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ABSOLUTE TAUBERIAN CONSTANTS FOR CESÀRO MEANS

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Abstract. This paper is concerned with introducing two inequalities of the form $\sum_{n=0}^{\infty} |\tau_n - a_n| \leqq KA$ and $\sum_{n=0}^{\infty} |\tau_n - a_n| \leqq K'B$, where $\tau_n = C_n^{(k)} - C_{n-1}^{(k)}$, $C_n^{(k)}$ denote the Cesàro transform of order k, K and K' are absolute Tauberian constants, $A = \sum_{n=0}^{\infty} |\Delta(na_n)| < \infty$, $B = \sum_{n=0}^{\infty} |\Delta((1/n)\sum_{\nu=1}^{n-1} \nu a_\nu)| < \infty$ and $\Delta u_k = u_k - u_{k+1}$. The constants K, K' will be determined.

1. **Introduction.** Let $\{s_n\}$ $(n \geqq 0)$ $(s_n = a_0 + a_1 + \cdots + a_n)$ be a sequence of real or complex numbers. Denote by t_n a linear transform T

$$t_n = \sum_k c_{n,k} s_k \quad (^1) \tag{1.1}$$

of s_k supposed convergent for all sufficiently large values of n. In various special cases, it has been found that theorems of the following type hold. Suppose that p, n are related in an appropriate way (usually the assumption is that $p/n \to \alpha$ as $n \to \infty$, where $\alpha > 0$ is a constant). Suppose that

$$\limsup_{n \to \infty} |na_n| < \infty. \tag{1.2}$$

Then there is a constant A such that

$$\limsup_{n \to \infty} |t_n - s_p| \leqq A \limsup_{n \to \infty} |na_n|. \tag{1.3}$$

There are also analogous results in which (1.1) is replaced by a sequence-to-function transformation. Usually the best possible value of the constant A has been determined.

Theorems of this type were first considered by Hadwiger [8] and have since been investigated by various authors; see for example Agnew ([1], [2]) and Jakimovski [10].

Some similar theorems have been obtained with (1.2) replaced by the weaker condition

$$\limsup_{n \to \infty} |\gamma_n| < \infty, \tag{1.4}$$

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(1) Unless otherwise indicated, the symbol $\sum$ stands for $\sum_0^\infty$.

where we write

$$\gamma_n = \frac{1}{n+1}\sum_{\nu=1}^{n} \nu a_\nu. \tag{1.5}$$

See, for example, Delange [5], Rajagopal [14], Meir [13] and Sherif ([15], [16]). Also, other theorems have been obtained with (1.2) replaced by a condition of Schmidt's type

$$\limsup_{p\to\infty} \underset{|p-q|\leqq \lambda p^{1/2}}{\text{maximum}} |s_q - s_p| \leqq \lambda L \qquad (\lambda > 0), \tag{1.6}$$

where $\limsup_{n\to\infty} |n^{1/2}a_n| = L < \infty$. See for example Anjaneyulu [3].

Denoting by $C_n^{(k)}$ the Cesàro transform of order k so that

$$C_n^{(k)} = \binom{n+k}{n}^{-1} \sum_{\nu=0}^{n} \binom{n-\nu+k}{n-\nu} a_\nu \qquad (k \geqq 0),$$

we introduce in this paper estimates of a new form for the *absolute* Cesàro summability defined by Fekete [6]. The corresponding Tauberian conditions to (1.2) and to (1.4) will be

$$\sum |\Delta(na_n)| < \infty, \tag{1.7}$$

and

$$\sum_n \left| \Delta\left(\frac{1}{n}\sum_{\nu=1}^{n-1} \nu a_\nu\right) \right| < \infty \quad (^2), \tag{1.8}$$

respectively, where we define Δu_k by

$$\Delta u_k = u_k - u_{k+1}. \tag{1.9}$$

The estimates will be of the forms

$$\sum |\tau_n - a_n| \leqq K \sum |\Delta(na_n)|, \tag{1.10}$$

$$\sum |\tau_n - a_n| \leqq K' \sum_n \left| \Delta\left(\frac{1}{n}\sum_{\nu=1}^{n-1} \nu a_\nu\right) \right|, \tag{1.11}$$

respectively, where

$$\tau_n = C_n^{(k)} - C_{n-1}^{(k)}; \tag{1.12}$$

K and K' are absolute Tauberian constants.

It has been proved by Hyslop [9] that, if $\sum a_n$ is absolutely Abel summable and if (1.8) holds, then $\sum a_n$ is absolutely convergent. Since *absolute* summability $|C, k|$ implies *absolute* Abel summability(3), this theorem includes the result:

(2) (1.8) can be stated in the form that the sequence $\{na_n\}$ is absolutely summable $|C, 1|$.

(3) See Fekete [7].

(A) *absolute* summability $|C, k|$ together with (1.8) implies *absolute* convergence. *A fortiori*, it includes the result:

(B) *absolute* summability $|C, k|$ together with (1.7) implies *absolute* convergence.

It will be noted that, just as the Tauberian constant theorems already cited include the familiar "o" Tauberian theorems, so Theorems 3.1 and 2.1 of the present paper include (A) and (B) respectively.

I have much pleasure in expressing my gratitude to Professor B. Kuttner for his valuable suggestions during the presentation of this paper.

2. THEOREM 2.1. *Suppose that* (1.7) *holds. Then, whether*

$$\sum |\tau_n| < \infty, \tag{2.1}$$

holds or not, (1.10) *holds, where for* $k \geqq 0$,

$$K = \Gamma'(k+1)/\Gamma(k+1)+\gamma, \tag{2.2}$$

(γ *is Euler's constant*).

This result is the best possible in the sense that (1.10) *becomes false if* K *is replaced by any smaller constant.*

For the proof of Theorem 2.1, we require the following lemmas.

LEMMA 2.1. *Let*

$$A_n = \sum_\nu \alpha_{n,\nu} b_\nu. \tag{2.3}$$

Suppose that

$$\sum_n |\alpha_{n,\nu}| \quad \text{is bounded}(^4). \tag{2.4}$$

Let

$$K = \sup_\nu \sum_n |\alpha_{n,\nu}|. \tag{2.5}$$

Then

$$\sum_n |A_n| \leqq K \sum_\nu |b_\nu| \tag{2.6}$$

and this constant is the best possible in the sense that (2.6) *becomes false if* K *is replaced by any smaller constant.*

Proof. $\sum |A_n| \leqq \sum_n \sum_\nu |\alpha_{n,\nu} b_\nu| = \sum_\nu |b_\nu| \sum_n |\alpha_{n,\nu}| \leqq K \sum_\nu |b_\nu|$. On the other hand, given $\varepsilon > 0$, there is a ν_0 say such that $\sum_n |\alpha_{n,\nu_0}| > K-\varepsilon$. The conclusion follows on taking $b_{\nu_0} = 1$; $b_\nu = 0$ $(\nu \neq \nu_0)$.

(4) It has been shown by Mears [12], K. Knopp and G. G. Lorentz [11] that for the transformation (2.3) to transform every absolutely convergent series into an absolutely convergent series, it is necessary and sufficient that (2.4) holds.

LEMMA 2.2. *Let*

$$\beta > \alpha+1. \tag{2.7}$$

Then

$$\sum_{n=\nu}^{\infty} \binom{n-\nu+\alpha}{n-\nu} \Big/ \binom{n+\beta}{n} = \beta\cdot\Gamma(\nu+1)\Gamma(\beta-1-\alpha)/\Gamma(\nu+\beta-\alpha). \tag{2.8}$$

Proof. The left-hand side of (2.8) is equal to

$$\begin{aligned}&\left[\binom{\nu+\beta}{\nu}\right]^{-1}\left\{1+\frac{(1+\alpha)(\nu+1)}{(\nu+\beta+1)}+\frac{(1+\alpha)(2+\alpha)(\nu+1)(\nu+2)}{2!(\nu+\beta+1)(\nu+\beta+2)}+\cdots\right\}\\ &\qquad= \left[\binom{\nu+\beta}{\nu}\right]^{-1}\cdot F\{(1+\alpha);(\nu+1);(\nu+\beta+1);1\},\end{aligned} \tag{2.9}$$

with the notation of Chapter I of Bailey's tract [4]. By Gauss' theorem of §1.3 of Bailey [4], (2.9) is equal to

$$\left[\binom{\nu+\beta}{\nu}\right]^{-1}\cdot\frac{\Gamma(\nu+\beta+1)\Gamma(\beta-1-\alpha)}{\Gamma(\nu+\beta-\alpha)\Gamma(\beta)}$$

from which the right-hand side of (2.8) is established.

We are now in position to prove Theorem 2.1. It is clear that $\tau_0=a_0$. But for $n\geqq 1$,

$$\begin{aligned}\tau_n &= \left[\binom{n+k}{n}\right]^{-1}\sum_{\nu=1}^{n}\binom{n-\nu+k}{n-\nu}a_\nu-\left[\binom{n-1+k}{n-1}\right]^{-1}\sum_{\nu=1}^{n-1}\binom{n-1-\nu+k}{n-1-\nu}a_\nu\\ &= \left[n\binom{n+k}{n}\right]^{-1}\sum_{\mu=1}^{n}\mu\binom{n-\mu+k-1}{n-\mu}a_\mu \qquad (2.10)\\ &= -\left[n\binom{n+k}{n}\right]^{-1}\sum_{\mu=1}^{n}\binom{n-\mu+k-1}{n-\mu}\sum_{\nu=0}^{\mu-1}\Delta(\nu a_\nu)\\ &= -\left[n\binom{n+k}{n}\right]^{-1}\sum_{\nu=0}^{n-1}\Delta(\nu a_\nu)\sum_{\mu=\nu+1}^{n}\binom{n-\mu+k-1}{n-\mu}\\ &= -\left[n\binom{n+k}{n}\right]^{-1}\sum_{\nu=0}^{n-1}\Delta(\nu a_\nu)\binom{n-\nu-1+k}{n-\nu-1}. \qquad (2.11)\end{aligned}$$

Also

$$a_n = \frac{1}{n}na_n = -\frac{1}{n}\sum_{\nu=0}^{n-1}\Delta(\nu a_\nu). \tag{2.12}$$

Thus, it follows from (2.11) and (2.12) that

$$\tau_n-a_n = \sum_{\nu=0}^{n-1}\Delta(\nu a_\nu)\left[\frac{1}{n}\left\{1-\binom{n-\nu-1+k}{n-\nu-1}\Big/\binom{n+k}{n}\right\}\right]. \tag{2.13}$$

Now, (2.13) is a transformation of the type considered in Lemma 2.1 and, for $n \geqq 1$,

$$
\begin{aligned}
\alpha_{n,\nu} &= 0 && \text{for } \nu > n, \\
&= (1/n)[\ -\tbinom{n-\nu-1-k}{n-\nu-1}/\tbinom{n+k}{n}] && \text{for } \nu \leqq n-1.
\end{aligned} \tag{2.14}
$$

Thus, the conditions of Lemma 2.1 are satisfied with

$$
K = \sup_{\nu} S_\nu \tag{2.15}
$$

where

$$
S_\nu = \sum_{n=\nu+1}^{\infty} \left| \frac{1}{n} \left\{ 1 - \binom{n-\nu-1+k}{n-\nu-1} \Big/ \binom{n+k}{n} \right\} \right|, \tag{2.16}
$$

provided that S_ν is bounded; next, we note that, since $k > 0$, $0 < \binom{n-\nu-1+k}{n-\nu-1} < \binom{n+k}{n}$, so that we may omit the modulus sign in (2.16). We now have

$$
\begin{aligned}
S_\nu - S_{\nu-1} &= \sum_{k=\nu}^{\infty} \left[\binom{n-\nu+k}{n-\nu} - \binom{n-\nu-1+k}{n-\nu-1} \right] \Big/ n \binom{n+k}{n} - \frac{1}{\nu} \\
&= \frac{1}{k+1} \sum_{n=\nu}^{\infty} \binom{n-\nu+k-1}{n-\nu} - \frac{1}{\nu}.
\end{aligned} \tag{2.17}
$$

Replacing n by $n+1$, we thus get

$$
S_\nu - S_{\nu-1} = \frac{1}{k+1} \sum_{n=\nu-1}^{\infty} \binom{n-\nu+k}{n+1-\nu} \Big/ \binom{n+k+1}{n} - \frac{1}{\nu}. \tag{2.18}
$$

Applying Lemma 2.2, we find that

$$
S_\nu - S_{\nu-1} = \Gamma(\nu)/\Gamma(\nu+1) - 1/\nu = 0. \tag{2.19}
$$

Thus,

$$
K = S_0 = \sum_{n=1}^{\infty} \frac{1}{n} \left\{ 1 - \binom{n-1+k}{n-1} \Big/ \binom{n+k}{n} \right\} = \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \frac{n}{n+k} \right). \tag{2.20}
$$

The conclusion thus follows from (2.20) and §12.16 of Whittaker and Watson [17].

3. THEOREM 3.1. *Suppose that* (1.8) *holds. Then whether* (2.1) *holds or not,* (1.11) *holds, where*

$$
\begin{aligned}
K' &= K && \textit{for } k \geqq 1, \\
&= -K+2 && \textit{for } 0 < k < 1.
\end{aligned} \tag{3.1}
$$

Proof. Write

$$
\phi_n = -\Delta\left(\frac{1}{n} \sum_{\nu=1}^{n-1} \nu a_\nu \right). \tag{3.2}
$$

Let

$$
u_n = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu a_\nu; \qquad \phi_n = u_n - u_{n-1}. \tag{3.3}
$$

Then,

$$na_n = (n+1)u_n - nu_{n-1} = u_n + n\phi_n = \sum_{\mu=1}^{n} \phi_\mu + n\phi_n, \tag{3.4}$$

i.e.

$$a_n = \frac{1}{n}\sum_{\mu=1}^{n} \phi_\mu + \phi_n. \tag{3.5}$$

Using (3.5), it follows from (2.10) that

$$\tau_n = \left[n\binom{n+k}{n}\right]^{-1} \sum_{\nu=1}^{n} \binom{n-\nu+k-1}{n-\nu}\left\{\sum_{\mu=1}^{\nu} \phi_\mu + \nu\phi_\nu\right\} = A+B \quad \text{(say)}. \tag{3.6}$$

But

$$A = \left[n\binom{n+k}{n}\right]^{-1} \sum_{\mu=1}^{n} \phi_\mu \sum_{\nu=\mu}^{n} \binom{n-\nu+k-1}{n-\nu} = \left[n\binom{n+k}{n}\right]^{-1} \sum_{\mu=1}^{n} \phi_\mu \binom{n-\mu+k}{n-\mu}.$$

Thus,

$$\tau_n = \left[n\binom{n+k}{n}\right]^{-1} \sum_{\nu=1}^{n} \left\{\binom{n-\nu+k}{n-\nu} + \binom{n-\nu+k-1}{n-\nu}\nu\right\}\phi_\nu. \tag{3.7}$$

It follows from (3.5) and (3.7) that

$$\begin{aligned} a_n - \tau_n &= \frac{1}{n}\sum_{\nu=1}^{n}\left[1 - \left\{\binom{n-\nu+k}{n-\nu} + \binom{n-\nu+k-1}{n-\nu}\nu\right\}\Big/\binom{n+k}{n}\right]\phi_\nu + \phi_n \\ &= \sum_{\nu=1}^{n} \alpha_{n,\nu}\phi_\nu \quad \text{(say)}, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \alpha_{n,\nu} &= 0 && \text{for } \nu > n, \\ &= (1/n)\left[1 - \left\{\tbinom{n-\nu+k}{n-\nu} + \nu\tbinom{n-\nu+k-1}{n-\nu}\right\}/\tbinom{n+k}{n}\right] && \text{for } \nu < n, \\ &= (1/n)\left[1-(n+1)/\tbinom{n+k}{n}\right]+1 && \text{for } \nu = n. \end{aligned}$$

Thus, the conditions of Lemma 2.1 are satisfied with

$$K' = \sup_\nu \psi_\nu, \tag{3.9}$$

where

$$\psi_\nu = \sum_{n=\nu}^{\infty} |\alpha_{n,\nu}|. \tag{3.10}$$

Write

$$b_{n,\nu} = \left(1 + \frac{\nu k}{n-\nu+k}\right)\binom{n-\nu+k}{n-\nu}\Big/\binom{n+k}{n} \qquad (0 \leqq \nu \leqq n),$$

so that

$$\begin{aligned} \alpha_{n,\nu} &= (1/n)(1-b_{n,\nu}) && \text{for } 0 \leqq \nu \leqq n-1, \\ &= (1/n)(1-b_{n,n}) + 1 && \text{for } \nu = n. \end{aligned}$$

Then

$$\psi_\nu = \sum_{n=\nu}^{\infty} \left| \frac{1}{n}(1-b_{n,\nu})+1 \right|. \tag{3.11}$$

We note that, for $0 \leqq \nu \leqq n-1$, $b_{n,\nu+1}/b_{n,\nu} = h_{n,\nu}/g_{n,\nu}$ where

$$h_{n,\nu} = (n-\nu)[n-\nu-1+(\nu+2)k], \qquad g_{n,\nu} = (n-\nu-1+k)[n-\nu+(\nu+1)k].$$

It is easily verified that $h_{n,\nu}-g_{n,\nu} = k(1-k)(\nu+1)$. Thus, for fixed n, $b_{n,\nu}$ is an increasing function of ν if $k<1$ and decreasing if $k>1$. But

$$b_{n,0} = 1. \tag{3.12}$$

Then, if $k \geqq 1$, $b_{n,\nu} \leqq 1$. Also, if $0<k<1$, $b_{n,\nu}>1$.

(i) $k \geqq 1$. Since $b_{n,\nu} \leqq 1$, we can omit the modulus sign in (3.11). We thus get

$$\psi_\nu = \sum_{n=\nu}^{\infty} \frac{1}{n}(1-b_{n,\nu})+1. \tag{3.13}$$

We deduce that

$$\psi_\nu = S_{\nu-1}-M_\nu+1, \tag{3.14}$$

where

$$\begin{aligned} M_\nu &= \sum_{n=\nu}^{\infty} \nu k\binom{n-\nu+k}{n-\nu}\Big/ n\binom{n+k}{n}(n-\nu+k) \qquad (0 \leqq \nu \leqq n) \\ &= \frac{\nu}{k+1}\sum_{n=\nu-1}^{\infty} \binom{n-\nu+k-1}{n-\nu}\Big/\binom{n+k}{n}. \end{aligned} \tag{3.15}$$

Replacing n by $n+1$, we thus get

$$M_\nu = \frac{\nu}{k+1}\sum_{n=\nu-1}^{\infty} \binom{n-\nu+k}{n+1-\nu}\Big/\binom{n+1+k}{n}.$$

Now, using Lemma 2.2, we have

$$M_\nu = 1. \tag{3.16}$$

Combining (2.15), (2.19), (3.9), (3.14) and (3.16), the result clearly follows.

(ii) $0<k<1$. Since $b_{n,\nu}>1$, then

$$\alpha_{n,\nu} < 0 \quad \text{for } 1 \leqq \nu < n-1. \tag{3.17}$$

But

$$\alpha_{n,n} = \frac{1}{n}\left[1-(n+1)\binom{n+k}{n}\right]+1 = \frac{(n+1)}{n}\left[1-1\Big/\binom{n+k}{n}\right] > 0. \tag{3.18}$$

Hence, it follows from (3.10), (3.17) and (3.18) that

$$\psi_\nu = -\sum_{n=\nu+1}^{\infty} \alpha_{n,\nu}+\alpha_{\nu,\nu} = -\sum_{n=\nu}^{\infty} \alpha_{n,\nu}+2\alpha_{\nu,\nu}.$$

The argument given in case (i) shows that $\sum_{n=\nu}^{\infty} \alpha_{n,\nu} = S_\nu$. Thus,

$$\psi_\nu = -S_\nu + 2\alpha_{\nu,\nu}. \tag{3.19}$$

Now, using (3.18), we find that

$$\alpha_{\nu,\nu} = \frac{(\nu+1)}{\nu}\left(1 - 1\bigg/\binom{\nu+k}{\nu}\right). \tag{3.20}$$

But, since $k<1$, $\binom{\nu+k}{\nu} < (\nu+1)$. Hence

$$1 - 1\bigg/\binom{\nu+k}{\nu} < 1 - 1/(\nu+1). \tag{3.21}$$

It thus follows from (3.20) and (3.21) that

$$\alpha_{\nu,\nu} < 1. \tag{3.22}$$

Since, $\alpha_{\nu,\nu} \to 1$ as $\nu \to \infty$, it follows from (3.22) that

$$\sup_\nu \alpha_{\nu,\nu} = 1. \tag{3.23}$$

Combining (3.9), (3.19) and (3.23), the final conclusion holds.

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