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ADDENDUM TO "ON A PROBLEM OF TURÁN ABOUT POLYNOMIALS WITH CURVED MAJORANTS"

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Let π_n denote the class of polynomials $p_n(x)=\sum_{k=0}^n a_k x^k$ of degree n which satisfy $|p_n(x)|\leqq(1-x^2)^{1/2}$ for $-1<x<1$. Given any x_0 in $[-1, 1]$, how large can $|p_n^{(k)}(x_0)|$ (the kth derivative at x_0) be if $p_n(x)$ belongs to the class π_n? In case $x_0=0$ the problem is equivalent to the problem of estimating $|a_k|$ if $p_n(x)=\sum_{k=0}^n a_k x^k \in \pi_n$. It has been shown [3] that if $p_n(x)=\sum_{k=0}^n a_k x^k \in \pi_n$ then

$$|a_1| \leqq n-1, \qquad |a_2| \leqq \{(n-1)^2+1\}/2.$$

Here we prove the following theorem which gives a sharp estimate for each of the coefficients.

THEOREM. *If $p_n(x)=\sum_{k=0}^n a_k x^k$ is a polynomial of degree n such that $|p_n(x)| \leqq(1-x^2)^{1/2}$ for $-1<x<1$ and $U_n(x)$ denotes the nth Chebyshev polynomial of the second kind, then, according as $n-k$ is even or odd, $|a_k|$ is bounded above by the absolute value of the coefficient of x^k in $e^{i\gamma}(1-x^2)U_{n-2}(x)$ or $e^{i\gamma}(1-x^2)U_{n-3}(x)$, respectively.*

The idea of proof comes from a paper of O. D. Kellogg [2].

Proof. Without loss of generality we may suppose $p_n(x)$ to be real for real x.

Taking first the case in which k, n are both even, we consider the polynomial

$$\tfrac{1}{2}\{p_n(x)+p_n(-x)\} = a_0+a_2x^2+a_4x^4+\cdots+a_{n-2}x^{n-2}+a_nx^n$$

and compare it with the polynomial

$$\begin{aligned}(1-x^2)U_{n-2}(x) &= -2^{n-2}x^n+\sum_{j=1}^{n/2-1}(-1)^{j-1}\left\{\frac{(n-1-j)!}{(j-1)!\,(n-2j)!}+\frac{1}{4}\frac{(n-2-j)!}{j!\,(n-2-2j)!}\right\} \\ &\qquad\qquad \times(2x)^{n-2j}+(-1)^{n/2-1} \\ &= \sum_{k=0}^{n} A_k x^k \quad \text{(say)}.\end{aligned}$$

If $-1<\lambda<1$, the difference

$$D(x, \lambda) = (1-x^2)U_{n-2}(x)-\lambda\{a_0+a_2x^2+a_4x^4+\cdots+a_{n-2}x^{n-2}+a_nx^n\}$$

is positive at all points of the interval $(-1, 1)$ where $(1-x^2)U_{n-2}(x)=(1-x^2)^{1/2}$ and negative where $(1-x^2)U_{n-2}(x)=-(1-x^2)^{1/2}$. It is readily seen that there are

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a total of $n-1$ points on the interval $(-1, 1)$ where $(1-x^2)U_{n-2}(x)$ is alternately equal to $(1-x^2)^{1/2}$, $-(1-x^2)^{1/2}$. Hence the polynomial $D(x, \lambda)$ has at least $n-2$ real zeros in $(-1, 1)$. Since it also vanishes at -1, $+1$ we conclude that all the zeros of $D(x, \lambda)$ are real and distinct. All the odd coefficients of $D(x, \lambda)$ being zero, none of the even coefficients can vanish; for then by Déscartes' rule of signs, the zeros of $D(x, \lambda)$ could not all be real. Thus, for every even $k \leqq n$ and for $-1<\lambda<1$, $|A_k-\lambda a_k| \neq 0$. This is possible only if $|a_k| \leqq |A_k|$ for $k=0, 2, 4, \ldots, n$.

In case k, n are both odd we apply the above reasoning to the polynomial

$$(1-x^2)U_{n-2}(x)-(\lambda/2)\{p(x)-p(-x)\}$$

(whose even coefficients are all zero) in order to get the desired conclusion.

If k is even and n is odd we consider

$$(1-x^2)U_{n-3}(x)-(\lambda/2)\{p(x)+p(-x)\}$$

whereas if k is odd and n is even we argue with

$$(1-x^2)U_{n-3}(x)-(\lambda/2)\{p(x)-p(-x)\}.$$

One can in fact show that only those polynomials given at the end of the statement of the theorem are extremal for the kth coefficient when $n-k$ is even or odd respectively.

We also observe that Theorem 2 of our paper [3] is an immediate consequence of a theorem of Levin which appears as Theorem 11.7.2 in [1]. If we set $f(z)=p_n(\cos z)$, $\omega(z)=e^{i(n-1)z}\sin z$ the conditions of Levin's theorem are satisfied with $\tau=n$, $\sigma=n$. Since differentiation is a B-operator we have

$$|(d/dx)p_n(\cos x)| \leqq |(d/dx)\{e^{i(n-1)x}\sin x\}|, \qquad -\infty<x<\infty,$$

which readily gives the desired result.

We take this opportunity to make it clear that z^* appearing on p. 448 of [3] is real and the polynomials $p_n(z)$ in Theorems C and 4 of that paper are supposed to have real coefficients.

References

1. R. P. Boas, Jr., *Entire functions*, Academic Press, New York, 1954. MR **16**, 914.
2. O. D. Kellogg, *On bounded polynomials in several variables*, Math. Z. **27** (1927), 55–64.
3. Q. I. Rahman, *On a problem of Turán about polynomials with curved majorants*, Trans. Amer. Math. Soc. **163** (1972), 447–455.

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