TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 168, June 1972

BOUNDED CONTINUOUS FUNCTIONS ON A COMPLETELY REGULAR SPACE

BY
F. DENNIS SENTILLES

Abstract. Three locally convex topologies on $C(X)$ are introduced and developed, and in particular shown to coincide with the strict topology on locally compact X and yield dual spaces consisting of tight, τ-additive and σ-additive functionals respectively for completely regular X.

The Riesz-Markov Representation Theorem says that any continuous linear functional F on the space of continuous functions on a compact Hausdorff space X with the topology of uniform convergence on X must have the form

$$F(f) = \int_X f\, d\mu$$

where μ is a bounded regular Borel measure on X. This yields a very satisfactory relationship between the topology on X, the space $C(X)$, a natural class of linear functionals on it, and those measures on X that measure at least the sets determined by the topology on X in the usual way, the Borel sets.

This kind of representation was subsequently extended to locally compact spaces: first to functionals on the space $C_0(X)$ of continuous functions vanishing at infinity, and then further, to the bounded continuous functions on X. The last result, due to R. C. Buck, demanded the use of a locally convex topology, the strict topology, rather than a norm topology. In both extensions the same satisfactory relationship between measure and topology was obtained.

In this paper we begin the development of locally convex topologies for $C(X)$ which extend this kind of representation to its last reasonable setting, completely regular Hausdorff spaces. This setting appears to be ultimate in the sense of Hewitt's example of a regular space upon which the only continuous functions are constants.

1. **Definitions and preliminaries.** The actual work of integral representation of linear forms has been done by other authors, going back to Aleksandrov [1] and, following his work, by Varadarajan [39], and later Knowles [21] and more recently Kirk [20] and Moran ([24], [25]). Our work relies heavily on theirs and will not extend the representations they have obtained but will relate these works to earlier

Received by the editors December 23, 1970 and, in revised form, November 21, 1971.

AMS 1969 *subject classifications.* Primary 4601, 4625; Secondary 2813, 4606.

Key words and phrases. Locally convex spaces, Mackey topology, strict topology, tight, τ-additive functionals, σ-additive functionals, measure compact, D spaces.

versions of the Riesz-Markov Theorem in the context of locally convex topologies on $C(X)$; $C(X)$ henceforth denotes the bounded continuous real-valued functions on a completely regular Hausdorff space X and, for $f \in C(X)$, $\|f\| = \sup\{|f(x)| : x \in X\}$. It is no restriction to consider only real-valued functions and functionals.

On the space $C(X)$ Varadarajan, and Knowles following him, distinguishes three classes of linear functionals. A real linear functional ϕ on $C(X)$ is said to be

(1) *tight*—if for any net $f_\alpha \in C(X)$ with $1 \geqq \|f_\alpha\|$ such that $f_\alpha \to 0$ uniformly on compacta in X, one has $\phi(f_\alpha) \to 0$,

(2) *τ-additive*—if for any net $f_\alpha \in C(X)$ such that $f_\alpha(x) \to 0$ for each $x \in X$ and $f_\gamma(x) \geqq f_\alpha(x)$ for $\alpha \geqq \gamma$ and all $x \in X$, one has $\phi(f_\alpha) \to 0$,

(3) *σ-additive*—if for any pointwise decreasing sequence $f_n \in C(X)$ with $f_n(x) \to 0$ for each $x \in X$, one has $\phi(f_n) \to 0$.

The collection of all functionals satisfying (1), (2), and (3) is denoted by M_t, M_τ and M_σ respectively and clearly $M_t \subset M_\tau \subset M_\sigma$. A net $\{f_\alpha\}$ satisfying the conditions in (2) will be called decreasing and this will be denoted by $f_\alpha \searrow 0$.

Each of the pairings (C, M_t), (C, M_τ) and (C, M_σ) are dual pairs in the sense of [30, p. 32] where $C = C(X)$. When X is locally compact, $M_t = M_\tau$ and there is a 1-1 correspondence between M_t and the bounded regular Borel measures on X [21, Theorem 25]. When X is compact, $M_t = M_\sigma$ and these are precisely the spaces of bounded linear forms on $C(X)$. We will define and investigate certain dual pair topologies [30, p. 34] of each of these dual pairings and tie these to the locally-convex topologies related to the aforementioned versions of the Riesz-Markov Theorem; the actual integral representation theory follows from the works mentioned above and is summarized below.

To begin this outline of existing representation theorems, we choose from the varied and varying definitions of Baire and Borel sets in a topological space T the following: The Baire sets are those sets in the σ-algebra $\mathrm{Ba}(T)$ generated by the zero-sets in T; the Borel sets are those sets in the σ-algebra $\mathrm{B}(T)$ generated by the open sets in T. (A zero set is a set of the form $f^{-1}(0)$, $f \in C(X)$; the set $X \setminus f^{-1}(0)$ is called a cozero set.) Clearly, $\mathrm{Ba}(T) \subset \mathrm{B}(T)$. A positive Borel (Baire) measure on T is a countably additive set function μ defined on $\mathrm{B}(T)$ ($\mathrm{Ba}(T)$) with values in $[0, \infty)$. A Borel (Baire) measure on X is the difference of two positive Borel (Baire) measures. Every positive Baire measure is known to be a regular Baire measure in that $\mu(E) = \sup\{\mu(Z) : Z \subset E, Z \text{ a zero set}\}$ for all $E \in \mathrm{Ba}(T)$. A positive Borel measure ν will be called *regular* if $\nu(E) = \sup\{\nu(C) : C \subset E, C \text{ a closed set}\}$, and will be called *compact regular* if $\nu(E) = \sup\{\nu(K) : K \subset E, K \text{ a compact set}\}$, both these requirements being for all $E \in \mathrm{B}(T)$.

The need for three classes of measures arises quite naturally, for these correspond exactly to the three classes of linear functionals just mentioned. This is set forth in 1.3 below and we discuss the matter in that context. If νX is the real compactification of X, $x \in \nu X \setminus X$ and $\bar{f}$ is the unique extension of $f \in C(X)$ to the Stone-Čech compactification βX of X, the functional $\phi(f) = \bar{f}(x)$ is seen to be σ-additive but not τ-additive nor tight. If $x \in X$, this same functional would be tight. It is not so easy to produce a τ-additive, nontight and non-σ-additive functional. Examples appear in [21]. Varadarajan [39] and Knowles [21] both assert that τ-additive functionals and their corresponding measures should be the main point of interest; [21, Theorem 4.3] is interesting in this regard. Finally, if one wishes a satisfactory relationship between the topology on X, functionals on $C(X)$ and measures on X, the restriction of tightness is too strong, for the compacta in X generate neither $\mathrm{Ba}(X)$ nor $\mathrm{B}(X)$, and for certain topological spaces (e.g., the rationals) may be a rather trivial class.

It is clear that the closed regular Borel measures on X are more closely related to the topology on X. However, used as representatives of a class of linear functions on $C(X)$, these are defective. Let $X=[1, \Omega] \times [1, \Omega]-\{(\Omega, \Omega)\}$, Ω being the first uncountable ordinal. Define ν_1, ν_2 on $\mathrm{B}(X)$ by $\nu_1(E)=1$ ($\nu_2(E)=1$) if E contains an unbounded closed subset of $\{\Omega\} \times [1, \Omega)$ ($[1, \Omega) \times \{\Omega\}$) and 0 otherwise. If $\nu=\nu_1-\nu_2$, then $\int_X f\, d\nu=0$ for all $f \in C(X)$, yet $\nu \neq 0$. However, if one restricts his interests to closed regular nonnegative Borel measures μ and ν such that $\phi(f)=\int_X f\, d\mu=\int_X f\, d\nu$ is τ-additive on $C(X)$, it follows from [20, Corollary 1.15] and 1.3 below that $\mu=\nu$. When X is normal one can show that, even without this assumption, $\mu=\nu$.

According to the representation theorem of Aleksandrov [39, p. 165], there is a 1-1 order preserving correspondence between the positive linear functionals on $C(X)$ and the positive, totally finite, finitely additive set functions μ defined on $\mathrm{Ba}(X)$. Using the Stone-Čech compactification βX of X and the Riesz-Markov Theorem, we, following Knowles, adopt the following notations. If ϕ is a positive linear functional on $C(X)$, let $\bar{\phi}(\bar{f})=\phi(f)$ be its unique extension to $C(\beta X)$ where $\bar{f} \in C(\beta X)$ and $\bar{f}=f$ on X. Then,

$$\phi(f) = \int_X f\, d\mu = \int_{\beta X} \bar{f}\, d\bar{\mu} = \int_{\beta X} \bar{f}\, d\bar{\nu} = \bar{\phi}(\bar{f})$$

where μ is a positive, totally finite Baire measure on X, $\bar{\mu}$ is a positive regular Baire measure on βX and $\bar{\nu}$ is a positive regular Borel measure on βX. On βX, *regular* of course implies *compact regular*.

From [21, Theorems 2.1, 2.4 and 2.5] one has

THEOREM 1.1. *The positive linear functional ϕ is*
(1) *σ-additive iff $\bar{\mu}(Z)=0$ for every zero set $Z \subset \beta X \setminus X$,*
(2) *τ-additive iff $\bar{\nu}(Q)=0$ for every compact set $Q \subset \beta X \setminus X$,*
(3) *tight iff τ-additive and X is $\bar{\nu}^*$ measurable, where*

$$\bar{\nu}^*(E) = \inf\{\bar{\nu}(U) : U \supset E, U \text{ open}\}$$

for any set E.

And in terms of sets and additivity, from [21] one has

THEOREM 1.2. *The positive linear functional* ϕ *is*

(1) σ*-additive iff* $\mu(Z_n) \to 0$ *for every sequence of zero sets* Z_n *decreasing to the null set,*

(2) τ*-additive iff* $\mu(Z_\alpha) \to 0$ *for every net of zero sets* Z_α *decreasing to the null set,*

(3) *tight iff for any* $\varepsilon > 0 \; \exists$ *a compact set* $K \subset X$ *such that* $\mu_*(X \backslash K) < \varepsilon$, *where* $\mu_*(E) = \sup \{\mu(Z) : Z \subset E, Z \text{ a zero set}\}$ *for all sets* E.

Finally [21, Theorems E, 2.1, 2.4 and 2.5],

THEOREM 1.3. *If* ϕ *is positive linear functional, then, if* ϕ *is*

(1) σ*-additive, then* $\bar{\mu}^*$ *induces a Baire measure on* X *agreeing with* μ *and hence* $\phi(f) = \int_X f \, d\mu$ *with* μ *a regular Baire measure,*

(2) τ*-additive, then* $\bar{\nu}$ *induces a regular Borel measure* ν *on* X *with* $\phi(f) = \int_X f \, d\nu$ *and* $\nu(U) = \mu_*(U)$ *for open sets* U *and hence agreeing with* μ *on Baire sets,*

(3) *tight, then* ν *is a compact-regular Borel measure.*

Our development of locally convex topologies on $C(X)$ is keyed to 1.1. We will cast our results in the context of M_t, M_τ and M_σ, and leave integral representation by a unique Borel measure as a (properly) measure theoretic problem, save for a few remarks in §9.

If ϕ is any real and bounded linear functional on $C(X)$, we define $\phi^+(f) = \sup \{\phi(g) : 0 \leqq g \leqq f\}$, $\phi^-(f) = -\inf \{\phi(h) : 0 \leqq h \leqq f\}$ for all $f \geqq 0$, extend these functions to all of $C(X)$ in the usual way, and set $|\phi|(f) = \phi^+(f) + \phi^-(f)$ for all $f \in C(X)$. Varadarajan proved that the conditions of σ-additivity, τ-additivity and tightness of ϕ are equivalent to the same for (1) $|\phi|$ or (2) ϕ^+ and ϕ^-, so that the above theorems tell it all for such functionals. Finally, we define $\|\phi\| = |\phi|(1)$.

Remaining notions of measure theory and locally convex topological vector spaces are found in [15] and [30], respectively, unless otherwise noted. In particular, if E is a locally convex space with dual E', then $\langle x, x' \rangle$ represents the value of $x' \in E'$ at $x \in E$. Finally, if $x \in X$ then δ_X represents the tight linear functional $\phi(f) = f(x)$.

2. **The strict, substrict and superstrict topologies on** $C(X)$. At least two other authors ([31] and [37]) have defined "strict" topologies for $C(X)$. van Rooij [31] uses the bounded, but not necessarily continuous, functions on X which vanish at infinity in the manner of Buck's work [3], for which he obtains M_t as the dual space. Summers [37] makes use of the nonnegative u.s.c. functions on X vanishing at infinity. In case X is locally compact both these topologies coincide with Buck's original strict topology [3], generated by the seminorms $p_\zeta(f) = \|f\zeta\|, f \in C(X)$, one for each $\zeta \in C_0(X)$. The difficulty that one faces in extending these definitions and concepts to the completely regular case is that $C_0(X)$ may be empty, and that tying the definition to the compacta in X also ties one to a dual space that need not be M_τ, much less M_σ.

We proceed as follows. For each compact set $Q \subset \beta X \backslash X$ let

$$C_Q = C_Q(X) = \{f \in C(X) : \bar{f} | Q = 0\}.$$

Then C_Q is a Banach algebra with approximate identity and $C(X)$ is a C_Q module. According to [34], C_Q defines a "strict" topology β_Q on $C(X)$, this topology generated by the seminorms $p_\zeta(f)=\|f\zeta\|$, $f\in C(X)$, one for each $\zeta\in C_Q$. Clearly, if $p_\zeta(f)=0$ for every $\zeta\in C_Q$ then $f\equiv 0$, so that β_Q is Hausdorff and [33], [34] and [35] apply.

We define the strict topology β on $C(X)$ to be the inductive limit topology Lin β_Q of the topologies β_Q taken over the family $\mathcal{Q}$ of all compact subsets Q of $\beta X\backslash X$ [30, p. 79]. Additionally, we define the superstrict topology β_1 to be the inductive limit topology Lin β_Z of the topologies β_Z taken over the family $\mathcal{Z}$ of all zero sets such that $Z\subset\beta X\backslash X$.

To define the subscript topology on $C(X)$ first let κ denote the compact-open topology on $C(X)$ and, for each $r>0$, let $B_r=\{f:\|f\|\leqq r\}$. The collection $\mathcal{U}=\{U: U$ is absolutely convex and absorbent and for $r>0\ \exists$ a κ-neighborhood V_r of 0 such that $U\cap B_r\supset V_r\cap B_r\}$ is, by [30, p. 10], a base for a locally convex topology on $C(X)$ which we will denote by β_0 and call the substrict topology. Clearly, β_0 is the finest locally convex topology agreeing with κ on the sets B_r and, by virtue of [7], β_0 is the strict topology of Buck on locally compact X. Finally, let $\mathcal{P}$ denote the topology of pointwise convergence on X and let $\|\ \ \|$ denote the norm topology defined by the norm $\|f\|$. If γ denotes any locally convex topology, let $W\in\gamma$ mean that W is an absolutely convex absorbent γ-neighborhood of 0.

THEOREM 2.1. (a) $\mathcal{P}\leqq\kappa\leqq\beta_0\leqq\beta\leqq\beta_1\leqq\|\ \ \|$.

(b) *All topologies in* (a) *are locally convex and Hausdorff.*

Proof. (a) Clearly $\mathcal{P}\leqq\kappa\leqq\beta_0$ and $\beta\leqq\beta_1\leqq\|\ \ \|$. Let $W\in\beta_0$. To show that $W\in\beta$ it suffices to show that $W\in\beta_Q$ for every $Q\in\mathcal{Q}$. According to [34, Theorem 2.2] it suffices to show that for a given $r>0$ there is a $V\in\beta_Q$ such that $W\cap B_r\supset V\cap B_r$. Since $W\in\beta_0$ there is a compact set $K\subset X$ such that $W\cap B_r\supset B_r\cap U$ where $U=\{f:|f(x)|\leqq 1$ for $x\in K\}$. Let $\zeta\in C(\beta X)$ such that $\|\zeta\|\leqq 1$, $\zeta\equiv 1$ on K and $\zeta\equiv 0$ on Q. Then $\zeta\in C_Q$ and if $f\in V=\{g:\|g\zeta\|\leqq\varepsilon\}$ and $f\in B_r$ then $|f(x)|\leqq\varepsilon$ on K and hence $f\in U\cap B_r$ which completes the proof that $\beta_0\leqq\beta$.

(b) Since $\mathcal{P}$ is Hausdorff so are all the others. That they are locally convex is straightforward.

As to a characterization of the β and β_1 neighborhoods, we have from [30, pp. 78–79]

THEOREM 2.2. (a) $W\in\beta$ $(W\in\beta_1)$ *iff* $W\in\beta_Q$ *for all* $Q\in\mathcal{Q}$ $(W\in\beta_Z$ *for all* $Z\in\mathcal{Z})$.

(b) $W\in\beta$ $(W\in\beta_1)$ *iff for each* $Q\in\mathcal{Q}$ $(Z\in\mathcal{Z})$ *there exists* $V_Q\in\beta_Q$ $(V_Z\in\beta_Z)$ *such that* $W\supset\langle\bigcup_{Q\in\mathcal{Q}}V_Q\rangle$ $(W\supset\langle\bigcup_{Z\in\mathcal{Z}}V_Z\rangle)$ *where* $\langle V\rangle$ *denotes the absolutely convex envelope of* V.

Initially, we have the following relations of these topologies to previously defined topologies:

THEOREM 2.3. (a) *The topologies* κ *through* $\| \quad \|$ *are equal iff* X *is compact.*

(b) *If* X *is locally compact, then* β *is the original strict topology of Buck* [3] *and, additionally,* $\beta=\beta_0$.

Proof. (a) This is clear.

(b) If X is locally compact, then $\beta X\backslash X \in \mathcal{Q}$ and $C_0(X)=C_{\beta X\backslash X}(X)$. Let $\bar{\beta}$ denote Buck's strict topology, generated by the seminorms $P_\zeta(f)=\|\zeta f\|$, $\zeta \in C_0(X)$. It is apparent that $\beta \leqq \bar{\beta}$. On the other hand, if $W \in \bar{\beta}$, then $W \supset \{f : \|f\zeta\| \leqq 1\}$ for some $\zeta \in C_0(X)$. Since $\zeta \in C_Q$ for all $Q \in \mathcal{Q}$, it follows that $W \in \beta_Q$ for every $Q \in \mathcal{Q}$ and hence that $W \in \beta$.

To see that $\beta=\beta_0$ in this case, note from [3] that $\beta=\kappa$ on the sets B_r and from [34] that β is the finest locally convex topology on $C(X)$ equal to β and hence κ on all sets B_r.

Turning to the other "strict" topologies found in the literature, let ω_1 denote the topology of van Rooij [31] determined by the seminorms $p_\zeta(f)=\|f\zeta\|$ where ζ is a bounded function on X such that $\{|\zeta| \geqq \varepsilon\}=\{x : |\zeta(x)| \geqq \varepsilon\}$ is compact for any $\varepsilon>0$, and let ω_2 denote the topology ω_v of Summers [37, §3] generated by the seminorms $P_v(f)=\|vf\|$ where v is a nonnegative u.s.c. function of X such that $\{v \geqq \varepsilon\}$ is compact for each $\varepsilon>0$. Finally, let m denote the mixed topology $\gamma[\kappa, \| \quad \|]$ as defined by Wiweger [42, §1, p. 50]. We have

THEOREM 2.4. (a) $\beta_0=m$ *and* β_0 *has a base of neighborhoods of the form*

$$W(K_i, a_i) = \bigcap_{i=1}^{\infty} \{f : \sup\{|f(x)| : x \in K_i\} \leqq a_i\}$$

where $0<a_i \to \infty$ *and* K_i *is a compact subset of* X.

(b) $\beta_0=\omega_1=\omega_2$.

Proof. (a) Clearly, $W(K_i, a_i) \in \beta_0$ for any choice of the sequences $\{K_i\}$ and $\{a_i\}$. According to [42, Example D, p. 65], the sets $W(K_i, a_i)$ form a base for the neighborhood system at 0 for the topology m and hence $m \leqq \beta_0$. On the other hand, since $\beta_0=\kappa$ on each set B_r, then β_0 satisfies the hypothesis of [42, 2.2.2] and hence $\beta_0 \leqq m$.

(b) Clearly, $\omega_2 \leqq \omega_1 \leqq \beta_0$. Suppose $W=W(K_i, a_i) \in \beta_0$ and let

$$w(x) = \sup\{f(x) : f \in W\} \geqq 0.$$

Choose N such that $a_n \geqq 1$ for $n \geqq N$ and let $a=\min\{1, a_1, a_2, \ldots, a_N\}$. Set $\zeta(x)=2/(w(x)+a)$ for $x \in \bigcup_{m=1}^{\infty} K_m$ and 0 for $x \notin \bigcup_{m=1}^{\infty} K_n$. Then ζ is bounded and u.s.c.

Hence $K=\{\zeta \geqq \varepsilon\}$ is closed. Choose M so that $1/a_n<\varepsilon/4$ for $n \geqq M$. Now $K \subset \{w \leqq 2/\varepsilon\}$ and $\{w \leqq 2/\varepsilon\} \subset \bigcup_{k=1}^{M} K_k$. For suppose there is an x with $w(x) \leqq 2/\varepsilon$ and $x \notin \bigcup_{k=1}^{M} K_k$. Choose $f \in C(X)$ such that $\|f\| \leqq 4/\varepsilon$, $f(x)=4/\varepsilon$ and $f\equiv 0$ on $\bigcup_{k=1}^{M} K_k$. Then $f \in W$ and hence $2/\varepsilon \geqq w(x) \geqq f(x)=4/\varepsilon$, an impossibility. Since K must then be a closed subset of a compact set, K itself is compact.

Finally, $x \in K_i$ implies $(w(x)+a)/2 \leqq a_i$ for all i and hence $\{f : \|f\zeta\| \leqq 1\} \subset W$. Consequently, $W \in \omega_2$ and $\beta_0 = \omega_1 = \omega_2$.

We will look again at the relationship between β_0, β and β_1 in the sequel. For the moment we again consider the original strict topology. If $Q \in \mathcal{Q}$, then $\beta X \backslash Q$ is a locally compact Hausdorff space and $C(\beta X \backslash Q)$ can be given the strict topology defined by the Banach algebra $C_0(\beta X \backslash Q)$. It is straightforward, and important in the light of the relatively well-known strict topology vis-à-vis a locally compact space, that

THEOREM 2.5. *$C(X)_{\beta_Q}$ is topologically isomorphic to $C(\beta X \backslash Q)$ with the strict topology defined by $C_0(\beta X \backslash Q)$.*

The topology β_Z is particularly nice. A topology ξ on a locally convex space E with dual E' is called the strong Mackey topology of the duality (E, E') iff w^*-compact (i.e. $\sigma(E', E)$ compact) subsets of E' are ξ-equicontinuous; in such a case ξ is the finest locally convex topology on E with dual E' [30, p. 62].

COROLLARY 2.6. *$C(X)_{\beta_Z}$ is a strong Mackey space for $Z \in \mathcal{Z}$. If $\zeta \in C(X)$ such that $\zeta(x)=0$ iff $x \in Z$, then $p(f)=\|\zeta f\|$ is a norm on $C(X)$ defining a topology equivalent to β_Z on each set B_r.*

Proof. For $Z \in \mathcal{Z}$, $\beta X \backslash Z$ is σ-compact locally compact and by [6, Theorem 2.6] $C(\beta X \backslash Z)$ with the strict topology is a strong Mackey space(1). By 2.5, so is $C(X)_{\beta_Z}$.

The functions ζ_n such that $\zeta_n \equiv 1$ on $\{x : |\zeta(x)| \geqq 1/n\}$ and 0 off $\{x : |\zeta(x)| > 1/n+1\}$ form an approximate identity for the algebra C_Z and $\|\zeta_n/\zeta\| \leqq n+1$. By [33, Theorem 3.3], p_ζ satisfies the conclusion of the theorem.

3. **The general inductive limit of strict topologies.** Certain arguments carry over verbatim from β to β_1. With the above as models and in the context of [34] we derive some general results.

Let X be a Banach space and suppose that, for each $\alpha \in \Lambda$, B_α is a Banach algebra having approximate identity $\{E_\lambda^\alpha\}$ with $\|E_\lambda^\alpha\| \leqq 1$, X a B_α-module, and $\|Tx\|=0$ for all $T \in B_\alpha$ implying $x=0$. According to [34] each β_α determines a strict topology β_α on X by way of the seminorms $x \to \|Tx\|$, $T \in B_\alpha$. Let $\beta = \mathrm{Lin}_{\alpha \in \Lambda} \beta_\alpha$ be the inductive limit of these. The absolutely convex set $W \in \beta$ iff $W \in \beta_\alpha$ for all $\alpha \in \Lambda$. Certain properties of each β_α carry over quite easily to β; others, such as a description of β-convergence, give evidence of great difficulty.

THEOREM 3.1. *If $W \subset X$ is absolutely convex and for each $r>0$ and $\alpha \in \Lambda$ there is a $V \in \beta_\alpha$ such that $V \cap B_r \subset W$, then $W \in \beta$. Hence β is the finest locally convex topology on X agreeing with itself on each set $B_r = \{x \in X : \|x\| \leqq r\}$.*

(1) Actually [6, Theorem 2.6] is valid for paracompact, locally compact spaces. In fact, w^*-countably compact subsets are equicontinuous in that context.

Proof. Let W be an absolutely convex absorbent set in X such that $W \supset V_r^\alpha \cap B_r$ where $V_r^\alpha \in \beta_\alpha$. Then $W \in \beta_\alpha$ by [34, Theorem 2.2], hence $W \in \beta$. The remainder of the proof follows readily.

COROLLARY 3.2. *The continuity of linear maps on X_β is determined on the norm bounded subsets of X and is equivalent to β-continuity there on.*

THEOREM 3.3. *A linear mapping $A: X \to Y$, where Y is a locally convex topological vector space, is continuous on X_β iff A is continuous on X_{β_α} for every α.*

Proof. For if V is a neighborhood of 0 in Y, then $A^{-1}(V) \in \beta$ iff $A^{-1}(V) \in \beta_\alpha$ for every α.

More generally,

THEOREM 3.4. *A collection of linear mappings $\mathscr{A}$ of X into Y is equicontinuous in X_β iff $\mathscr{A}$ is equicontinuous in X_{β_α} for every α.*

From 3.4 we obtain an important result concerning the strong Mackey topology for the duality (X_β, X'_β). Precisely,

COROLLARY 3.5. (a) *As subsets of X' (the dual of X in the norm topology (see [35, Theorem 4.1])), $X'_\beta = \bigcap_\alpha X'_{\beta_\alpha}$.*

(b) *β is the strong Mackey topology of the duality (X, X'_β) if β_α is the strong Mackey topology of the duality (X, X'_{β_α}) for each α* [2].

Proof. (a) If $F \in X'_\beta$, then $F \in X'_{\beta_\alpha}$ for some α and hence F is bounded and we can view $F \in X'$ according to [35, Theorem 4.1]. By 3.3, $F \in X'_{\beta_\alpha}$ for every α. Conversely, $F \in \bigcap_\alpha X'_{\beta_\alpha}$ implies $F \in X_\beta$ again by 3.3.

(b) If β_α is its strong Mackey topology for all α then every $\sigma(X'_\beta, X)$-compact set in X'_β is β_α-equicontinuous for each α and hence is β-equicontinuous.

THEOREM 3.6. (a) *X'_β is a Banach space in the norm*

$$\|x'\| = \sup \{|\langle x, x'\rangle| : \|x\| \leqq 1, x \in X\}.$$

(b) *When the β-bounded sets are norm bounded, this norm generates the strong topology on X'_β.*

Proof. It is apparent from 3.2 that (a) holds. Since the norm bounded sets in X are β_α-bounded for some α and hence β-bounded, (b) follows readily under the stated hypothesis.

4. **Duality.** In this section we establish that the dual spaces of $C(X)$ with the β_0, β and β_1 topologies are respectively M_t, M_τ and M_σ and thus link these topologies with the extensive work of Varadarajan [39] and the later integral representation work of Knowles [21].

---

(2) We take this opportunity to point out that throughout [33] the topology τ-represents the strong Mackey topology on X_β and that without this interpretation (carelessly overlooked by the author) [33, Theorem 2.1] is not valid.

It follows readily from the definition of β_0 and from 3.1 that all these topologies possess the important property first discovered by Dorroh [8].

THEOREM 4.1. *The topologies β_0, β and β_1 are the strongest locally convex topologies agreeing with their respective selves on each set B_r and the continuity of linear maps in each of these topologies is thus determined on the sets B_r* (3).

THEOREM 4.2. *Let ϕ be a bounded linear functional on $C(X)$, and let ξ represent any one of the topologies β_0, β or β_1. The following are equivalent*: (a) *ϕ is ξ-continuous*, (b) *$|\phi|$ is ξ-continuous*, (c) *ϕ^+ and ϕ^- are ξ-continuous.*

The proof will be a good deal clearer if we retrieve the following special result in the locally compact setting; this lemma can also be observed as a Corollary to [35, Theorem 4.1].

LEMMA. *If Y is locally compact and Hausdorff and ϕ is a positive linear functional on $C(Y)$ and if $\{\xi_\alpha\}$ is an approximate identity of norm one for $C_0(Y)$, then ϕ is β-continuous iff for each $\varepsilon>0$ there is an α_0 such that $\phi(1-\xi_{\alpha_0})<\varepsilon$.*

Proof. If ϕ is β-continuous, then $\phi(1-\xi_\alpha)\to 0$ since $\xi_\alpha \xrightarrow{\beta} 1$. Conversely, for each α let $\phi_\alpha(f)=\phi(f\xi_\alpha)$. Each ϕ_α is β-continuous and, since ϕ is bounded by $\phi(1)$, the existence of an α_0 such that $\phi(1-\xi_{\alpha_0})<\varepsilon$ implies that ϕ is uniformly approximated by the net $\{\phi_\alpha\}$ on the sets B_r and so by 4.1 is β-continuous.

Proof of 4.2. We prove (a) $\to$ (c) $\to$ (b) $\to$ (a) in the context of β-continuity. It will be observed that β_1-continuity is a bit easier. For (a) $\to$ (c) it suffices to show that ϕ^+ is β-continuous and, in turn by 3.3, that ϕ^+ is β_Q-continuous for each Q.

The collection of functions $\{\psi_P : \psi_P \in C(X), \bar{\psi}_P\equiv 1$ on the compact set $P\subset\beta X$, 0 on Q, $0\leqq\bar{\psi}_P(x)\leqq 1$ for all $x\in\beta X\}$ is an approximate identity for C_Q. It suffices to show that $\phi^+(1-\psi_P)\to 0$ where the compact sets P in βX disjoint from Q are ordered by $P\geqq P'$ iff $P\supset P'$.

Since ϕ is β_Q-continuous, then given $\varepsilon>0$ there is a compact set P_0 such that if $P\geqq P_0$ then $|\phi(g-\psi_P g)|<\varepsilon$ for all $\|g\|\leqq 1$, $g\in C(X)$. If $0\leqq h\leqq 1-\psi_P$ we claim that $\phi(h)\leqq\varepsilon$ and hence that $\phi^+(1-\psi_P)\leqq\varepsilon$ for any $P\geqq P_0$.

Let $h_n=h/(1+1/n-\psi_{P_0})$. Then $0\leqq h_n\leqq 1$ and, if $\zeta\in C_Q$, then $|\zeta[h_n(1-\psi_{P_0})-h]|$ $=|\zeta_P[(-h/n)/(1/n+(1-\psi_{P_0}))]|\leqq|\zeta|/n$ and hence $h_n(1-\psi_{P_0})\to h$ in the topology β_Q.

Since $0\leqq h_n\leqq 1$, we have $|\phi(h_n-\psi_{P_0}h_n)|<\varepsilon$ and hence that $|\phi(h)|\leqq\varepsilon$. Thus $\phi^+(1-\psi_{P_0})\leqq\varepsilon$ and ϕ^+ is β_Q- and hence β-continuous.

Since $\phi^-=\phi^+-\phi$ it follows that ϕ^- is also β-continuous. Clearly (c) $\to$ (b), and by the lemma above applied to each space $\beta X\backslash Q$, (b) $\to$ (a).

(3) Actually, because $m=\gamma[\kappa, \|\ \|]$ is the finest *linear* topology agreeing with κ on each set B_r it follows from 2.3(b), 2.4(a), 2.5 and a reworking of the proof of 3.1 that the words "locally convex" may be replaced by "linear."

In the case of β_1-continuity, we take as an approximate identity in C_Z the sequence defined in 2.6 and proceed analogously. For β_0 we appeal to the definition of β_0 and [39, Theorem 9].

THEOREM 4.3. *Let ϕ be a positive linear functional on $C(X)$. Then*
(a) *ϕ is β_0-continuous iff ϕ is tight,*
(b) *ϕ is β-continuous iff ϕ is τ-additive,*
(c) *ϕ is β_1-continuous iff ϕ is σ-additive.*

Proof. For (b) write $\phi(f)=\int_{\beta X}\bar{f}\,d\bar{\nu}$ where $\bar{\nu}$ is the regular Borel measure on βX mentioned in 1.1. According to 3.5(a), $\bar{\nu}\in C'_{\beta_Q}$ as a subset of $C(X)'$ for all Q. But this readily implies that $\bar{\nu}(Q)=0$ for all $Q\in\mathcal{Q}$ and hence by 1.1 that ϕ is τ-additive. Conversely, if ϕ is τ-additive then $\bar{\nu}(Q)=0$ for all $Q\in\mathcal{Q}$. If Q is given and $r>0$, then there is an open set $U\supset Q$ such that $\bar{\nu}(U)<1/2r$. Letting $\zeta\equiv 2\bar{\nu}(\beta X\setminus U)+1$ on $\beta X\setminus U$, 0 on Q and $0\leqq\zeta\leqq 2\bar{\nu}(\beta X\setminus U)+1$, we have $\zeta\in C_Q$ and, for $f\in B_r\cap\{g : \|g\zeta\|\leqq 1\}$,

$$|\phi(f)| = \left|\int_{\beta X}\bar{f}\,d\bar{\nu}\right| \leqq \int_{\beta X\setminus U}|\bar{f}|\,d\bar{\nu}+\int_U|\bar{f}|\,d\bar{\nu} \leqq \frac{1}{2\bar{\nu}(\beta X\setminus U)+1}\,\bar{\nu}(\beta X\setminus U)+r(1/2r) \leqq 1.$$

Hence $\phi^{-1}(-1,1)\supset B_r\cap\{g : \|g\zeta\|\leqq 1\}$ and hence, by 3.2, ϕ is β-continuous.

The proof of (c) can be made similarly, but the latter part can be more clearly seen as follows. If Z is a zero set in $\beta X\setminus X$, and $\bar{f}(x)=0$ iff $x\in Z$ and $f_n=(|f|/\|f\|)^{1/n}$, then $\{f_n\}$ is an approximate identity for C_Z, and since $1-f_n\searrow 0$, $\phi(1-f_n)\to 0$ when ϕ is σ-additive. From this one immediately obtains the β_Z-continuity of ϕ.

Part (a) follows from the definitions of β_0 and tightness and is essentially van Rooij's [31] result.

THEOREM 4.4. *β_0, β and β_1 are topologies of the dual pairs (C, M_t), (C, M_τ) and (C, M_σ) respectively.*

Proof. Combine 4.3 with 4.2 and [39, Theorems 7, 8, 9].

THEOREM 4.5. *$C(X)_{\beta_1}$ is a strong Mackey space and β_1 is the finest locally convex topology of the dual pair (C, M_σ).*

Proof. By 2.6 and 3.5(b), β_1 is the strong Mackey topology of the dual pair (C, M_σ) and hence β_1 is the Mackey topology of this duality.

The question of when the strict topology β is itself its strong Mackey topology has been of long standing interest ([3], [6], [33]). In the next section we will obtain some further answers to this question with a summation in §9. To close out this section we obtain for these topologies other analogues of the results in [3] and [35].

LEMMA 4.6. *The spaces M_σ, M_τ and M_t are complete in the norm*

$$\|\phi\| = \sup\{|\phi(f)| : f\in B_1\} = |\phi|(1).$$

Proof. This follows readily from 4.1.

THEOREM 4.7. *The topologies β_0, β, β_1 and $\|\ \ \|$ have the same bounded sets.*

Proof. If B is β_0 bounded, then B is pointwise bounded as a set of linear functionals on the Banach space M_t and hence $\sup_{f\in B}\sup_{x\in X}|f(x)|<\infty$ since the functionals δ_x for $x\in X$ belong to M_t with norm 1. Hence B is $\|\ \ \|$-bounded and the proof is complete since $\beta_0\leqq\beta\leqq\beta_1\leqq\|\ \ \|$.

The next theorem says that in all cases where β_0, β and β_1 are of interest, these topologies fail to possess certain desirable properties!

THEOREM 4.8. *The following are equivalent*:

(a) *is compact* (*pseudocompact*),

(b) β (β_1) *is barrelled,*

(c) β (β_1) *is bornological,*

(d) β (β_1) *is metrizable,*

(e) β (β_1) *is normable.*

Proof. We consider first the topology β. If X is compact, then $\mathcal{Q}=\{\theta\}$, and $C_\theta=C$ where θ is the null set. Since $1\in C_\theta$, then $\{f:\ \|f\|\leqq 1\}\in\beta$ and (e) holds. Clearly (e) implies (d) and (d) implies (c) as is well known. Given (c), it follows from [30, p. 82] that β is the finest locally convex topology on $C(X)$ having the same bounded sets as β. But the β and $\|\ \ \|$ bounded sets are identical and the norm topology is the finest locally convex topology on $C(X)$ having the same bounded sets as $\|\ \ \|$. Hence $\beta=\|\ \ \|$ and so must be barelled. Finally, if β is barelled, then B_1 must be a neighborhood of 0 being weakly closed. If X were not compact, then picking $x\in\beta X\backslash X$, we would have $B_1\in\beta_{\{x\}}$, a clear impossibility. Hence (a) holds.

For the topology β_1, if X is pseudocompact, then $\mathscr{Z}=\{\theta\}$ and the implications (a) $\to$ (e) $\to$ (d) $\to$ (c) $\to$ (b) hold as above. Given B_1, again $B_1\in\beta_1$ and if X were not pseudocompact, then there would be a $Z\subset\beta X\backslash X$ with $Z\neq\theta$ and we would have $B_1\in\beta_Z$, an equally clear impossibility.

It is also clear that 4.8 is true with β replaced by β_0. Note also that 4.8 implies that $\beta<\beta_1$ when X is pseudocompact and not compact.

In revising this paper we want to note related results found in several preprints of other authors that have been received since submission of this work. These will be injected at the close of each section that follows. The first is due to Giles [12] who introduced the topology β_0 and obtains, among others, 2.1, 2.3(b), 4.3(a), 4.4, 4.7 and 5.12 as these involve β_0; Giles does not always assume complete regularity or the Hausdorff property for his results. A second article by Cooper [7] includes our 2.4(a) for locally compact X. A third article by Hoffmann-Jørgensen [16] introduces a topology on $C(X)$ which by virtue of our 2.4(a) is seen to be β_0. His work contains our principal results on β_0 as well as those of [12] and further results to be noted in the sequel (some interesting results on $\sigma(C, M_t)$ compactness also appear in [16]). Finally, and very recently, a preprint by Fremlin, Garling and Haydon [11] was received which likewise introduces β_0, as well as two additional topologies which we will discuss at the close of §6. Finally all these authors include a Stone-Weierstrass theorem for $C(X)_{\beta_0}$. We observe that a theorem of the Stone-Weierstrass type for $C(X)_{\beta_0}$ can also be drawn from [27].

5. **Convergence.** The continuity of linear maps on $C(X)$ with any of the topologies β_0, β and β_1 is, in the sense of 3.3, easily determined. Continuity into $C(X)$ with these topologies is another matter entirely, except in the case of β_0 where convergence can be referred to uniform convergence on a distinguishable class of subsets of X, the compacta. Any analogous description for either of β or β_1 would be of interest(4).

As is well known, any locally convex topology γ on a vector space E is the topology of uniform convergence on the equicontinuous subsets of the dual E', the sets W^0 where $W \in \gamma$ [30, p. 48] and $W^0 = \{x' \in E' : |\langle x, x'\rangle| \leqq 1 \text{ for all } x \in W\}$(5). Consequently a description of the equicontinuous sets gives in this sense a description of convergence. For the topology β_0 we have the analogue of the descriptions found in [6] and [35] of the β-equicontinuous sets(6).

THEOREM 5.1. *For H a subset of M_t the following are equivalent*:

(a) *H is β_0-equicontinuous.*

(b) *$|H| = \{|\phi| : \phi \in H\}$ is β_0-equicontinuous.*

(c) *H is bounded in the norm on M_t and for every $\varepsilon > 0$ there is a compact set K such that $|\phi|(|f|) \leqq \varepsilon$ for all $f \in B_1$ with $f \equiv 0$ on K.*

(d) *Identifying H with the collection of compact-regular Borel measures μ on X which uniquely represent each $\phi \in H$* (1.3), *then* (1) $\sup_{\mu \in H} |\mu|(X) < \infty$, *and* (2) *if $\varepsilon > 0$, then there is a compact set $K \subset X$ such that $|\mu|(X \backslash K) < \varepsilon$ for all $\mu \in H$, where $|\mu|$ is the total variation of μ.*

Proof. We show that (a) $\rightarrow$ (c) $\rightarrow$ (d) $\rightarrow$ (b) $\rightarrow$ (a).

Given (a), it is clear that H is bounded in norm since the norm bounded set B_1 in $C(X)$ is β_0-bounded. Given $\varepsilon > 0$, the supposition that H be β_0-equicontinuous means that there is a compact set $K \subset X$ such that

$$\{f : |f(x)| \leqq 1 \text{ for all } x \in K\} \cap B_{1/\varepsilon} \subset \tfrac{1}{4} H^0.$$

Hence if $f \equiv 0$ on K, and $\|f\| \leqq 1$, then $(1/\varepsilon) f \in \frac{1}{2} H^0$ or $|\phi(f)| < \varepsilon/2$ for all $\phi \in H$. It follows that $|\phi|(|f|) = \phi^+(|f|) + \phi^-(|f|) \leqq \varepsilon$ and (c) holds.

Given (c), (1) of (d) follows because H is uniformly bounded and $|\mu|(X) = \|\phi\|$. For (2), we have $|\mu|(X \backslash K) = \sup \{|\mu|(C) : C \subset X \backslash K, C \text{ compact}\}$. Given $C \subset X \backslash K$, C compact, let $f \in C(X)$ be zero on K and 1 on C with $0 \leqq f \leqq 1$. Then, $|\mu|(C) \leqq \int_X f\, d|\mu| = |\phi|(f) \leqq \varepsilon$ for all $\phi \in H$. Hence (2) of (d) holds.

Assuming (d) and given $r > 0$, let $\varepsilon = 1/2r$ and let $a = \sup |\mu|(X)$. Suppose that $W = \{f : |f(x)| \leqq 1/2a \text{ for } x \in K\}$ and that $g \in W \cap B_r$. Then, $|\phi|(|g|) = \int_X |g|\, d|\mu|$

(4) For some very recent results, in terms of localization topologies, the reader is referred to Wheeler [41].

(5) Note that for $H \subset E'$, $H^0 = \{x \in E : |\langle x, x'\rangle| \leqq 1 \text{ for all } x' \in H\}$.

(6) This result also appears independently in [11] and [16].

$=\int_K |g|\, d|\mu| + \int_{X\setminus K} |g|\, d|\mu| \leqq 1$. Hence $W \cap B_r \subset |H|^0$ and since $W \in \beta_0$ and $|H|^0$ is absolutely convex and absorbent it follows from 4.1 that $|H|$ is β_0-equicontinuous.

Clearly (b) implies (a), to complete the proof. The equivalence of (a) and (d) is another form of the result in [4, Proposition 2].

Unfortunately, a similar characterization of β- or β_1-equicontinuity in terms of X is not apparent. In terms of βX one can use 5.1, 2.3(b) and 2.5. The aim of this section is to say as much as we can about β_0, β and β_1 convergence in terms of the w^*-compact, rather than the equicontinuous subsets of the dual space. The key idea is the use of Dini's Theorem, which allows us to confine matters to the w^*-compact sets of positive linear functionals.

Suppose that $H \subset M_\tau^+$ is w^*-compact and suppose that $f_\alpha \in C(X)$ and $f_\alpha \searrow 0$ on X. The functions $\langle f_\alpha, \phi\rangle \searrow 0$ for $\phi \in H$ and are continuous in ϕ with the w^*-topology on H. The convergence must then be uniform since H is w^*-compact.

If $Q \in \mathcal{Q}$, let $\mathcal{D} = \{\zeta \in C(X) : 0 \leqq \zeta \leqq 1, \zeta \equiv 0 \text{ on } Q\}$. Direct $\mathcal{D}$ by $\zeta \geqq \zeta'$ iff $\zeta(x) \geqq \zeta'(x)$ for all $x \in X$ so that $\mathcal{D}$ is a directed set and let $f_\zeta = 1-\zeta$ for $\zeta \in \mathcal{D}$. Then $f_\zeta \searrow 0$ on X and hence uniformly on H. Let $r > 0$ and let $a = \sup\{\|\phi\| : \phi \in H\}$ and choose ζ_0 such that $\phi(1-\zeta_0) < 1/2r$ for all $\phi \in H$ and let $V = \{f : \|f\zeta_0\| \leqq 1/2a\}$. Then $V \in \beta_Q$ and if $f \in V \cap B_r$ then $|\phi(f)| \leqq \phi(|f|) \leqq \|f\|\phi(1-\zeta_0) + \phi(f\zeta_0) \leqq 1$, $H^0 \supset V \cap B_r$ and, by 3.1, $H^0 \in \beta$, and hence H is β-equicontinuous. Clearly one can replace β by β_1 and M_τ by M_σ and obtain the same result. The remainder of this section is largely based on these ideas. We have proven

THEOREM 5.2. *The w^*-compact subsets of M_τ^+ (M_σ^+) are β- (β_1-)equicontinuous.*

The topology β_0 must now be brought back into the mix. A set $H \subset M_t$ is called *tight* if it satisfies 5.1(d) or, in other words, is β_0-equicontinuous. It has long been an outstanding problem to characterize the tight subsets of M_t, the original investigations being undertaken by Aleksandrov [1] and Prohorov [28] followed by Le Cam [4], Varadarajan [39] and Conway [6]. A sequential version is Dieudonné's well-known theorem [39, p. 198] for sequential convergence in M_t. The conjecture was that w^*-compact subsets of M_t^+ must be tight, and while Varadarajan [39] had a result in this direction, Billingsley [2] asserts that the "proof contains a lacuna." Recent work ([26], [11], and [16]) has shed quite a bit of light on this question and is summarized at the close of this section.

THEOREM 5.3. *Every w^*-compact subset of M_t^+ is tight iff β_0 is the topology of uniform convergence on the w^*-compact subsets of M_t^+.*

Proof. Let γ denote the topology of uniform convergence on the w^*-compact subsets of M_t^+. From 5.6, $\gamma \leqq \beta_0$ if we assume every w^*-compact set in M_t^+ to be tight. Now β_0 has a base of neighborhoods of the form

$$W(K_i, a_i) = \{f \in C(X) : |f(x)| \leqq a_i \text{ for all } x \in K_i\}$$

where $a_i \to \infty$. Letting $G = \bigcup_{n=1}^{\infty} (1/a_n)\{\delta_x : x \in K_n\}$ it follows that $G \subset M_t^+ \cap W(K_i, a_i)^0$ and hence that the w^*-closure H of G is w^*-compact in M_t^+ [30, p. 61]. Since $W(K_i, a_i) \supset H^0 = \{f \in C(X) : |\langle f, \phi \rangle| \leqq 1 \text{ for all } \phi \in H\} \in \gamma$, then $\beta_0 = \gamma$.

Conversely, $\beta_0 = \gamma$ implies that every w^*-compact set in M_t^+ is tight by 5.1.

Let us call X a T-space iff w^*-compact subsets of M_t^+ are tight. We have the following known result:

THEOREM 5.4 *If X is either locally compact or a topologically complete metric space, then X is a T-space.*

Proof. If X is locally compact, then $\beta = \beta_0$ and X is a T-space by 5.2.

If X is a topologically complete metric space then, given $\phi \in M_t^+$, one has $\phi(f) = \int_X f\, d\mu$, with μ a compact-regular Borel measure on X, and it follows that μ has separable support. If $H \subset M_t^+$ is w^*-compact and $\varepsilon > 0$, then

$$H_1 = \{\phi \in H : \phi(1) \geqq \varepsilon\}$$

is w^*-compact and it readily follows that $J = \{\phi/\|\phi\| : \phi \in H_1\}$ is also w^*-compact. From [2, p. 240], J is tight and hence there is a compact set K such that $(1/\|\phi\|)\mu(X \backslash K) < \varepsilon/a$ where $a = \sup\{\|\phi\| : \phi \in H_1\}$ and $\phi(f) = \int_X f\, d\mu$. Hence $\mu(X \backslash K) < \varepsilon$ for all $\mu \in H$.

Summarizing these results, we have

THEOREM 5.5. (a) β (β_1) *is the topology of uniform convergence on the w^*-compact subsets of M_τ^+ (M_σ^+).*

(b) β_0 *is the topology of uniform convergence on the w^*-compact subsets of M_t^+ iff X is a T-space.*

Proof. (a) Let γ denote the topology of uniform convergence on the w^*-compact subsets of M_τ^+. By 5.2, $\gamma \leqq \beta$.

For the converse, let $W \in \beta$. We can suppose that $W = \overline{V}$ where $V = \langle \bigcap_{Q \in \mathcal{Q}} V_{\xi_Q} \rangle$ and $\xi_Q \in C_Q$, $\xi_Q \geqq 0$, for by [30, p. 12], β has a base of β-closed neighborhoods.

Let $H = \{\phi^+ : \phi \in W^0\}$. Then, for each Q, $\frac{1}{2}H \subset V_{\xi_Q}^0$ and hence $\frac{1}{2}H \subset W^0 = \bigcup_{Q \in \mathcal{Q}} V_{\xi_Q}^0$ so that H is β-equicontinuous. But then W^0 is a subset of $K - K$ where $K \subset M_\tau^+$ is the w^*-closure of H and is therefore w^*-compact. From this it follows that $W = W^{00} \in \gamma$.

The proof for β_1 is entirely the same.

(b) This is simply a restatement of 5.3. Hence the proof.

These results are indicated by earlier work. The result in [4, Theorem 5] is closely related to 5.5(a) because of 4.5, 5.1, 2.3 and 2.5.

As an immediate corollary we have

THEOREM 5.6. $\beta = \beta_1$ *iff $M_\tau = M_\sigma$ and in this case β is the strong Mackey topology of the duality (C, M_τ). Under these conditions X must be realcompact.*

That X must be realcompact follows from [21, 3.2] or our earlier observation in §1. Moran [23] has pointed out that the converse of [21, 3.2] is not valid and goes on to study spaces for which $M_\sigma = M_t$ or $M_\sigma = M_\tau$ in his papers [24] and [25]. We will make more detailed references to these in the sequel. In any case, 5.5 is another partial answer to Buck's longstanding question of equality between the Mackey and strict topologies. In case of the equality $M_\sigma = M_t$ we have

THEOREM 5.7. *If $M_\sigma = M_t$, then $\beta = \beta_1$ is the strong Mackey topology of the dual pair (C, M_t).*

If X is a topologically complete separable metric space it follows from [2, p. 10] that $M_\sigma = M_t$. The hypothesis of separability is closely related to the problem of measure. It is also evident that $M_t = M_\tau$ when X is σ-compact. Regarding tight sets of measures and Prohorov's Theorem we have

THEOREM 5.8. (a) *$\beta = \beta_0$ iff $M_t = M_\tau$ and X is a T-space.*

(b) *$\beta = \beta_0 = \beta_1$ iff $M_t = M_\sigma$ and X is a T-space.*

(c) *If X is a T-space and $M_t = M_\sigma$ then w^*-compact subsets of M_t are tight.*

Part (c), of course, follows from (b), 5.5 and 5.6 and is an improvement over Varadarajan [39, Theorems 29 and 30, p. 205].

Since the equality of β_0 and β_1 seems to be a rather strong topological requirement on X, we have the rather surprising directive from 5.8(b), that when $\beta_0 \neq \beta$, then w^*-compact sets can be tight only when topological restrictions on X are not strong enough to obtain $M_\sigma = M_t$. Similar thoughts follow from 5.8(a). Finally, we point out that Varadarajan [39, p. 200] characterizes the w^*-compact sets in M_σ^+.

In the light of 5.7 it is relevant to point out

THEOREM 5.9. *β_1 is finer than the topology of uniform convergence on the pseudocompact subsets of X.*

Proof. Let $P \subset X$ be pseudocompact and let $W = \{f \in C(X) : |f(x)| \leqq 1 \text{ for all } x \in P\}$. If $Z \in \mathscr{Z}$, then $\bar{P} \cap Z = \theta$ since every continuous function on P is bounded. Hence there is a $\zeta \in C_Z$ such that $\zeta \equiv 1$ on $\bar{P}$ and $W \supset \{f : \|f\zeta\| \leqq 1\} \in \beta_Z$. Hence $W \in \beta_1$ and the proof is complete.

Let us consider convergence in terms of subsets of X. Let $P_\gamma = \{\phi \in M_\gamma^+ : \phi(1) = 1\}$, where γ represents t, τ or σ, be the probability measures in M_γ and give P_γ the relative w^*-topology $\sigma(M_\gamma, C)$. As is well known, the mapping $x \to \delta_x$ is a homeomorphism of X into P_t and the w^*-closure of $\{\delta_x : x \in X\}$ in the space M of all positive linear functionals on $C(X)$ is homeomorphic to βX, while the realcompactification νX of X is this same closure restricted to M_σ. Our next theorem characterizes β and β_1 convergence within the bounded sets B_r, $r > 0$, in this context and extends [3, Theorem 1(v)] to β and β_1.

THEOREM 5.9. *If $\|f_\alpha\| \leqq r$, then $f_\alpha \to 0$ β (β_1) iff $\langle f_\alpha, \phi \rangle \to 0$ uniformly for ϕ belonging to any compact subset of P_τ (P_σ).*

For the proof we need the

LEMMA 5.10. *If H is a w^*-compact subset in M_t^+, M_τ^+, or M_σ^+ respectively and $r>0$, then there is a compact subset J of P_t, P_τ or P_σ respectively such that $B_r \cap J \subset B_r \cap H$.*

Proof. Given H, let $H_1=\{\phi \in H : \|\phi\| \geqq 1/r\}=\{\phi \in H : \phi(1) \geqq 1/r\}$. Evidently, H_1 is w^*-compact. Let $J=\{\phi/\|\phi\| : \phi \in H_1\} \subset P$, where $P=P_t$, P_τ or P_σ as the case may be. Since $\|\phi\|=\phi(1)$, J is compact in P. Let $a=\sup\{\|\phi\| : \phi \in H\}$. If $f \in B_r \cap (1/a)J^0$, then for $\phi \in H\backslash H_1$, $|\phi(f)| \leqq \phi(\|f\|) \leqq 1$, while for $\phi \in H_1$, $1 \geqq |a\phi(f)/\|\phi\|| = a|\phi(f)|/\|\phi\|$ and hence $|\phi(f)| \leqq 1$. Thus $f \in B_r \cap H^0$.

Proof of 5.9. Let γ denote the topology of uniform convergence on the compact subsets of P_γ in the cases $\gamma=\tau$ or σ. For $P=P_\tau$ we have $\gamma \leqq \beta$. If $W \in \beta$ and B_r is given, then by 5.5 there is a w^*-compact $H \subset M_\tau^+$ such that $H^0 \subset W$. Choosing J according to the lemma, one has $B_r \cap J^0 \subset W$. Hence $\gamma \geqq \beta$ on B_r. The proof for β_1 is exactly the same.

It is also easy to see

THEOREM 5.11. *β_0 is the topology of uniform convergence on the subsets of P_t when X is a T-space.*

A number of new results on the T-space question have been discovered recently: [26], [16], [11] and [36]. Topsøe [36] has an especially interesting characterization of w^*-compactness versus tightness in M_t. The remaining authors have, apparently, independently shown that a hemicompact k-space is a T-space, as is any complete metric space [16]. In general, Mosiman [26] has shown that the property of being a T-space is preserved under countable products and intersections and is inherited on closed subspaces. A union of open subspaces is shown to be a T-space and the property is preserved under continuous maps whose inverse images of compact sets are compact. Similar results are found in [16]. Wheeler [26] has studied those spaces in which $\beta_0=\beta=\beta_1$ centered around conditions under which X is a T-space and X is strongly measure compact ($M_t=M_\sigma$ [25]). An example is given therein of a metric space X for which $M_t=M_\sigma$, yet X is not a T-space. Fernique's example [10] of a separable Hilbert space with weak topology is a σ-compact non-T-space. Varadarajan [39] gives an example of a countable non-T-space. A much more complete discussion can be found in [26, §4] and in [11].

6. **Another view of strict topologies: Dini's Theorem.** We have seen how Dini's classical convergence theorem comes into play in establishing the convergence theorems for the topologies β_0, β and β_1 in §5. In §4 we saw that these topologies generalize the norm topology on $C(X)$, X compact, in that the dual of $C(X)_{\beta_0}$, $C(X)_\beta$ and $C(X)_{\beta_1}$ is the dual of $C(X)_{\|\ \|}$ when X is compact. In this section we show that β and β_1 are extensions of the supremum norm topology for compact X to completely regular X in the sense of Dini's Theorem: If the net $\{f_\alpha\} \searrow 0$ on X, then $\|f_\alpha\| \rightarrow 0$ when X is compact.

Not surprisingly this matter involves the order structure on $C(X)$ as well as the locally convex linear structure of the norm and strict topologies. Let

$$C^+ = \{f \in C(X) : f \geqq 0\}.$$

Our principal reference is [32, pp. 203–230]. By 2.1, C^+ is β_0, β and β_1 closed and $C(X)$ with any of these is an ordered topological vector space. But much more is true.

In terms of [32], C^+ is a normal cone for a topology γ on $C(X)$ iff γ has a base of neighborhoods at 0 of absolutely convex, absorbent sets W with the property $f, g \in W$ and $f \leqq h \leqq g$ implies $h \in W$. If additionally γ has a base of neighborhoods W with the property $|f| \leqq |g|$ and $g \in W$ implies $f \in W$, then $C(X)_\gamma$ is said to be locally solid.

THEOREM 6.1. *Let ξ represent any one of the topologies β_0, β or β_1.*

(a) *$C(X)_\xi$ is locally solid.*

(b) *C^+ is a normal cone for ξ and the lattice operations $f \to f^+$, $f \to |f|$, $(f, g) \to \max(f, g)$ are all ξ-continuous.*

(c) *A ξ-equicontinuous set $A \subset C(X)'_\xi$ is contained in some set $B - B$ where $B \subset C(X)'^{+}_\xi = \{\phi \in C(X)'_\xi : \phi \geqq 0\}$ is ξ-equicontinuous.*

(d) *ξ is the topology of uniform convergence on the ξ-equicontinuous subsets of $C(X)'^{+}_\xi$.*

Proof. (a) By [32, p. 234], (a) is equivalent to (b).

(b) For $\xi = \beta_0$, we have a base of neighborhoods of the type $W(K_i, a_i)$ and hence (a) holds for β_0 and hence (b) holds. For $\xi = \beta$ or β_1, C^+ is first of all a normal cone because ξ has a base of neighborhoods of the form H^0 where $H \subset C(X)'^{+}_\xi$ is w^*-compact by 5.5. If $f, g \in H^0$ and $f \leqq h \leqq g$, then $|\phi(h)| \leqq \max(|\phi(f)|, |\phi(h)|) \leqq 1$ for $\phi \in H$ and hence $h \in H^0$.

According to [32, p. 234], the continuity of the map $f \to |f|$ implies the continuity of the remaining mappings in (b). Let $W \in \beta$ and let $|W| = \{f : |f| \in W\}$. We claim $|W| \in \beta$. For this it suffices to show that $|W|$ is absolutely convex, absorbent and $|W| \in \beta_Q$ for each $Q \in \mathcal{Q}$. If $f, g \in |W|$ and $|a| + |b| \leqq 1$, then $0 \leqq |af + bg| \leqq |a|\ |f| + |b|\ |g| \in W$. We can choose W with the property $0 \leqq h \leqq k \in W$ implies $h \in W$. Since $0 \in W$ and $|af + bg| \in W$ then $af + bg \in W$ and hence $|W|$ is absolutely convex. Clearly $|W|$ is absorbent.

Let $Q \in \mathcal{Q}$. Since $W \in \beta_Q$ there is a $\zeta \in C_Q$, $\zeta \geqq 0$ such that $V = \{f : \|f\zeta\| \leqq 1\} \subset W$. Now $f \in V$ implies $|f| \in V \subset W$. Hence $V \subset |W|$ and thus $W \in \beta_Q$.

The proof for $\xi = \beta_1$ is exactly the same.

(c) and (d). These follow from (b) and [32, p. 219].

Incidentally the relationship of (a) to (c) is not trivial, being essentially founded on a category type argument used for example in the proof in [32, p. 66].

From this point on we deal solely with β and β_1. It follows easily from 5.5 and the techniques used for 5.2 that a Dini type theorem holds for these topologies.

THEOREM 6.2. (a) *If* $f_\alpha \in C(X)$ *and* $f_\alpha \searrow 0$, *then* $f_\alpha \xrightarrow{\beta} 0$.
(b) *If* $f_n \in C(X)$ *and* $f_n \searrow 0$, *then* $f_n \xrightarrow{\beta_1} 0$.

Conversely,

THEOREM 6.3. β_1 *is the finest locally convex topology* γ *on* $C(X)$ *such that* $f_n \searrow 0$ *implies* $f_n \xrightarrow{\gamma} 0$.

Proof. Let $\mathscr{U}=\{A\subset C(X) : A$ is absolutely convex, absorbent and, for all sequences $f_n \searrow 0$, one has $f_n \in A$ eventually$\}$. It follows from [19, p. 10] that $\mathscr{U}$ is a base at 0 for a locally convex topology γ on $C(X)$.

It readily follows from the definitions that $C(X)'_\gamma = M_\sigma$. By 4.5, $\gamma \leqq \beta_1$ and, by 6.2(b), $\beta_1 \leqq \gamma$. Hence the proof.

Hence β_1 generalizes Dini's Theorem for sequences in the best possible way. For nets we must turn to β. It is clear that if β is Mackey, for example, if $M_\sigma = M_\tau$, then the proof of 6.3 can be repeated for β, and β is the finest locally convex topology on $C(X)$ generalizing Dini's Theorem for nets. More generally we have

THEOREM 6.4. *Let* $\mathscr{U}$ *be the collection of all absolutely convex sets* $W\subset C(X)$ *such that* (a) $f\leqq h\leqq g$, f, $g\in W$ *imply* $h\in W$, *and* (b) *if* $f_\alpha \searrow 0$, *then* $f_\alpha \in W$ *eventually. Then* $\mathscr{U}$ *is a base at* 0 *for a locally convex topology* γ *for which* C^+ *is a normal cone. Moreover*, $\gamma=\beta$.

Proof. Again by [19, p. 10], $\mathscr{U}$ does indeed define a locally convex topology on $C(X)$ since the sets in $\mathscr{U}$ are absorbent by (b). From 6.1 and 6.2, $\beta \leqq \gamma$. To obtain $\gamma \leqq \beta$ we first note that (a) makes C^+ a normal cone for γ. Hence by [21, p. 219], γ is the topology of uniform convergence on the γ-equicontinuous subsets of $C(X)'^+_\gamma$. But, as is apparent, $C(X)'_\gamma = M_\tau$ and if $K\subset M_\tau^+$ is γ-equicontinuous, then the w^*-closure H of K is w^*-compact [19, p. 61], $H\subset M_\tau^+$ and hence $K^0 \supset H^0 \in \beta$ by 5.2. Thus, $\gamma \leqq \beta$.

Theorems 6.3 and 6.4 allow us to relate our work to the recent work of Fremlin, Garling and Haydon [11], and in turn, to use their work to improve our 6.4. These authors introduce topologies T_τ and T_σ on $C(X)$ which yield M_τ and M_σ as respective dual spaces. The topology T_σ is shown to have as base at 0 all absolutely convex sets U such that $f_n \searrow 0$ implies that $f_n \in U$ eventually. Hence $T_\sigma = \beta_1$ by 6.3. The topology T_τ is shown to be (1) locally solid [11, Proposition 3] and (2) to have as base at 0 all absolutely convex U such that $f_\alpha \searrow 0$ implies $f_\alpha \in U$ eventually. Hence from 6.4, $\beta = T_\tau$. But (2) means that T_τ (and hence β) is the finest locally convex topology extending Dini's Theorem to decreasing nets of functions. Hence, in the sense of Dini's Theorem, β is the exact analogue of β_1.

7. **Completeness.** Let us call X a k_f space if bounded functions f, continuous on each compact subset K of X, must be continuous on X. Clearly a k-space, that is, one in which a set C is closed iff $C \cap K$ is closed for all compact K, is a k_f space; among k-spaces are found the locally compact and first countable spaces [18, p. 231].

The completeness of $C(X)_{\beta_0}$ is fairly straightforward.

THEOREM 7.1. *$C(X)_{\beta_0}$ is complete iff X is a k_f space.*

Proof. For the β_0-Cauchy net $\{f_\alpha\}$ in $C(X)$, let $f(x)=\lim f_\alpha(x)$ for each $x \in X$.

Supposing that $|f(x_n)| \geqq n^2$, we see that $H=\{(1/n)\delta_{x_n} : n=1, 2, \ldots\}$ is β_0-equicontinuous (by 5.1) and hence there is an α_0 such that $|\langle f_\alpha - f_{\alpha'}, \phi\rangle| \leqq 1$ for all α, $\alpha' \geqq \alpha_0$ and $\phi \in H$. It follows that $n \leqq 1+|f_{\alpha_0}(x_n)|/n$, a contradiction. Hence f is bounded.

To see that f is continuous, we note that given a compact K in X, the nėt $\{f_\alpha\}$ is uniformly Cauchy on K, and hence on X.

Finally $f_\alpha \xrightarrow{\beta_0} f$, for given $W(k_i, a_i) \in \beta_0$ there is an α_0 such that α, $\alpha' \geqq \alpha_0$ implies $f_\alpha - f'_\alpha \in W(k_i, a_i)$. Since $f_\alpha \xrightarrow{\mathscr{P}} f$ and $W(k_i, a_i)$ is $\mathscr{P}$ closed, $f_\alpha - f \in W(k_i, a_i)$ for all $\alpha \geqq \alpha_0$.

Conversely, by the Tietze Extension Theorem there is for each compact set K in X a function f_K agreeing with f on K and $\|f_K\| \leqq \|f\|$. The net $\{f_K\}$ under the natural ordering on K is β_0-Cauchy. Hence, $f \in C(X)$.

I wish to thank Robert F. Wheeler for pointing out the converse of 7.1 and to note that this result was discovered independently in [16] where it is also shown that $C(X)_{\beta_0}$ is complete iff it is quasi-complete. This latter result is also true for $C(X)_\beta$ and $C(X)_{\beta_1}$ due to a result of Raikov [29].

THEOREM 7.2. *$C(X)_\xi$ is complete iff $C(X)_\xi$ is quasi-complete, where $\xi=\beta_0$, β or β_1.*

Proof. This follows immediately from 4.1 and Raikov [29].

Further results on the completeness of $C(X)_\beta$ or $C(X)_{\beta_1}$ have been difficult to obtain.

THEOREM 7.3. *If X is a k_f space, then $C(X)_\beta$ and $C(X)_{\beta_1}$ are sequentially complete.*

Proof. The pointwise limit f of the β- (β_1-)Cauchy sequence $\{f_n\}$ is bounded and continuous as above. By [19, p. 10] β (β_1) has a base of β-closed (β_1-closed) absolutely convex sets W, which are in turn weakly closed (i.e., $\sigma(C, M_\tau)$, $\sigma(C, M_\sigma)$ closed) by [19, p. 34]. If $f_n - f_m \in W$ for $m, n \geqq N$ it follows from 1.3, the dominated convergence theorem and 4.4, that $f_n - f \in W$ for $n \geqq N$. Hence the proof.

From [19, p. 107] a locally convex space complete under some locally convex topology is complete under any finer topology of the same dual pair. Consequently,

THEOREM 7.4. *If X is a k_f-space and $M_t = M_\tau$ ($M_t = M_\sigma$), then $C(X)_\beta$ ($C(X)_{\beta_1}$) is complete.*

Both 7.2 and 7.3 appear independently in [11] in terms of $T_\tau = \beta$, $T_\sigma = \beta_1$. Moreover, using the result of Varadarajan [39, Theorem 13] that M_τ^+ is metrizable for metric X, it is shown in [11, Theorem 7] that $C(X)_\beta$ is complete for such an X, and furthermore that the completeness of $C(X)_{\beta_1}$ is equivalent to any one of (a) $M_\tau = M_\sigma$, (b) $\beta = \beta_1$, or, most interestingly, (c) X has no discrete set of measurable cardinality.

It is also conjectured in [11] that $C(X)_\beta$ is complete for any k-space X. Finally, Wheeler [26] in his study of β-simple spaces (those X for which $\beta_0=\beta=\beta_1$) gives an example of a β-simple space for which $C(X)_\beta$ is neither complete nor sequentially complete.

8. **Sequential continuity on** $C(X)$. In [34] consideration was given to the β-sequentially continuous linear functionals on $C(X)$. In this section we investigate this matter further and identify such linear functionals.

Let γ represent any one of the topologies β_0, β or β_1 and let $\mathscr{U}_\gamma=\{W\subset C(X) : W$ is absolutely convex and if $f_n \xrightarrow{\gamma} 0$ then $f_n \in W$ eventually$\}$. By [30, p. 10], $\mathscr{U}_\gamma$ is a base for a locally convex topology on $C(X)$. For $\gamma=\beta_0$, β or β_1 we denote these topologies by β_0^+, β^+ and β_1^+. This follows the notation of Webb [40] who along with Dudley [9] has considered topologies so defined more generally.

THEOREM 8.1. (a) $\beta_1=\beta_1^+$.

(b) $\beta\leqq\beta^+\leqq\beta_1$.

(c) $\beta_0\leqq\beta_0^+\leqq\beta_1$.

Proof. (a) Clearly $\beta_1\leqq\beta_1^+$ and hence $M_\sigma=C(X)'_{\beta_1}\subset C(X)'_{\beta_1^+}$. On the other hand, if $\phi\in C(X)_{\beta_1'^+}$ and $f_n \searrow 0$, then $f_n\to 0$ in the topology β_1 and hence in β_1^+. Consequently, $\phi(f_n)\to 0$ and hence $\phi\in M_\sigma$.

By 4.5, $\beta_1=\beta_1^+$.

(b) Clearly, $\beta^+\leqq\beta_1^+=\beta_1$.

(c) Again, $\beta_0^+\leqq\beta_1^+=\beta_1$.

THEOREM 8.2. $M_\sigma=C(X)'_{\beta_0^+}=C(X)'_{\beta^+}=C(X)'_{\beta_1^+}$.

Proof. We have $C(X)'_{\beta_0^+}\subset M_\sigma$ from 8.1. If $\phi\in M_\sigma$, then $\phi(f)=\int_X f\,d\mu$ where μ is a regular countably additive measure on Baire sets. If $f_n \xrightarrow{\beta_0} 0$, then, by 4.7, $\{f_n\}$ is bounded and hence, by the dominated convergence theorem and 2.1, $\phi(f_n)\to 0$. Hence $M_\sigma=C(X)'_{\beta_0^+}$. The proof for the remaining case is the same.

COROLLARY 8.3. (a) $\beta=\beta^+$ *iff* $\beta=\beta_1$.

(b) *If X is a T-space, then* $\beta_0=\beta_0^+$ *iff* $\beta_0=\beta_1$.

(c) *If* $\beta_0^+=\beta$, *then* $\beta_0^+=\beta_1$.

COROLLARY 8.4. *If X is a T-space and $M_t=M_\sigma$, then $\beta_0=\beta_0^+=\beta=\beta^+=\beta_1$ and β_0-continuity is equivalent to β_1-sequential continuity for linear mappings. If $M_\tau=M_\sigma$, then $\beta=\beta^+=\beta_1$ and the same conclusion holds for β.*

We do not have necessary and sufficient conditions in order that $\beta^+=\beta_1$. Wheeler [46] has shown that $\beta_0^+=\beta^+\geqq\beta$ for k_f spaces and that $\beta_0^+=\beta^+=\beta_1$ for D-spaces X (see §9). Regarding 8.3(c) note that, for $X=[1,\Omega)$, $\beta_0^+=\beta_1$ is the sup norm topology, while, by 4.8, $\beta<\beta_1$ since X is not compact.

Webb [40] calls a locally convex space E sequentially barrelled if any w^*-null sequence in E' is equicontinuous. Recall that β and β_1 are never barrelled in the interesting cases. Moran [25] calls the space X *metacompact* if every open cover has an open point finite refinement: we have

THEOREM 8.5. (a) $C(X)_{\beta_1}$ *is sequentially barrelled.*

(b) *The same is true of* $C(X)_\beta$ *if* X *is metacompact and normal.*

Proof. (a) That $C(X)_{\beta_1}$ is sequentially barrelled follows from 4.5.

(b) Suppose $\phi_n \in M_\tau$ and $\phi_n \xrightarrow{w^*} 0$. By 1.3 and 4.3, $\phi_n(f)=\int_X f\,d\mu_n$. According to [25, 6.4], if $\{Z_\alpha\}$ is a decreasing net of zero sets in X, then $|\mu_n|(Z_\alpha)\searrow 0$ uniformly in n.

Let $H=\{\phi_n : n=1,2,\ldots\}$. According to 3.1 it suffices to show that given $Q\in\mathscr{Q}$ and $r>0$ there is a $V\in\beta_Q$ such that $V\cap B_r\subset H^0$. Let $\mathscr{D}$ be as preceding 5.2, $U_\zeta=\{x : 1-1/4ra<\zeta(x)\}$ where $a=\sup\{\|\phi_n\| : n=1,2,\ldots\}$ and $Z_\zeta=X\backslash U_\zeta$. Then $Z_\zeta\searrow\theta$ and hence there is a ζ_0 such that $\zeta\geqq\zeta_0$ implies $|\mu_n|(Z_\zeta)<1/8r$ for all n. Hence

$$\int_X (1-\zeta_0)\,d|\mu_n| \leqq \int_{U_{\zeta_0}} (1-\zeta_0)\,d|\mu_n| + \int_{Z_{\zeta_0}} (1-\zeta_0)\,d|\mu_n| \leqq \tfrac{1}{2}r.$$

If $g\in B_r\cap\{f : \|f\zeta_0\|\leqq 1/2a\}$, then we easily obtain $g\in H^0$. Hence the proof.

Unfortunately [25, 6.4] requires the additional hypothesis of normality on X. The Dieudonné plank and measures $\mu_n=\delta_{(n+1,\Omega)}-\delta_{(n,\Omega)}$ provide a counterexample. We also note that if $C(X)_\beta$ is sequentially barrelled, then X must be sequentially closed in βX.

Turning to β_0, we have

THEOREM 8.6. *If* X *is locally compact, metacompact and normal, or metrically topologically complete, then* $C(X)_{\beta_0}$ *is sequentially barrelled.*

Proof. In the first case the result follows from 2.3 and 8.5(b). In the second we observe that a sequence of tight measures is supported on a complete separable subspace T, and the Prohorov Theorem [2] then implies that the sequence is a tight set of measures on T and hence on X and, hence, by 5.1(a), β_0-equicontinuous.

The w^*-sequentially barrelled property is closely related to w^*-completeness of the dual space. From 8.1(a), 8.5 and [25, 6.3], one has Aleksandrov's well-known result (8.7(a)).

THEOREM 8.7. (a) M_σ *is* w^**-sequentially complete.*

(b) *If* X *is metacompact and normal, then* M_τ *is* w^**-sequentially complete.*

An alternate proof of the latter part of 8.6 can be drawn from [26]. As is well known X is G_δ in βX when X is a complete metric space. By [21, p. 148], $M_t=M_\tau$. From [26, Corollary 4.4], X is a T-space. By 5.8(a), $\beta_0=\beta$ and the conclusion follows from 8.5(b).

9. **Integral representation.** In revising this section I want to thank Robert F. Wheeler for a number of conversations and helpful observations. The principal

aim of this paper was to obtain a theory of integral representation of linear functionals on $C(X)$ within the context of locally convex spaces. In particular, one is interested in the finest locally converse topology for which a given representation holds. In our viewpoint, a secondary goal should be a theory which allows the statement of results which unify the theory for both locally compact and for metrizable spaces. Our first result does hold for σ-compact locally compact spaces or complete separable metric spaces.

THEOREM 9.1. *If X is either σ-compact locally compact or a complete separable metric space, then*

(a) $\beta_0=\beta=\beta_1$.

(b) *$C(X)_\beta$ is a complete locally convex Hausdorff topological vector space.*

(c) *$\phi \in C(X)'_\beta$ iff there is a unique compact regular Borel measure μ on X such that $\phi(f)=\int_X f\,d\mu$.*

(d) *β is the finest locally convex topology of the dual pair $(C, M_t)=(C, M_\tau)=(C, M_\sigma)$.*

These assertions follow immediately from the remark preceding 5.7, along with 1.3, 2.1, 4.3, 5.4, 4.4, 4.7, and 7.1. Recalling that continuity for linear maps in any one of these topologies is determined on the bounded sets B_r, $r>0$, we note from [34] (or the proof of 2.6) that β is given by a norm on the sets B_r when X is σ-compact, locally compact. We do not know if this remains true for separable metric spaces but do note that this condition (for β_0) implies that X must be the closure of a countable union of compacta, and, conversely, if X has this property, then there is a norm on $C(X)$ yielding a topology on $C(X)$ coarser than β_0 on each B_r.

The result of Conway [6] combined with the recent conclusions in [11] allow an extension of 9.1 which does not involve any dependence on our results for β_1.

THEOREM 9.2. *If X is a locally compact paracompact space or a complete metric space, then* 9.1 *remains true with the omission of the topology β_1 and the dual pair (C, M_σ).*

Proof. By 5.4, X is a T-space and as noted in the proof of 8.6, $M_t=M_\tau$. Hence by 5.8(a), $\beta=\beta_0$. The analogue of (d) follows, in the respective cases, from [6] and [11, Theorem 4]; (b) follows from 7.1; and (c) from 1.3 and 4.3.

All further results we have been able to obtain linking integral representation with locally convex topologies γ for which $C(X)_\gamma$ is a strong Mackey space (or, somewhat weaker, for which 9.1(d) holds) ultimately involve conditions which imply that $M_\sigma=M_\tau$, or equivalently, $\beta=\beta_1$ (and then make use of 5.7). Such spaces X are called measure compact and have been studied most notably by Moran ([23], [24], [25]) and more recently by Mosiman and Wheeler [26]. Varadarajan [39, p. 175] has some earlier results, notably that a Lindelöf space is measure compact. Some very interesting and important related results are due to Granirer [14].

The outstanding point of all these studies is that the equality $M_\tau = M_\sigma$ involves both topological and cardinality assumptions on X. Our aim in the remainder of this section is to incorporate the main conclusion of Moran [25] within our results, and to do the same with a prominent theorem due to Katetov for which we provide an alternate proof based on the work of Granirer [14]. We will otherwise only state what can be readily drawn from these works and then conjecture further.

We first summarize the matter of unique integral representation and the properties of the representing measure; the theorem is but a summary of the work in [20] and [21].

THEOREM 9.3. (a) *ϕ is a positive β-continuous linear functional on $C(X)$ iff there is a nonnegative closed regular Borel measure μ on X such that*

(i) *$\phi(f)=\int_X f\,d\mu$ for all $f\in C(X)$,*

(ii) *$\mu(G_\alpha)\to 0$ for every net of closed sets G_α decreasing to the null set,*

(iii) *$\mu(O)=\sup\{\mu(U) : U\subset O,\ U$ a cozero set$\}$ for any open set O. Moreover, μ is unique.*

(b) *If X is metacompact, then the measure μ above has a closed Lindelöf subspace as a support, and $\mu(O)=\sup\{\mu(Z) : Z\subset 0\}$ for any open set O.*

(c) *If X is normal and μ and γ are closed regular Borel measures on X such that $\int_X f\,d\mu=\int_X f\,d\nu$ for all $f\in C(X)$, then $\mu=\nu$.*

Proof. (a) This relies essentially on Kirk [20]. From 4.3, 1.2 and 1.3 we obtain a measure $\mu\in M_\tau^+$ satisfying (i). The measure μ is then a net additive content [20, p. 333] on X; (ii) and (iii) then follow from [20, 1.9, 1.13 and 1.14]. The uniqueness of μ then follows from [20, 1.15].

(b) This is due to Moran [25, 5.1, 5.3].

(c) If A is closed choose $B\subset X\backslash A$, B closed such that $|\mu|((X\backslash A)\backslash B)<\varepsilon/2$ and $|\nu|((X\backslash A)\backslash B)<\varepsilon/2$. Using normality, $|\nu(A)-\mu(A)|<\varepsilon$ and it follows that $\mu=\nu$ on all Borel sets.

We turn now to the matter of measure compactness and the equality $\beta=\tau(C, M_\tau)$ where $\tau(C, M_\tau)$ is the strong Mackey topology of the duality (C, M_τ). The cardinal of a set D is said to be of measure zero if any measure defined on 2^D has a countable subset of D for its support. Equivalently, $\mu\in M_\sigma^+(D)$ implies $\mu\in M_t^+(D)$ when D has the discrete topology. If d is a continuous pseudo-metric (CPM) on X and $D\subset X$ and there is an α such that, for any $x, y\in D$, $x\neq y$ one has $d(x,y)\geqq\alpha$, then D is called d-discrete. If every d-discrete subset of X, for any CPM d, has cardinal of measure zero, then X is called a D-space. Granirer [14] has accomplished some exquisite characterizations of D-spaces, and points out that it is consistent with Zermelo-Fraenkel set theory to assume that every X is a D-space. The discrete reals form a D-space if the continuum hypothesis holds. Granirer [14, p. 8] also notes that Gödel's constructibility axiom along with Zermelo-Fraenkel set theory implies that every cardinal is of measure 0. Finally, Granirer [14] lets DM_σ^+

$=\{\mu \in M_\sigma^+ :$ there is a D-subset S of X such that $\mu(E)=\mu(X)$ for every Baire set $E \supset S\}$, and $DM_\sigma = DM_\sigma^+ - DM_\sigma^+$; X is a D-space iff $M_\sigma = DM_\sigma$.

The next theorem is due to Katetov. The proof is our own and avoids a key difficulty of his proof by an appeal to the work of Granirer.

THEOREM 9.4. *A paracompact D-space is measure compact, whence* $\beta = \tau(C, M_\tau)$.

Proof. Let $\nu \in M_\sigma^+$, $\nu \neq 0$. We can show that $M_\sigma = M_\tau$ by showing that the support of ν is nonempty [25, p. 509].

If support ν is empty, then for each $x \in X$ there is a cozero set U_x containing x such that $\mu(U_x)=0$. Let $\mathscr{U}=\{U_x : x \in X\}$ and let $\mathscr{V}$ be a locally finite refinement of $\mathscr{U}$. Let $\mathscr{A}$ be a partition of unity, $A=\{\sum_{V\in\mathscr{V}'} f_V : \mathscr{V} \supset \mathscr{V}' \text{ finite}\}$ subordinate to $\mathscr{V}$ [18, p. 171].

The collection of functions in $\mathscr{A}$ is uniformly bounded and equicontinuous in $C(X)$ since $\mathscr{V}$ is locally finite. There is a net $\nu_\alpha \in M_t^+$ of measures of the form $\sum_{i=1}^n a_i \delta_{x_i}$, such that $\nu_\alpha \to \nu$, $\sigma(M_\sigma, C)$. Since $\nu \neq 0$, we can suppose that $\|\nu_\alpha\| \geqq \delta > 0$ for all α and some δ. According to Granirer [14, Theorem 2] there is an α_0 such that $|\langle \sum_{V\in\mathscr{V}'} f_V, \nu_{\alpha_0} - \nu\rangle| < \delta/2$ for all finite $\mathscr{V}' \subset \mathscr{V}$. But $\langle \sum_{V\in\mathscr{V}'} f_V, \nu\rangle = 0$, $\sum_{V\in\mathscr{V}'} f_V \nearrow 1$ on X and hence $|\langle 1, \nu_{\alpha_0}\rangle| = \|\nu_{\alpha_0}\| < \delta/2$, a contradiction.

Since a discrete set D has cardinal of measure zero iff $M_\tau(D) = M_\sigma(D)$, it follows readily from Varadarajan [39, p. 177] that a paracompact, measure compact space must be a D-space.

The topological restrictions of 9.4 can be decreased if the cardinality restrictions are increased, as Moran [25, 4.3] has shown. While a paracompact space is normal and metacompact, the converse is false due to an example of Michael discussed in [25, §7].

THEOREM 9.5. *If X is normal, metacompact and every closed discrete subset of X has semireducible cardinal* [25, p. 510], *then X is measure compact and* $\beta = \tau(C, M_\tau)$.

Regarding the equality $\beta = \tau(C, M_\tau)$, our enthusiasm for the above results is tempered by the fact that $\beta = \tau(C, M_\tau)$ for the discrete reals (9.2) and that the above cardinality assumptions cannot be shown to hold therein without the continuum hypothesis. If one seeks only topological conditions for the equality $\beta = \tau(C, M_\tau)$, one then encounters the example of Moran [25, §7] of a nonnormal measure compact space, dependent on the continuum hypothesis. Moran [25, §7] also takes note of a measure compact space which is not metacompact. While we conjecture that metacompactness and normality imply that $\beta = \tau(C, M_\tau)$, this topological condition cannot be necessary. Our overall conjecture is that $\beta = \tau(C, M_\tau)$ iff $M_\tau = DM_\sigma$. We have been able to show that $M_\tau = DM_\sigma$ for paracompact X, and note then that $M_\tau = DM_\sigma$ for the discrete reals. Most recently, Wheeler [41] has attempted this kind of attack using instead the space of measures of separable support of Dudley.

Bibliography

1. A. D. Aleksandrov, *Additive set-functions in abstract spaces*, a) **8 (50)** (1940), 307–348; b) ibid. **9 (51)** (1941), 563–628; c) Mat. Sb. **13 (55)** (1943), 169–238. MR **2**, 315; MR **3**, 207; MR **6**, 275.

2. P. Billingsley, *Convergence of probability measures*, Wiley, New York, 1968.

3. R. C. Buck, *Bounded continuous functions on a locally compact space*, Michigan Math. J. **5** (1958), 95–104. MR **21** #4350.

4. L. Le Cam, *Convergence in distribution of stochastic processes*, Univ. California Publ. Statist. **2** (1957), 207–236. MR **19**, 128.

5. J. B. Conway, *The strict topology and compactness in the space of measures*, Bull. Amer Math. Soc. **72** (1966), 75–78. MR **32** #4509.

6. ———, *The strict topology and compactness in the space of measures*, Trans. Amer. Math. Soc. **126** (1967), 474–486. MR **34** #6503.

7. J. B. Cooper, *The strict topology and spaces with mixed topology*, Proc. Amer. Math. Soc. **30** (1971), 583–592.

8. J. R. Dorroh, *The localization of the strict topology via bounded sets*, Proc. Amer. Math. Soc. **20** (1969), 413–414. MR **38** #3721.

9. R. M. Dudley, *On sequential convergence*, Trans. Amer. Math. Soc. **112** (1964), 483–507. MR **30** #5266.

10. X. Fernique, *Processus linéaires, processus généralisés*, Ann. Inst. Fourier (Grenoble) **17** (1967), fasc. 1, 1–92. MR **36** #4628.

11. D. H. Fremlin, D. J. H. Garling and R. G. Haydon, *On measures on topological spaces* (to appear).

12. R. Giles, *A generalization of the strict topology*, Trans. Amer. Math. Soc. **161** (1971), 467–474.

13. L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR **22** #6994.

14. E. Granirer, *On Baire measures on D-topological spaces*, Fund. Math. **60** (1967), 1–22. MR **34** #8165.

15. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N. J., 1950; Russian transl., IL Moscow, 1953. MR **11**, 504; MR **16**, 22.

16. J. Hoffmann-Jørgensen, *A generalization of the strict topology and its applications to compactness in $C(X)$ and $M(X)$*, Preprint Series 1969/70, no. 32, Matematisk Institut, Aarhus Universitet.

17. M. Katetov, *Measures in fully normal spaces*, Fund. Math. **38** (1951), 73–84. MR **14**, 27.

18. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955. MR **16**, 1136.

19. R. B. Kirk, *Measures in topological spaces and B-compactness*, Nederl. Akad. Wetensch. Proc. Ser. A **72** = Indag. Math. **31** (1969), 172–183. MR **39** #7410.

20. ———, *Locally compact, B-compact spaces*, Nederl. Akad. Wetensch. Proc. Ser. A **72** = Indag. Math. **31** (1969), 333–344. MR **41** #9201.

21. J. D. Knowles, *Measures on topological spaces*, Proc. London Math. Soc. (3) **17** (1967), 139–156. MR **34** #444.

22. J. Marik, *The Baire and Borel measure*, Czechoslovak Math. J. **7 (82)** (1957), 248–253. MR **19**, 535.

23. W. Moran, *The additivity of measures on completely regular spaces*, J. London Math. Soc. **43** (1968), 633–639. MR **37** #4225.

24. ———, *Measures and mappings on topological spaces*, Proc. London Math. Soc. (3) **19** (1969), 493–508.

25. ———, *Measures on metacompact spaces*, Proc. London Math. Soc. (3) **20** (1970), 507–524.

26. S. E. Mosiman and R. F. Wheeler, *The strict topology in a completely regular setting: Relations to topological measure theory* (submitted).

27. C. Portenier, *Théorème de Stone-Weierstrass et espaces de type* (*C*), C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A872–A874. MR **41** #7407.

28. Ju. V. Prohorov, *Convergence of random processes and limit theorems in probability theory*, Teor. Verojatnost. i Primenen. **1** (1956), 177–238. (Russian) MR **18**, 943.

29. D. A. Raĭkov, *A criterion of completeness of locally convex spaces*, Uspehi Mat. Nauk **14** (1959), no. 1 (85), 223–229. (Russian) MR **21** #3747.

30. A. P. Robertson and W. Robertson, *Topological vector spaces*, Cambridge Tracts in Math. and Math. Phys., no. 53, Cambridge Univ. Press, New York, 1964. MR **28** #5318.

31. A. C. M. van Rooij, *Tight functionals and the strict topology*, Kyungpook Math. J. **7** (1967), 41–43. MR **37** #3336.

32. H. H. Schaefer, *Topological vector spaces*, Macmillan, New York, 1966. MR **33** #1689.

33. F. D. Sentilles, *Conditions for equality of the Mackey and strict topologies*, Bull. Amer. Math. Soc. **76** (1970), 107–112. MR **40** #7768.

34. ———, *The strict topology on bounded sets*, Pacific J. Math. **34** (1970), 529–540.

35. F. D. Sentilles and D. C. Taylor, *Factorizations in Banach algebras and the general strict topology*, Trans. Amer. Math. Soc. **142** (1969), 141–152. MR **40** #703.

36. F. Topsøe, *Compactness in spaces of measures*, Studia Math. **36** (1970), 195–212. MR **42** #3245.

37. W. H. Summers, *The general complex bounded case of the strict weighted approximation property*, Math. Ann. **192** (1971), 90–98.

38. ———, *Separability in the strict and substrict topologies* (submitted).

39. V. S. Varadarajan, *Measures on topological spaces*, Mat. Sb. **55** (**97**) (1961), 35–100; English transl., Amer. Math. Soc. Transl. (2) **48** (1965), 161–228. MR **26** #6342.

40. J. H. Webb, *Sequential convergence in locally convex spaces*, Proc. Cambridge Philos. Soc. **64** (1968), 341–364. MR **36** #5652.

41. R. F. Wheeler, *Characterizations of the strict topology for completely regular spaces* (submitted).

42. A. Wiweger, *Linear spaces with mixed topology*, Studia Math. **20** (1961), 47–68. MR **24** #A3490.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI 65201