ADDENDUM TO "ON A PROBLEM OF TURÁN ABOUT POLYNOMIALS WITH CURVED MAJORANTS"

BY

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Let $\pi_n$ denote the class of polynomials $p_n(x) = \sum_{k=0}^{n} a_k x^k$ of degree $n$ which satisfy $|p_n(x)| \leq (1 - x^2)^{1/2}$ for $-1 < x < 1$. Given any $x_0$ in $[-1, 1]$, how large can $|p_n^{(k)}(x_0)|$ (the $k$th derivative at $x_0$) be if $p_n(x)$ belongs to the class $\pi_n$? In case $x_0 = 0$ the problem is equivalent to the problem of estimating $|a_k|$ if $p_n(x) = \sum_{k=0}^{n} a_k x^k \in \pi_n$. It has been shown [3] that if $p_n(x) = \sum_{k=0}^{n} a_k x^k \in \pi_n$ then

$$|a_1| \leq n - 1, \quad |a_2| \leq (n - 1)^2 + 1/2.$$ 

Here we prove the following theorem which gives a sharp estimate for each of the coefficients.

**Theorem.** If $p_n(x) = \sum_{k=0}^{n} a_k x^k$ is a polynomial of degree $n$ such that $|p_n(x)| \leq (1 - x^2)^{1/2}$ for $-1 < x < 1$ and $U_n(x)$ denotes the $n$th Chebyshev polynomial of the second kind, then, according as $n - k$ is even or odd, $|a_k|$ is bounded above by the absolute value of the coefficient of $x^k$ in $e^{iy}(1 - x^2)U_{n-2}(x)$ or $e^{iy}(1 - x^2)U_{n-3}(x)$, respectively.

The idea of proof comes from a paper of O. D. Kellogg [2].

**Proof.** Without loss of generality we may suppose $p_n(x)$ to be real for real $x$.

Taking first the case in which $k, n$ are both even, we consider the polynomial

$$\frac{1}{2}(p_n(x) + p_n(-x)) = a_0 + a_2 x^2 + a_4 x^4 + \cdots + a_{n-2} x^{n-2} + a_n x^n$$

and compare it with the polynomial

$$(1 - x^2)U_{n-2}(x) = -2^{n-2} x^n + \sum_{j=1}^{\frac{n-1}{2}} (-1)^{j-1} \left[ \frac{(n-1-j)!}{(j-1)! (n-2j)!} \right. + \frac{1}{4} \frac{(n-2-j)!}{j(n-2-2j)!} \left. \right]$$

$$\times (2x)^{n-2j} + (-1)^{n/2 - 1}$$

$$= \sum_{k=0}^{n} A_k x^k \quad \text{(say)}.$$ 

If $-1 < \lambda < 1$, the difference

$$D(x, \lambda) = (1 - x^2)U_{n-2}(x) - \lambda\{a_0 + a_2 x^2 + a_4 x^4 + \cdots + a_{n-2} x^{n-2} + a_n x^n\}$$

is positive at all points of the interval $(-1, 1)$ where $(1 - x^2)U_{n-2}(x) = (1 - x^2)^{1/2}$ and negative where $(1 - x^2)U_{n-2}(x) = -(1 - x^2)^{1/2}$. It is readily seen that there are

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a total of \( n - 1 \) points on the interval \((-1, 1)\) where \((1 - x^2)U_{n-2}(x)\) is alternately equal to \((1 - x^2)^{1/2}, -(1 - x^2)^{1/2}\). Hence the polynomial \( D(x, \lambda) \) has at least \( n - 2 \) real zeros in \((-1, 1)\). Since it also vanishes at \(-1, +1\) we conclude that all the zeros of \( D(x, \lambda) \) are real and distinct. All the odd coefficients of \( D(x, \lambda) \) being zero, none of the even coefficients can vanish; for then by Descartes' rule of signs, the zeros of \( D(x, \lambda) \) could not all be real. Thus, for every even \( k \leq n \) and for \(-1 < \lambda < 1\), \( |A_k - \lambda a_k| \neq 0 \). This is possible only if \(|a_k| \leq |A_k|\) for \( k = 0, 2, 4, \ldots, n \).

In case \( k, n \) are both odd we apply the above reasoning to the polynomial

\[
(1 - x^2)U_{n-2}(x) - (\lambda/2)(p(x) - p(-x))
\]

(whose even coefficients are all zero) in order to get the desired conclusion.

If \( k \) is even and \( n \) is odd we consider

\[
(1 - x^2)U_{n-2}(x) - (\lambda/2)(p(x) + p(-x))
\]

whereas if \( k \) is odd and \( n \) is even we argue with

\[
(1 - x^2)U_{n-2}(x) - (\lambda/2)(p(x) - p(-x)).
\]

One can in fact show that only those polynomials given at the end of the statement of the theorem are extremal for the \( k \)th coefficient when \( n - k \) is even or odd respectively.

We also observe that Theorem 2 of our paper [3] is an immediate consequence of a theorem of Levin which appears as Theorem 11.7.2 in [1]. If we set \( f(z) = p_n(\cos z) \), \( \omega(z) = e^{i(n-1/2)z} \sin z \) the conditions of Levin's theorem are satisfied with \( \tau = n, \sigma = n \). Since differentiation is a \( B \)-operator we have

\[
|d/dx)p_n(\cos x)| \leq |(d/dx)[e^{i(n-1/2)x} \sin x]|, \quad -\infty < x < \infty,
\]

which readily gives the desired result.

We take this opportunity to make it clear that \( z^* \) appearing on p. 448 of [3] is real and the polynomials \( p_n(z) \) in Theorems C and 4 of that paper are supposed to have real coefficients.

**References**