

## ABSOLUTE TAUBERIAN CONSTANTS FOR CESÀRO MEANS

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**Abstract.** This paper is concerned with introducing two inequalities of the form  $\sum_{n=0}^{\infty} |\tau_n - a_n| \leq KA$  and  $\sum_{n=0}^{\infty} |\tau_n - a_n| \leq K'B$ , where  $\tau_n = C_n^{(k)} - C_{n-1}^{(k)}$ ,  $C_n^{(k)}$  denote the Cesàro transform of order  $k$ ,  $K$  and  $K'$  are absolute Tauberian constants,  $A = \sum_{n=0}^{\infty} |\Delta(na_n)| < \infty$ ,  $B = \sum_{n=0}^{\infty} |\Delta((1/n) \sum_{v=1}^n va_v)| < \infty$  and  $\Delta u_k = u_k - u_{k+1}$ . The constants  $K, K'$  will be determined.

1. **Introduction.** Let  $\{s_n\}$  ( $n \geq 0$ ) ( $s_n = a_0 + a_1 + \dots + a_n$ ) be a sequence of real or complex numbers. Denote by  $t_n$  a linear transform  $T$

$$(1.1) \quad t_n = \sum_k c_{n,k} s_k \quad (1)$$

of  $s_k$  supposed convergent for all sufficiently large values of  $n$ . In various special cases, it has been found that theorems of the following type hold. Suppose that  $p, n$  are related in an appropriate way (usually the assumption is that  $p/n \rightarrow \alpha$  as  $n \rightarrow \infty$ , where  $\alpha > 0$  is a constant). Suppose that

$$(1.2) \quad \limsup_{n \rightarrow \infty} |na_n| < \infty.$$

Then there is a constant  $A$  such that

$$(1.3) \quad \limsup_{n \rightarrow \infty} |t_n - s_p| \leq A \limsup_{n \rightarrow \infty} |na_n|.$$

There are also analogous results in which (1.1) is replaced by a sequence-to-function transformation. Usually the best possible value of the constant  $A$  has been determined.

Theorems of this type were first considered by Hadwiger [8] and have since been investigated by various authors; see for example Agnew ([1], [2]) and Jakimovski [10].

Some similar theorems have been obtained with (1.2) replaced by the weaker condition

$$(1.4) \quad \limsup_{n \rightarrow \infty} |\gamma_n| < \infty,$$

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(1) Unless otherwise indicated, the symbol  $\sum$  stands for  $\sum_0^{\infty}$ .

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where we write

$$(1.5) \quad \gamma_n = \frac{1}{n+1} \sum_{\nu=1}^n \nu a_\nu.$$

See, for example, Delange [5], Rajagopal [14], Meir [13] and Sherif ([15], [16]). Also, other theorems have been obtained with (1.2) replaced by a condition of Schmidt's type

$$(1.6) \quad \limsup_{p \rightarrow \infty} \max_{|p-q| \leq \lambda p^{1/2}} |s_q - s_p| \leq \lambda L \quad (\lambda > 0),$$

where  $\limsup_{n \rightarrow \infty} |n^{1/2} a_n| = L < \infty$ . See for example Anjaneyulu [3].

Denoting by  $C_n^{(k)}$  the Cesàro transform of order  $k$  so that

$$C_n^{(k)} = \binom{n+k}{n}^{-1} \sum_{\nu=0}^n \binom{n-\nu+k}{n-\nu} a_\nu \quad (k \geq 0),$$

we introduce in this paper estimates of a new form for the *absolute* Cesàro summability defined by Fekete [6]. The corresponding Tauberian conditions to (1.2) and to (1.4) will be

$$(1.7) \quad \sum |\Delta(na_n)| < \infty,$$

and

$$(1.8) \quad \sum_n \left| \Delta \left( \frac{1}{n} \sum_{\nu=1}^{n-1} \nu a_\nu \right) \right| < \infty \quad (2),$$

respectively, where we define  $\Delta u_k$  by

$$(1.9) \quad \Delta u_k = u_k - u_{k+1}.$$

The estimates will be of the forms

$$(1.10) \quad \sum |\tau_n - a_n| \leq K \sum |\Delta(na_n)|,$$

$$(1.11) \quad \sum |\tau_n - a_n| \leq K' \sum_n \left| \Delta \left( \frac{1}{n} \sum_{\nu=1}^{n-1} \nu a_\nu \right) \right|,$$

respectively, where

$$(1.12) \quad \tau_n = C_n^{(k)} - C_{n-1}^{(k)};$$

$K$  and  $K'$  are absolute Tauberian constants.

It has been proved by Hyslop [9] that, if  $\sum a_n$  is absolutely Abel summable and if (1.8) holds, then  $\sum a_n$  is absolutely convergent. Since *absolute* summability  $|C, k|$  implies *absolute* Abel summability<sup>(2)</sup>, this theorem includes the result:

<sup>(2)</sup> (1.8) can be stated in the form that the sequence  $\{na_n\}$  is absolutely summable  $|C, 1|$ .

<sup>(3)</sup> See Fekete [7].

(A) *absolute summability*  $|C, k|$  together with (1.8) implies *absolute convergence*. *A fortiori*, it includes the result:

(B) *absolute summability*  $|C, k|$  together with (1.7) implies *absolute convergence*.

It will be noted that, just as the Tauberian constant theorems already cited include the familiar “*o*” Tauberian theorems, so Theorems 3.1 and 2.1 of the present paper include (A) and (B) respectively.

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2. THEOREM 2.1. *Suppose that (1.7) holds. Then, whether*

$$(2.1) \quad \sum |\tau_n| < \infty,$$

*holds or not, (1.10) holds, where for  $k \geq 0$ ,*

$$(2.2) \quad K = \Gamma'(k+1)/\Gamma(k+1) + \gamma,$$

*( $\gamma$  is Euler’s constant).*

*This result is the best possible in the sense that (1.10) becomes false if  $K$  is replaced by any smaller constant.*

For the proof of Theorem 2.1, we require the following lemmas.

LEMMA 2.1. *Let*

$$(2.3) \quad A_n = \sum_{\nu} \alpha_{n,\nu} b_{\nu}.$$

*Suppose that*

$$(2.4) \quad \sum_n |\alpha_{n,\nu}| \text{ is bounded}^{(4)}.$$

*Let*

$$(2.5) \quad K = \sup_{\nu} \sum_n |\alpha_{n,\nu}|.$$

*Then*

$$(2.6) \quad \sum_n |A_n| \leq K \sum_{\nu} |b_{\nu}|$$

*and this constant is the best possible in the sense that (2.6) becomes false if  $K$  is replaced by any smaller constant.*

**Proof.**  $\sum |A_n| \leq \sum_n \sum_{\nu} |\alpha_{n,\nu} b_{\nu}| = \sum_{\nu} |b_{\nu}| \sum_n |\alpha_{n,\nu}| \leq K \sum_{\nu} |b_{\nu}|$ . On the other hand, given  $\epsilon > 0$ , there is a  $\nu_0$  say such that  $\sum_n |\alpha_{n,\nu_0}| > K - \epsilon$ . The conclusion follows on taking  $b_{\nu_0} = 1$ ;  $b_{\nu} = 0$  ( $\nu \neq \nu_0$ ).

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<sup>(4)</sup> It has been shown by Mears [12], K. Knopp and G. G. Lorentz [11] that for the transformation (2.3) to transform every absolutely convergent series into an absolutely convergent series, it is necessary and sufficient that (2.4) holds.

LEMMA 2.2. *Let*

$$(2.7) \quad \beta > \alpha + 1.$$

*Then*

$$(2.8) \quad \sum_{n=\nu}^{\infty} \binom{n-\nu+\alpha}{n-\nu} / \binom{n+\beta}{n} = \beta \cdot \Gamma(\nu+1)\Gamma(\beta-1-\alpha) / \Gamma(\nu+\beta-\alpha).$$

**Proof.** The left-hand side of (2.8) is equal to

$$(2.9) \quad \left[ \binom{\nu+\beta}{\nu} \right]^{-1} \left\{ 1 + \frac{(1+\alpha)(\nu+1)}{(\nu+\beta+1)} + \frac{(1+\alpha)(2+\alpha)(\nu+1)(\nu+2)}{2!(\nu+\beta+1)(\nu+\beta+2)} + \dots \right\} \\ = \left[ \binom{\nu+\beta}{\nu} \right]^{-1} \cdot F\{(1+\alpha); (\nu+1); (\nu+\beta+1); 1\},$$

with the notation of Chapter I of Bailey's tract [4]. By Gauss' theorem of §1.3 of Bailey [4], (2.9) is equal to

$$\left[ \binom{\nu+\beta}{\nu} \right]^{-1} \cdot \frac{\Gamma(\nu+\beta+1)\Gamma(\beta-1-\alpha)}{\Gamma(\nu+\beta-\alpha)\Gamma(\beta)}$$

from which the right-hand side of (2.8) is established.

We are now in position to prove Theorem 2.1. It is clear that  $\tau_0 = a_0$ . But for  $n \geq 1$ ,

$$(2.10) \quad \tau_n = \left[ \binom{n+k}{n} \right]^{-1} \sum_{\nu=1}^n \binom{n-\nu+k}{n-\nu} a_\nu - \left[ \binom{n-1+k}{n-1} \right]^{-1} \sum_{\nu=1}^{n-1} \binom{n-1-\nu+k}{n-1-\nu} a_\nu \\ = \left[ n \binom{n+k}{n} \right]^{-1} \sum_{\mu=1}^n \mu \binom{n-\mu+k-1}{n-\mu} a_\mu$$

$$(2.11) \quad = - \left[ n \binom{n+k}{n} \right]^{-1} \sum_{\mu=1}^n \binom{n-\mu+k-1}{n-\mu} \sum_{\nu=0}^{\mu-1} \Delta(\nu a_\nu) \\ = - \left[ n \binom{n+k}{n} \right]^{-1} \sum_{\nu=0}^{n-1} \Delta(\nu a_\nu) \sum_{\mu=\nu+1}^n \binom{n-\mu+k-1}{n-\mu} \\ = - \left[ n \binom{n+k}{n} \right]^{-1} \sum_{\nu=0}^{n-1} \Delta(\nu a_\nu) \binom{n-\nu-1+k}{n-\nu-1}.$$

Also

$$(2.12) \quad a_n = \frac{1}{n} n a_n = - \frac{1}{n} \sum_{\nu=0}^{n-1} \Delta(\nu a_\nu).$$

Thus, it follows from (2.11) and (2.12) that

$$(2.13) \quad \tau_n - a_n = \sum_{\nu=0}^{n-1} \Delta(\nu a_\nu) \left[ \frac{1}{n} \left\{ 1 - \binom{n-\nu-1+k}{n-\nu-1} / \binom{n+k}{n} \right\} \right].$$

Now, (2.13) is a transformation of the type considered in Lemma 2.1 and, for  $n \geq 1$ ,

$$(2.14) \quad \begin{aligned} \alpha_{n,\nu} &= 0 && \text{for } \nu > n, \\ &= (1/n) [ -\binom{n-\nu-1-k}{n-\nu-1} / \binom{n+k}{n} ] && \text{for } \nu \leq n-1. \end{aligned}$$

Thus, the conditions of Lemma 2.1 are satisfied with

$$(2.15) \quad K = \sup_{\nu} S_{\nu}$$

where

$$(2.16) \quad S_{\nu} = \sum_{n=\nu+1}^{\infty} \left| \frac{1}{n} \left\{ 1 - \frac{\binom{n-\nu-1+k}{n-\nu-1}}{\binom{n+k}{n}} \right\} \right|,$$

provided that  $S_{\nu}$  is bounded; next, we note that, since  $k > 0$ ,  $0 < \binom{n-\nu-1+k}{n-\nu-1} < \binom{n+k}{n}$ , so that we may omit the modulus sign in (2.16). We now have

$$(2.17) \quad \begin{aligned} S_{\nu} - S_{\nu-1} &= \sum_{k=\nu}^{\infty} \left[ \frac{\binom{n-\nu+k}{n-\nu} - \binom{n-\nu-1+k}{n-\nu-1}}{n \binom{n+k}{n}} - \frac{1}{\nu} \right] \\ &= \frac{1}{k+1} \sum_{n=\nu}^{\infty} \frac{\binom{n-\nu+k-1}{n-\nu}}{n} - \frac{1}{\nu}. \end{aligned}$$

Replacing  $n$  by  $n+1$ , we thus get

$$(2.18) \quad S_{\nu} - S_{\nu-1} = \frac{1}{k+1} \sum_{n=\nu-1}^{\infty} \frac{\binom{n-\nu+k}{n+1-\nu}}{\binom{n+k}{n}} - \frac{1}{\nu}.$$

Applying Lemma 2.2, we find that

$$(2.19) \quad S_{\nu} - S_{\nu-1} = \Gamma(\nu) / \Gamma(\nu+1) - 1/\nu = 0.$$

Thus,

$$(2.20) \quad K = S_0 = \sum_{n=1}^{\infty} \frac{1}{n} \left\{ 1 - \frac{\binom{n-1+k}{n-1}}{\binom{n+k}{n}} \right\} = \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 - \frac{n}{n+k} \right).$$

The conclusion thus follows from (2.20) and §12.16 of Whittaker and Watson [17].

**3. THEOREM 3.1.** *Suppose that (1.8) holds. Then whether (2.1) holds or not, (1.11) holds, where*

$$(3.1) \quad \begin{aligned} K' &= K && \text{for } k \geq 1, \\ &= -K+2 && \text{for } 0 < k < 1. \end{aligned}$$

**Proof.** Write

$$(3.2) \quad \phi_n = -\Delta \left( \frac{1}{n} \sum_{\nu=1}^{n-1} \nu a_{\nu} \right).$$

Let

$$(3.3) \quad u_n = \frac{1}{n+1} \sum_{\nu=1}^n \nu a_{\nu}; \quad \phi_n = u_n - u_{n-1}.$$

Then,

$$(3.4) \quad na_n = (n+1)u_n - mu_{n-1} = u_n + n\phi_n = \sum_{\mu=1}^n \phi_\mu + n\phi_n,$$

i.e.

$$(3.5) \quad a_n = \frac{1}{n} \sum_{\mu=1}^n \phi_\mu + \phi_n.$$

Using (3.5), it follows from (2.10) that

$$(3.6) \quad \tau_n = \left[ n \binom{n+k}{n} \right]^{-1} \sum_{\nu=1}^n \binom{n-\nu+k-1}{n-\nu} \left\{ \sum_{\mu=1}^{\nu} \phi_\mu + \nu\phi_\nu \right\} = A + B \quad (\text{say}).$$

But

$$A = \left[ n \binom{n+k}{n} \right]^{-1} \sum_{\mu=1}^n \phi_\mu \sum_{\nu=\mu}^n \binom{n-\nu+k-1}{n-\nu} = \left[ n \binom{n+k}{n} \right]^{-1} \sum_{\mu=1}^n \phi_\mu \binom{n-\mu+k}{n-\mu}.$$

Thus,

$$(3.7) \quad \tau_n = \left[ n \binom{n+k}{n} \right]^{-1} \sum_{\nu=1}^n \left\{ \binom{n-\nu+k}{n-\nu} + \binom{n-\nu+k-1}{n-\nu} \right\} \nu \phi_\nu.$$

It follows from (3.5) and (3.7) that

$$(3.8) \quad \begin{aligned} a_n - \tau_n &= \frac{1}{n} \sum_{\nu=1}^n \left[ 1 - \left\{ \binom{n-\nu+k}{n-\nu} + \binom{n-\nu+k-1}{n-\nu} \right\} / \binom{n+k}{n} \right] \nu \phi_\nu + \phi_n \\ &= \sum_{\nu=1}^n \alpha_{n,\nu} \phi_\nu \quad (\text{say}), \end{aligned}$$

where

$$\begin{aligned} \alpha_{n,\nu} &= 0 && \text{for } \nu > n, \\ &= (1/n) [1 - \{ \binom{n-\nu+k}{n-\nu} + \binom{n-\nu+k-1}{n-\nu} \} / \binom{n+k}{n}] && \text{for } \nu < n, \\ &= (1/n) [1 - (n+1) / \binom{n+k}{n}] + 1 && \text{for } \nu = n. \end{aligned}$$

Thus, the conditions of Lemma 2.1 are satisfied with

$$(3.9) \quad K' = \sup_{\nu} \psi_{\nu},$$

where

$$(3.10) \quad \psi_{\nu} = \sum_{n=\nu}^{\infty} |\alpha_{n,\nu}|.$$

Write

$$b_{n,\nu} = \left( 1 + \frac{\nu k}{n-\nu+k} \right) \binom{n-\nu+k}{n-\nu} / \binom{n+k}{n} \quad (0 \leq \nu \leq n),$$

so that

$$\begin{aligned} \alpha_{n,\nu} &= (1/n)(1 - b_{n,\nu}) && \text{for } 0 \leq \nu \leq n-1, \\ &= (1/n)(1 - b_{n,n}) + 1 && \text{for } \nu = n. \end{aligned}$$

Then

$$(3.11) \quad \psi_\nu = \sum_{n=\nu}^{\infty} \left| \frac{1}{n} (1 - b_{n,\nu}) + 1 \right|.$$

We note that, for  $0 \leq \nu \leq n-1$ ,  $b_{n,\nu+1}/b_{n,\nu} = h_{n,\nu}/g_{n,\nu}$  where

$$h_{n,\nu} = (n-\nu)[n-\nu-1+(\nu+2)k], \quad g_{n,\nu} = (n-\nu-1+k)[n-\nu+(\nu+1)k].$$

It is easily verified that  $h_{n,\nu} - g_{n,\nu} = k(1-k)(\nu+1)$ . Thus, for fixed  $n$ ,  $b_{n,\nu}$  is an increasing function of  $\nu$  if  $k < 1$  and decreasing if  $k > 1$ . But

$$(3.12) \quad b_{n,0} = 1.$$

Then, if  $k \geq 1$ ,  $b_{n,\nu} \leq 1$ . Also, if  $0 < k < 1$ ,  $b_{n,\nu} > 1$ .

(i)  $k \geq 1$ . Since  $b_{n,\nu} \leq 1$ , we can omit the modulus sign in (3.11). We thus get

$$(3.13) \quad \psi_\nu = \sum_{n=\nu}^{\infty} \frac{1}{n} (1 - b_{n,\nu}) + 1.$$

We deduce that

$$(3.14) \quad \psi_\nu = S_{\nu-1} - M_\nu + 1,$$

where

$$(3.15) \quad \begin{aligned} M_\nu &= \sum_{n=\nu}^{\infty} \nu k \binom{n-\nu+k}{n-\nu} / n \binom{n+k}{n} (n-\nu+k) \quad (0 \leq \nu \leq n) \\ &= \frac{\nu}{k+1} \sum_{n=\nu-1}^{\infty} \binom{n-\nu+k-1}{n-\nu} / \binom{n+k}{n}. \end{aligned}$$

Replacing  $n$  by  $n+1$ , we thus get

$$M_\nu = \frac{\nu}{k+1} \sum_{n=\nu-1}^{\infty} \binom{n-\nu+k}{n+1-\nu} / \binom{n+1+k}{n}.$$

Now, using Lemma 2.2, we have

$$(3.16) \quad M_\nu = 1.$$

Combining (2.15), (2.19), (3.9), (3.14) and (3.16), the result clearly follows.

(ii)  $0 < k < 1$ . Since  $b_{n,\nu} > 1$ , then

$$(3.17) \quad \alpha_{n,\nu} < 0 \quad \text{for } 1 \leq \nu < n-1.$$

But

$$(3.18) \quad \alpha_{n,n} = \frac{1}{n} \left[ 1 - (n+1) \binom{n+k}{n} \right] + 1 = \frac{(n+1)}{n} \left[ 1 - 1 / \binom{n+k}{n} \right] > 0.$$

Hence, it follows from (3.10), (3.17) and (3.18) that

$$\psi_\nu = - \sum_{n=\nu+1}^{\infty} \alpha_{n,\nu} + \alpha_{\nu,\nu} = - \sum_{n=\nu}^{\infty} \alpha_{n,\nu} + 2\alpha_{\nu,\nu}.$$

The argument given in case (i) shows that  $\sum_{n=v}^{\infty} \alpha_{n,v} = S_v$ . Thus,

$$(3.19) \quad \psi_v = -S_v + 2\alpha_{v,v}.$$

Now, using (3.18), we find that

$$(3.20) \quad \alpha_{v,v} = \frac{(\nu+1)}{\nu} \left( 1 - 1 / \binom{\nu+k}{\nu} \right).$$

But, since  $k < 1$ ,  $\binom{\nu+k}{\nu} < (\nu+1)$ . Hence

$$(3.21) \quad 1 - 1 / \binom{\nu+k}{\nu} < 1 - 1/(\nu+1).$$

It thus follows from (3.20) and (3.21) that

$$(3.22) \quad \alpha_{v,v} < 1.$$

Since,  $\alpha_{v,v} \rightarrow 1$  as  $\nu \rightarrow \infty$ , it follows from (3.22) that

$$(3.23) \quad \sup_v \alpha_{v,v} = 1.$$

Combining (3.9), (3.19) and (3.23), the final conclusion holds.

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