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# EQUIVARIANT BORDISM AND SMITH THEORY. III

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**Abstract.** The bordism classes of a manifold with involution is determined by means of Wu type classes in the Smith cohomology.

1. **Introduction.** Being given a closed differentiable manifold $M^n$ with a differentiable involution $t\colon M^n \to M^n$, it was shown in [3] that the bordism class of the involution $(M, t)$ is determined by characteristic numbers from the Smith theory of $(M, t)$. The object of this note is to show that the bordism class may be determined from the Smith theory of $(M, t)$ directly.

Roughly this is analogous to a theorem of Wu [5]. For a closed manifold $M^n$, the Wu classes $v_i \in H^i(M; Z_2)$ are defined by $\langle v_i \cup x, [M]\rangle = \langle Sq^i x, [M]\rangle$ for all $x$ in $H^{n-i}(M; Z_2)$, and then the Stiefel-Whitney classes of $M$ are given by $w_i = \sum_j Sq^{i-j} v_j$. By Thom's theorem [4], the characteristic numbers $\langle w_{i_1} \cup \cdots \cup w_{i_r}, [M]\rangle$ with $i_1 + \cdots + i_r = n$ determine the bordism class of $M$. In the case of an involution, analogous Wu classes arise from operations in the Smith theory.

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2. **Smith theory characteristic classes.** Being given a space $X$ and a continuous involution $t\colon X \to X$, let $F \subset X$ be the fixed point set of $t$ and $X/t$ the orbit space. The Smith theory cohomology of the involution $(X, t)$ is then $H^i_{Z_2}(X, t; Z_2) = H^i(X/t, F; Z_2) \oplus H^i(F; Z_2)$.

Provided the involution $(X, t)$ is sufficiently nice, i.e. if $F$ is an equivariant strong deformation retract of a neighborhood in $X$, the projection $\pi_X\colon X \times EZ_2 \to X$ with $EZ_2$ a universal space for $Z_2$ induces an isomorphism

$$\pi_X^*\colon H^*(X/t, F; Z_2) \xrightarrow{\cong} H^*(X \times EZ_2/Z_2, F \times BZ_2; Z_2),$$

where $BZ_2$ is a classifying space for $Z_2$ and $F \times BZ_2 = F \times EZ_2/Z_2$. This will be assumed throughout, being true for a differentiable involution.

Now let $c' \in H^1(X \times EZ_2/Z_2; Z_2)$ be the characteristic class of the double cover $X \times EZ_2 \to X \times EZ_2/Z_2$, and $c \in H^1(F \times BZ_2; Z_2)$ the induced class. The cup product with $c'$ defines the Gysin homomorphism

$$\mu\colon H^i(X/t, F; Z_2) \to H^{i+1}(X/t, F; Z_2).$$

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Letting $\delta: H^i(F; Z_2) \to H^{i+1}(X/t, F; Z_2)$ be the coboundary one then defines homomorphisms

$$\phi_{k+1}: H^i(F; Z_2) \to H^{i+k+1}(X/t, F; Z_2)$$

by $\phi_{k+1} = \mu^k \delta$.

Since the coboundary homomorphism

$$\delta': H^*(F \times BZ_2; Z_2) \to H^*(X \times EZ_2/Z_2, F \times BZ_2; Z_2)$$

is a $H^*(X \times EZ_2/Z_2; Z_2)$ module homomorphism, $\phi_{k+1}(\alpha) = \delta'(\pi^*(\alpha) \cup c^k)$.

Letting $I: H^i(F; Z_2) \to H^i_{Z_2}(X, t; Z_2)$ and $J: H^i(X/t, F; Z_2) \to H^i_{Z_2}(X, t; Z_2)$ be the inclusions of the summands, one defines homomorphisms

$$\psi_k: H^i(F; Z_2) \to H^{i+k}_{Z_2}(X, t; Z_2)$$

by $\psi_0 = I$, $\psi_k = J\phi_k$ if $k \geqq 1$.

Now suppose $(X, t)$ behaves like a differentiable involution on an $n$-dimensional manifold. Specifically, suppose there is a fundamental class $[X, t] \in H^{Z_2}_n(X, t; Z_2)$ and the fixed set $F$ is the disjoint union of subspaces $F^k$ $(0 \leqq k \leqq n)$, so that there is a fundamental class $[F^k] \in H_k(F^k; Z_2)$ with respect to which $F^k$ satisfies Poincaré duality.

The formulas $\langle v_i \cup x, [F^k]\rangle = \langle Sq^i x, [F^k]\rangle$ for all $x \in H^{k-i}(F^k; Z_2)$ and $w_i = \sum_j Sq^{i-j} v_j$ then define Stiefel-Whitney classes $w_i \in H^i(F; Z_2)$, following Wu. Further, there are classes $\alpha_i \in H^i(F; Z_2)$ defined on $F^k$ by $\langle \alpha_i \cup x, [F^k]\rangle = \langle \psi_{n-k+i} x, [X, t]\rangle$ for all $x \in H^{k-i}(F^k; Z_2)$.

For any pair of sequences $(i_1, \ldots, i_r)$, $(j_1, \ldots, j_s)$, one may then compute the number $\langle w_{i_1} \cup \cdots \cup w_{i_r} \cup \alpha_{j_1} \cup \cdots \cup \alpha_{j_s}, [F]\rangle \in Z_2$, which is the value on $[F^k]$ where $k = i_1 + \cdots + i_r + j_1 + \cdots + j_s$. These will be called the *Wu-Stiefel-Whitney numbers* of the involution $(X, t)$.

The main result of this paper is

PROPOSITION. *If $(M^n, t)$ is a differentiable involution on a closed n-manifold, then $(M^n, t)$ bounds if and only if all of the Wu-Stiefel-Whitney numbers of $(M^n, t)$ are zero.*

By the results of Conner and Floyd [2, Chapter IV], the involution bounds if and only if all of the characteristic numbers $\langle w_{i_1} \cup \cdots \cup w_{i_r} \cup w'_{j_1} \cup \cdots \cup w'_{j_s}, [F]\rangle$ vanish, where $w'_i$ is the $i$th Stiefel-Whitney class of the normal bundle $\nu$ of $F$ in $M$. Hence it suffices to show

LEMMA. *The class $\alpha_i \in H^i(F; Z_2)$ is the ith Stiefel-Whitney class of the complement of $\nu$ and is given by $w'_i + P_i(w'_1, \ldots, w'_{i-1})$ for a universal polynomial $P_i$.*

Thus, the Stiefel-Whitney classes of the normal bundle of the fixed set in $M$ are determined by the Wu classes $\alpha_i$ or the operations $\psi_k$.

3. **Proof of the Lemma.** Now suppose $X=M^n$ is a closed $n$-dimensional manifold and $t$ is a differentiable involution. The fixed set $F$ is then a disjoint union of closed manifolds, and $F^k$ is the union of the $k$-dimensional components of $F$. Let $\nu^{n-k}$ be the normal bundle of $F^k$ in $M$ and identify a neighborhood of $F$ in $M$ with the disjoint union of the disc bundles $D(\nu^{n-k})$, with $t$ agreeing with multiplication by $-1$ in the discs.

Let $\overline{M}$ be obtained from $M$ by deleting the interiors of the sets $D(\nu^{n-k})$, so that $\overline{M}$ is a manifold with boundary, whose boundary $\partial\overline{M}$ is the union of the sphere bundles $S(\nu^{n-k})$. (*Note.* $D(\nu^0)=F^n$ and $S(\nu^0)$ is empty.) The involution $t$ acts freely on $\overline{M}$, so that $\overline{M}/t$ is a manifold with boundary, whose boundary is $(\partial\overline{M})/t$ which is the union of the real projective space bundles $RP(\nu^{n-k})$. Let $p\colon RP(\nu^{n-k})\to F^k$ denote the projection.

Since $F^k$ is an equivariant strong deformation retract of $D(\nu^{n-k})$, one has $H^*(M/t, F; Z_2)\cong H^*(M/t, \bigcup_k D(\nu^{n-k})/t; Z_2)$ and by excision this is

$$H^*(\overline{M}/t, \partial\overline{M}/t; Z_2).$$

Henceforth, these groups will be identified. In particular, the coboundary homomorphism

$$\delta'\colon H^*(F\times BZ_2; Z_2)\to H^*(M/t, F; Z_2)$$

is identical with the composite of

$$(p\times q)^*\colon H^*(F\times BZ_2; Z_2)\to H^*(\partial\overline{M}/t; Z_2),$$

where $p$ is projection on $F$ and $q$ classifies the double cover $\partial\overline{M}\to\partial\overline{M}/t$ or $S(\nu^{n-k})\to RP(\nu^{n-k})$, and the coboundary homomorphism

$$\delta''\colon H^*(\partial\overline{M}/t; Z_2)\to H^*(\overline{M}/t, \partial\overline{M}/t; Z_2).$$

Also, in $H_n^{Z_2}(M, t; Z_2)\cong H_n(\overline{M}/t, \partial\overline{M}/t; Z_2)\oplus H_n(F^n; Z_2)$, the fundamental class of $[M, t]$ is the sum of the fundamental classes $[\overline{M}/t, \partial\overline{M}/t]$ and $[F^n]$.

Now recall that $H^*(RP(\nu^{n-k}); Z_2)$ is the free $H^*(F^k; Z_2)$ module via $p^*$ on $1, c, \ldots, c^{n-k-1}$, where $c=(p\times q)^*(c)$, and the algebra structure is given by the relation

$$c^{n-k}+c^{n-k-1}p^*w_1'+\cdots+c^{n-k-r}p^*w_r'+\cdots+p^*w_{n-k}' = 0.$$

In particular, for $a\in H^*(F^k; Z_2)$

$$\begin{aligned}\langle p^*a\cup c^j, [RP(\nu^{n-k})]\rangle &= 0, & j &< n-k-1,\\ &= \langle a, [F^k]\rangle, & j &= n-k-1.\end{aligned}$$

One may now use the relation $c^{n-k}=c^{n-k-1}p^*w_1'+\cdots+p^*w_{n-k}'$ inductively to write

$$\begin{aligned}c^{n-k+i-1} &= c^{i-1}(c^{n-k}) = c^{i-1}(c^{n-k-1}p^*w_1'+\cdots+c^{n-k-i}p^*w_i'+\cdots)\\ &= c^{n-k+i-2}p^*w_1'+\cdots+c^{n-k-1}p^*w_i'+\text{terms } c^jp^*(x)\\ &\qquad\qquad \text{with } j < n-k-1,\end{aligned}$$

so that if $c^{n-j}=c^{n-k-1}p^*(x_j)+\text{terms } c^q p^*(y), q<n-k-1$, then $x_i=w_1'x_{i-1}+\cdots+w_{i-1}'x_1+w_i'$, which is the inductive formula for the $i$th Stiefel-Whitney class of the complement of $\nu$, and expresses $x_i$ inductively as $x_i=w_i'+P_i(w_1',\ldots,w_{i-1}')$.

Now let $k<n$ and $x\in H^{k-i}(F^k;Z_2)$. Then

$$\begin{aligned}\langle\alpha_i\cup x,[F^k]\rangle &= \langle\psi_{n-k+i}x,[M,t]\rangle = \langle J\phi_{n-k+i}x,[M,t]\rangle\\ &= \langle\phi_{n-k+i}x,[\overline{M}/t,\partial\overline{M}/t]\rangle\\ &= \langle\delta'(\pi_F^*(x)\cup c^{n-k+i-1}),[\overline{M}/t,\partial\overline{M}/t]\rangle\\ &= \langle\delta''(p^*(x)\cup c^{n-k+i-1}),[\overline{M}/t,\partial\overline{M}/t]\rangle\\ &= \langle p^*(x)\cup c^{n-k+i-1},[\partial\overline{M}/t]\rangle\\ &= \langle p^*(x)\cup c^{n-k+i-1},[RP(\nu^{n-k})]\rangle\\ &= \Big\langle c^{n-k-1}p^*(x\cup(w_i'+P_i))+\sum_{j<n-k-1}c^jp^*(xy),[RP(\nu^{n-k})]\Big\rangle\\ &= \langle(w_i'+P_i)\cup x,[F^k]\rangle\end{aligned}$$

so $\alpha_i=w_i'+P_i(w_1',\ldots,w_{i-1}')$ in $F^k$ for $k<n$.

For $k=n$, the homomorphism $\delta\colon H^*(F^n;Z_2)\to H^*(M/t,F;Z_2)$ is zero, so $\psi_i$ is zero on $H^*(F^n;Z_2)$ for $i>0$. Thus $\alpha_i=0=w_i'+P_i(w_1',\ldots,w_{i-1}')$ for $i>0$. For $i=0$ and $x\in H^n(F^n;Z_2)$,

$$\langle\alpha_0\cup x,[F^n]\rangle = \langle\psi_0x,[M,t]\rangle = \langle Ix,[M,t]\rangle = \langle x,[F^n]\rangle$$

so $\alpha_0=1=w_0'$.

This completes the proof of the lemma.

One may now note that the classes defined actually give all of the Smith theory characteristic classes of the tangent bundle. Specifically, let $(BO_n, S)$ be the classifying space for $n$-plane bundles with involution, $i\colon BO_n\to BO_{n+1}$ the classifying map for $\gamma_n\oplus 1$, the Whitney sum with a trivial bundle with trivial action, and $\tau\colon M^n\to BO_n$ the equivariant classifying map for the tangent bundle of $M$.

Now $BO_n$ is the limit of actions on Grassmannians, so may be assumed nice near the fixed set, with the fixed set $FBO_n=\bigcup_{j+k=n}BO_j\times BO_k$, where the $+1$ eigenbundle over $BO_j\times BO_k$ has dimension $j$, and the $-1$ eigenbundle has dimension $k$. Then under $i$, $BO_j\times BO_k$ is sent into $BO_{j+1}\times BO_k$. The classifying map $\tau$ sends $F^q$ into $BO_q\times BO_{n-q}$ with the classes $w_i(F^q)$ and $\alpha_j$ completely determining the cohomology homomorphism.

Now examining the summand $H^*(BO_n/Z_2, FBO_n;Z_2)$ in the Smith theory of $(BO_n, S)$, one may first form the product with $EZ_2$. Then $BO_n\times EZ_2/Z_2$ is homotopy equivalent to $BO_n\times BZ_2$, since both classify $n$-plane bundles with involution over a double cover, and the inclusion of the fixed component $BO_n\times BO_0$ induces a homotopy equivalence $BO_n\times BO_0\times BZ_2\to BO_n\times EZ_2/Z_2$. Thus, the homomorphism

$$\delta'\colon \bigoplus_{j+k=n;k\neq 0} H^*(BO_j\times BO_k\times BZ_2;Z_2)\to H^*(BO_n/Z_2,FBO_n;Z_2)$$

is an isomorphism. This is compatible with $i^*$.

Thus, under the tangent map $\tau: M \to BO_n$, or its stabilization to $BO$, the induced homomorphism on Smith theory is determined by the classes $w_i(F^q)$ and $\alpha_j$, for every Smith theory class is obtained from products of these by applying the homomorphisms $\psi_k$ and summing.

4. **Remarks.** The results obtained obviously give

COROLLARY. *If $f: (M^n, t) \to (N^n, s)$ is an equivariant map of differentiable involutions on closed n-manifolds inducing isomorphisms on Smith theory, then $(M^n, t)$ and $(N^n, s)$ are bordant as involutions.*

This generalizes a result obtained in a discussion with Professor Conner:

PROPOSITION. *If two differentiable involutions $(M^n, t)$ and $(N^n, s)$ have the same equivariant homotopy type then they are bordant as involutions.*

**Proof (Direct argument).** Let $f: (M^n, t) \to (N^n, s)$ be an equivariant homotopy equivalence. Then $f: M \to N$ is a homotopy equivalence and $f: F_M \to F_N$ is a homotopy equivalence. Then $f^*(w') = f^*(w(N)/w(F_N)) = w(M)/w(F_M) = w'$ and $f^*(w(F_N)) = w(F_M)$ so the involutions have the same Conner-Floyd characteristic numbers, and so are bordant.

Finally, the interested reader is encouraged to look at Professor Conner's paper [1]. While there is no direct overlap, there are definite interrelations with this paper. It may clarify the relation between Smith theory and equivariant bordism.

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