$ be an open 3-cell so that $x_1 \subseteq W \subseteq h^{-1}(p(U'))$. Let $V = \pi^{-1}h(W)$. Then $V - f^{-1}(x_1)$ is homeomorphic by $\pi^{-1}h$ to $W - \{x_1\}$ which is homeomorphic to $S^2 \times (0, 1)$.

(ii) As in part (i) let $x_1, x_2, x_3, \ldots$ be the points of $N^3$ whose inverses are not cellular. We can find pairwise disjoint closed neighborhoods $K_1, K_2, K_3, \ldots$ of $f^{-1}(x_1), f^{-1}(x_2), f^{-1}(x_3), \ldots$ respectively so that $K_i - f^{-1}(x_i)$ is homeomorphic to $S^2 \times (0, 1]$. Then each $K_i$ is a 3-manifold with 2-sphere boundary, and $\pi|_{K_i}$ is a
boundary preserving map of $K_i$ onto a 3-cell. Furthermore, $\pi|_{M^3 - \bigcup_{i=1}^n K_i}$ is a homeomorphism. Thus $Q^3$ can be obtained by cutting $K_1, K_2, K_3, \ldots$ out of $M^3$, and replacing each by a 3-cell.

**Corollary 4.** Let $M^3$ and $N^3$ be compact 3-manifolds, possibly with boundary. Let $f: M^3 \to N^3$ be a boundary preserving, onto map such that $\{x \in N^3 : f^{-1}(x) \text{ is not cellular in } M^3\}$ is a finite set. If $M^3$ is homeomorphic to $N^3$, then $f^{-1}(x)$ is cellular for every $x \in N^3$.

**Proof.** By Corollary 3, part (ii), there are closed 3-manifolds $*K^3_0, \ldots, *K^3_n$ such that

$$M^3 = N^3 \# *K^3_0 \# \cdots \# *K^3_n.$$ 

By a corollary to the Grushko-Neumann Theorem (see p. 192 of [10]), the rank of $\pi_1(M^3)$ is equal to the sum of the ranks of $\pi_1(N^3), \pi_1(K^3_0), \ldots, \pi_1(K^3_n)$. Therefore

$$\pi_1(*K^3_0) = \cdots = \pi_1(*K^3_n) = 1,$$

and each $*K^3_i$ ($i=0, \ldots, n$) is a homotopy 3-sphere.

If $M^3$ is closed and orientable, we use the unique decomposition theorem of Milnor [14] to show that $*K^3_0, \ldots, *K^3_n$ are all 3-spheres. This shows that $f^{-1}(x)$ is cellular for every $x \in N^3$.

If $M^3$ is orientable with boundary, we can sew a cube-with-handles onto each boundary component of $M^3$ to obtain a closed manifold $M^3_0$. The homeomorphism from $M^3$ to $N^3$ induces a similar sewing of cubes-with-handles onto $\text{Bd } N^3$ to give a closed 3-manifold $N^3_0$ which is homeomorphic to $M^3_0$. We have

$$M^3_0 = N^3_0 \# *K^3_0 \# \cdots \# *K^3_n$$

and the argument for the closed orientable case applies.

If $M^3$ is nonorientable, we apply the previous argument to the orientable double covering of $M^3$.

**Corollary 5.** Let $M^3$ and $N^3$ be compact (i.e., closed) 3-manifolds. Let $f: M^3 \to N^3$ be an onto map such that $\{x \in N^3 : f^{-1}(x) \text{ is not cellular in } M^3\}$ is finite, and let $g: N^3 \to M^3$ be an onto map such that $\{x \in M^3 : g^{-1}(x) \text{ is not cellular in } N^3\}$ is finite. Then $M^3$ is homeomorphic to $N^3$.

**Proof.** By Corollary 2, we have $M^3 = N^3 \# *K^3_0 \# \cdots \# *K^3_n$ and $N^3 = M^3 \# *Q^3_0 \# \cdots \# *Q^3_m$. By a corollary to the Grushko-Neumann Theorem (p. 192 of [10]) we see that $*K^3_0, \ldots, *K^3_n, *Q^3_0, \ldots, *Q^3_m$ are all homotopy 3-spheres. This implies that all of the point inverses of $f$ and $g$ have property $UV^\infty$. Then Corollary 5 follows from Corollary 2.3 of [11].

**VI. On Haken's finiteness theorem.** In [5], Wolfgang Haken stated a finiteness theorem for incompressible surfaces in a compact 3-manifold $M^3$. We are interested here only in the special case of the theorem where the surfaces are closed: this
case is stated as Theorem C. Some difficulties arise with Haken’s proof in the case where $M^3$ is not irreducible. Haken’s proof is correct and can be simplified considerably in the case where $M^3$ is irreducible. We give here an argument due to John Hempel to show that the finiteness theorem holds in the case where $M^3$ may not be irreducible. Haken intended to prove Kneser’s Theorem [7] as a special case of the finiteness theorem; our argument uses Kneser’s Theorem. The previous results of this paper depend on the finiteness theorem directly through Theorem 2 of [12].

In this section we will be working in the piecewise-linear category. A surface is a 2-manifold. If $F^2$ is a surface in a 3-manifold $M^3$, and if $F^2$ is not a 2-sphere, then $F^2$ is incompressible in $M^3$ if every simple closed curve in $F^2$ that bounds an (open) disk in $M^3 - F^2$ also bounds a disk in $F^2$. A 2-sphere is incompressible in $M^3$ if it does not bound a 3-cell in $M^3$. A 3-manifold $M^3$ is irreducible if every 2-sphere in $M^3$ bounds a 3-cell in $M^3$.

Two surfaces $F^2_0$ and $F^2_1$ in a 3-manifold $M^3$ are parallel in $M^3$ if there is an embedding $\alpha: F^2_0 \times [0, 1] \to M^3$ such that $\alpha_0: F^2_0 \to M^3$ is the inclusion map, and $\alpha_1: F^2_1 \to M^3$ takes $F^2_0$ homeomorphically onto $F^2_1$. If $F^2_1, \ldots, F^2_n$ are disjoint surfaces in a 3-manifold $M^3$, and if $L^3$ is the closure of a complementary domain of $M^3 - \bigcup_{i=1}^n F^2_i$, then $L^3$ is a parallelity component if, for some $i=1, \ldots, n$, there is a homeomorphism $h: F^2_i \times [0, 1] \to L^3$ such that $h_0: F^2_i \to L^3$ is the inclusion map, and $h_1: F^2_i \to L^3$ takes $F^2_0$ homeomorphically onto $F^2_1$ for some $j=1, \ldots, n, j \neq i$.

If $C^3$ is a 3-manifold, possibly with boundary, we define $\hat{C}^3$ to be the 3-manifold, possibly with boundary, obtained from $C^3$ by capping off each 2-sphere boundary component of $C^3$ with a 3-cell.

If $B^3$ is a 3-cell, and if $B^3_1, \ldots, B^3_k$ are disjoint polyhedral 3-cells in $\text{Int} B^3$, then we call the manifold-with-boundary $B^3 - (\bigcup_{i=1}^k \text{Int} B^3_i)$ a punctured 3-cell.

**Lemma A.** If $F^2$ is an incompressible surface in the product $M^2 \times [0, 1]$, where $M^2$ is a compact 2-manifold, then $F^2$ is parallel to $M^2 \times \{0\}$ and $M^2 \times \{1\}$.

This lemma is stated and proved by Haken on pp. 91–96 of [5].

**Lemma B.** If $C^3$ is a 3-manifold, possibly with boundary, and $\hat{C}^3$ is irreducible, then the finiteness theorem holds for $C^3$. In other words, there is an integer $n = n(C^3)$ such that if $F^2_1, \ldots, F^2_{n+1}$ are $n + 1$ disjoint incompressible polyhedral surfaces in $C^3$, then two of these surfaces are parallel.

**Proof.** We have assumed the finiteness theorem for irreducible 3-manifolds, so there is an integer $n = n(C^3)$ such that if there are more than $n(C^3)$ disjoint incompressible surfaces in $\hat{C}^3$, then two of them are parallel. There are disjoint 3-cells $B^3_1, \ldots, B^3_k$ such that $C^3 = \hat{C}^3 - (\bigcup_{i=1}^k \text{Int} B^3_i)$. Let $n = n(C^3) = n(\hat{C}^3) + 2k$. Let $F^2_1, \ldots, F^2_{n+1}$ be $n + 1$ disjoint incompressible surfaces in $C^3$. Then $n - k + 1$ of these surfaces are incompressible in $\hat{C}^3$. There are $k + 1$ distinct pairs from $F^2_1, \ldots, F^2_{n+1}$ which are parallel in $\hat{C}^3$. (We say that the pair $(F^2_i, F^2_j)$ is distinct from the pair...
Theorem C. Let $M^3$ be a compact 3-manifold, possibly with boundary. Then there is an integer $n_0 = n(M^3)$ such that if $F_1, \ldots, F_{n_0 + 1}$ are $n_0 + 1$ disjoint polyhedral-incompressible surfaces in $M^3$, then two of these surfaces are parallel.

Proof. Let $\Sigma = \{S_2, \ldots, S_f\}$ be a disjoint collection of 2-spheres in $M^3$. Let $N_1, \ldots, N_s$ be disjoint regular neighborhoods of $S_2, \ldots, S_f$ respectively. Let $C_1, \ldots, C_s$ be the components of $\text{Cl} (M^3 - \bigcup_{i=1}^s N_i)$. (The $C_i$'s are determined up to homeomorphism by the $S_i$'s and do not depend on the choice of the $N_i$'s. Note that $k$ may not equal $l$ since some of the $S_i$'s may not separate $M^3$.) We will call $\Sigma$ a complete system of 2-spheres in $M^3$ if $C_1, \ldots, C_s$ are each irreducible.

We will let $n(M^3, \Sigma) = \sum_{i=1}^s n(C_i)$ where $n(C_i)$ is defined in Lemma B.

Kneser's Theorem [7] shows that there is a complete system $\Sigma_0$ of 2-spheres in $M^3$. We will assume $\Sigma_0$ is a fixed complete system and we will let $n_0 = n(M^3, \Sigma_0)$.

Let $F_2, \ldots, F_{n_0 + 1}$ be disjoint incompressible surfaces in $M^3$. Let $F^2 = \bigcup_{i=1}^{n_0 + 1} F_i$. Suppose $\Sigma = \{S_2, \ldots, S_f\}$ is a complete system of 2-spheres in $M^3$, each of which is in general position with respect to $F^2$, and suppose that $n(M^3, \Sigma) = n_0$. Let $m(M^3, \Sigma, F^2)$ be the number of components of $(\bigcup_{i=1}^s S_i) \cap F^2$. (Each of these components is a simple closed curve.) We can suppose $m(M^3, \Sigma, F^2)$ is minimal over all such complete systems of 2-spheres in $M^3$. Theorem C will be proved if $m(M^3, \Sigma, F^2)$ is zero. For then there will be more than $n(C_i)$ of the surfaces $F_2, \ldots, F_{n_0 + 1}$ in one of the components $C_i$, and two of these surfaces must be parallel in $C_i$ by Lemma B. (Let $N_1, \ldots, N_s$ and $C_1, \ldots, C_s$ be defined as before.)

So we suppose that $m(M^3, \Sigma, F^2) > 0$. Any simple closed curve of $\bigcup_{i=1}^s S_i$ must bound a disk in $F^2$, since $F^2$ is incompressible. Therefore, we can choose an "innermost" (on $F^2$) simple closed curve $J$ of $\bigcup_{i=1}^s S_i$ in $F^2$; suppose $J \subset S_{i_1} \cap F^2$ for some $i_1 = 1, \ldots, l$ and $s_1 = 1, \ldots, n_0 + 1$. Let $D^2$ be the disk that $J$ bounds in $F^2$. Then $D^2$ is contained in some $C_q$ (where $q = 1, \ldots, k$) except for a regular neighborhood of $\partial D^2$.

Let $E_1^2$ and $E_2^2$ be the two disks bounded by $J$ in $S_{i_1}$. We can push each of the 2-spheres $E_1^2 \cup D^2$ and $E_2^2 \cup D^2$ to one side so that they each miss $D^2$, and so that they are each contained in $C_q$. Then one of these 2-spheres must be in the boundary of a punctured cube $P^3$ in $C_q$ since $C_q$ is irreducible. Let $S_{i_1}'$ be the 2-sphere that is not in the boundary of $P^3$, and let $\Sigma' = \{S_{i_1}, S_{i_1}', S_{i_2}, S_{i_2}', \ldots, S_{i_k}\}$. We will show that $\Sigma'$ is a complete system of 2-spheres in $M^3$, that $n(M^3, \Sigma') = n_0$, and that $m(M^3, \Sigma', F^2) < m(M^3, \Sigma, F^2)$.

Let $C_1 (i = 1, \ldots, k)$ be the component of $\partial (M^3 - \bigcup_{i=1}^s N_i)$ on the "other side" of $S_{i_1}$. (If $S_{i_1}$ does not separate $M^3$, then $C_1$ may equal $C_1$.) If we choose a small regular neighborhood $N_{i_1}'$ of $S_{i_1}$ and let $N_{i_1}' \cap D^2 = \emptyset$ and let $N_{i_1}' = N_{i_1}$ for
i ≠ r, we can define $C_q^3$ and $C_r^3$ to be components of $\text{Cl}(M^3 - \bigcup_{i=1}^l N_i^3)$. A subdisk $D^2_\delta$ of $D^2$ is a spanning disk of $C^3_\delta$ and if we remove the interior of a regular neighborhood of $D^2_\delta$, this separates $C^3_\delta$ into two components, one homeomorphic to $C^3_\delta$, and the other homeomorphic to the punctured cube $P^3$. Thus $C^3_\delta$ is homeomorphic to $C^3_\delta$. Furthermore, $C^3_\delta$ is homeomorphic to the manifold obtained by sewing $P^3$ to $C^3_r$ along a disk on the boundary of each. Thus $C^3_\delta$ is homeomorphic to $C^3_r$. We also have $n(C^3_\delta) + n(C^3_r) = n(C^3_\delta) + n(C^3_r)$ since the 2-sphere boundary components of $C^3_\delta \cap P^3$ which were removed from $C^3_\delta$ to obtain $C^3_\delta$ were added to $C^3_r$ to obtain $C^3_r$. Thus $n(M^3, \Sigma^3) = n_0$.

Since $S^2 \cap D^2 = \emptyset$, $m(M^3, \Sigma^3, F^3) < m(M^3, \Sigma, F^3)$, and this contradicts our assumption that $m(M^3, \Sigma, F^3)$ was minimal.

**BIBLIOGRAPHY**


Department of Mathematics, University of Utah, Salt Lake City, Utah 84112

Current address: Department of Mathematics, Western Michigan University, Kalamazoo, Michigan 49001

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use