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ONE-PARAMETER INVERSE SEMIGROUPS

BY
CARL EBERHART AND JOHN SELDEN

Abstract. This is the second in a projected series of three papers, the aim of which is the complete description of the closure of any one-parameter inverse semigroup in a locally compact topological inverse semigroup. In it we characterize all one-parameter inverse semigroups. In order to accomplish this, we construct the free one-parameter inverse semigroups and then describe their congruences.

0. Let G be a subgroup of the multiplicative group of positive real numbers and let P denote the subsemigroup of G consisting of all $x \in G$ with $x \geqq 1$. Denote by $\mathscr{C}_P$ the class of all inverse semigroups H for which there is a homomorphism $f: P \to H$ such that $f(P)$ generates H (no proper inverse subsemigroup of H contains $f(P)$). We shall call such semigroups H *one-parameter inverse semigroups* and denote by $\mathscr{C} = \bigcup_P \mathscr{C}_P$ the class of all one-parameter inverse semigroups.

The class $\mathscr{C}$ contains well-known semigroups. For example, each homomorphic image of a subgroup of R, the positive real numbers, is a member of $\mathscr{C}$. Also the bicyclic semigroup B is a member of $\mathscr{C}$, as is seen by noting that B is generated by a copy of the nonnegative integers. Indeed, if H is any elementary inverse semigroup, then H^1 is generated by a homomorphic image of the nonnegative integers, and so is a one-parameter inverse semigroup.

The main purpose of this paper is to describe all one-parameter inverse semigroups. In the process of doing this, we shall construct what we term the *free one-parameter inverse semigroups* F_P, one for each subgroup G of R and its associated semigroup P. The semigroup F_P is the only inverse semigroup (up to isomorphism) generated by a subsemigroup isomorphic with P which has the property that each homomorphism $f: P \to S$, an inverse semigroup, extends uniquely to a homomorphism $\bar{f}: F_P \to S$. In particular, every $H \in \mathscr{C}_P$ is a homomorphic image of F_P. We thus adopt the point of view that by describing F_P and the lattice of congruences of F_P for arbitrary P, we will have described all one-parameter inverse semigroups.

We shall assume a certain familiarity with the algebraic theory of semigroups, particularly inverse semigroups. (See Clifford and Preston [1].)

The existence and uniqueness of F_P is a consequence of a theorem due to McAlister [3, Theorem 33]. We were greatly aided in the actual description of F_P

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by two results of Gluskin on elementary inverse semigroups [2, p. 24]. For the description of the congruences on F_P, the results of Reilly and Scheiblich in [4] proved useful.

Although this paper is primarily algebraic in nature, there is a natural topology on F_P with respect to which F_P is a topological inverse semigroup. This fact, together with several other comments of a topological nature, are included in remarks throughout the paper.

1. **The free inverse semigroup on a set X.** In this section we shall review some theory which has already been obtained by McAlister in [3].

If S is an inverse semigroup generated by a subset X, then we say that S is *freely generated* by X provided each function from X into an inverse semigroup extends to a homomorphism on S. One shows easily, using the fact that homomorphisms on inverse semigroups take inverses to inverses, that if S is freely generated by X, then each function from X into an inverse semigroup T extends to a unique homomorphism from S into T.

1.1. THEOREM. *For any nonvoid X there is one and only one inverse semigroup (up to isomorphism) I_X freely generated by X.*

Although it is not our intention to investigate them here, we remark that many interesting questions arise concerning the structure of I_X and its lattice of congruences. For example, it is not difficult to show that the smallest group congruence on I_X has the free group on X as its quotient semigroup.

Now let P be a fixed semigroup. Consider the class of pairs (f, S) where S is an inverse semigroup and f is a homomorphism from P into S so that $f(P)$ generates S. Define two pairs (f, S) and (g, T) to be equivalent provided there is an isomorphism $\phi: S \xrightarrow{\text{onto}} T$ so that $\phi f = g$. This is easily seen to be an equivalence relation on pairs. We call a pair (f, S) a *free pair* provided given any pair (g, T) there is a homomorphism $\phi: S \to T$ such that $\phi f = g$. It follows from the fact that two homomorphisms on an inverse semigroup which agree on a generating set are identical, that the homomorphism ϕ above is unique.

The next theorem establishes the existence and uniqueness of a free pair (f, S).

1.2. THEOREM. *There is an inverse semigroup S and a homomorphism $f: P \to S$ such that (f, S) is a free pair. Furthermore any two free pairs are equivalent. The homomorphism f is* 1-1 *if and only if P is embeddable in an inverse semigroup.*

In case f is 1-1 we identify P with $f(P)$ and call S the inverse semigroup freely generated by the subsemigroup P and denote S by F_P. Note that F_P is characterized by the property that any homomorphism from P into an inverse semigroup extends to a unique homomorphism on F_P. In particular, any inverse semigroup generated by a homomorphic image of P is isomorphic with a quotient semigroup of F_P.

2. **The free one-parameter inverse semigroups F_P.** Let G be a fixed subgroup of R and let $P=\{x \in G \mid x \geqq 1\}$, $P_0=P\backslash\{1\}$. In this section we shall describe fully the structure of the semigroups F_P and F_{P_0} freely generated by the subsemigroups P and P_0 respectively.

First we construct a homomorphic image B_P of F_P which is a generalization of the bicyclic semigroup B. This construction is similar to the one found on p. 107 of Vol. 2 of [1]. Let $B_P=P\times P$ with the following operation:

$$(x, y)(z, w) = (xz/y \wedge z, yw/y \wedge z)$$

where $y \wedge z=\min\{y, z\}$. It is easily checked that the product of two elements of B_P is an element of B_P. In fact we have the following consequence of Theorems 8.43 and 8.44 of Vol. 2 of [1]:

2.1. Theorem. *B_P is a bisimple inverse semigroup which is generated by $P_0 \times 1$.*

2.2. Theorem. *The real number 1 is the identity for F_P. Furthermore F_{P_0} does not have an identity and in fact is isomorphic with $F_P\backslash\{1\}$. Thus F_P is obtained from F_{P_0} by adjoining an identity.*

Proof. Since 1 is the identity of P and P generates F_P, 1 is the identity of F_P. Let S denote the inverse subsemigroup of F_P generated by P_0, and let f be a homomorphism from P_0 into an inverse semigroup T. We assume T has an identity e, for otherwise we could adjoin it. Then f extends to a homomorphism $g: P \to T$ by defining $g(1)=e$. Now g extends to a homomorphism $\bar{g}: F_P \to T$, and $\bar{g}|S$ is clearly the sought extension of f to S. Thus S is freely generated by P_0; that is, $S=F_{P_0}$. Now suppose S has an identity i. Then there exist $x_1, x_2, \ldots, x_n$ in P_0 such that $i=x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$ where $j_k \in \{1, -1\}$ for $k=1, 2, \ldots, n$. Thus $x_1^{j_1}x_1^{-j_1}=x_1^{j_1}x_1^{-j_1}\cdot i=i$ and hence, for some $x \in P_0$, $i=xx^{-1}$ or $i=x^{-1}x$. Suppose that $i=xx^{-1}$. Let $f: P_0 \to P_0 \times 1 \subseteq B_P$ be given by $f(t)=(t, 1)$. Then f extends to a homomorphism $\bar{f}: S \to B_P$. Further $\bar{f}(S)=B_P$ since $P_0 \times 1$ generates B_P. Hence $\bar{f}(i)$ is an identity for B_P and so $\bar{f}(i)=(1, 1)$. But $\bar{f}(i)=\bar{f}(xx^{-1})=\bar{f}(x)\bar{f}(x)^{-1}=(x, 1)(1, x)=(x, x)$ and $x\neq 1$. From this contradiction we conclude that $S=F_{P_0}$ does not have an identity. In particular $1 \notin S$. Suppose $x \in F_P\backslash\{1\}$. Then there exist elements $x_1, x_2, \ldots, x_n$ of P so that $x=x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$ where $j_k \in \{1, -1\}$ for $k=1, 2, \ldots, n$. In fact we may assume that $x_k \in P_0$ for $k=1, 2, \ldots, n$ (this is true for at least one value of k since $x\neq 1$). Thus $x \in S$ and we have shown that $S=F_P\backslash\{1\}$. This completes the proof of this theorem.

An elementary inverse semigroup is defined to be an inverse semigroup generated by a single element. An elementary inverse semigroup may or may not be a one-parameter inverse semigroup depending on whether it has an identity; however we do have the following corollary.

2.3. Corollary. *Suppose the given subgroup G of R is cyclic. Then F_{P_0} is an elementary inverse semigroup with the property that every elementary inverse semigroup is a homomorphic image of F_{P_0}.*

Proof. This follows from 2.2 together with the fact that a homomorphism on the positive integers is determined by its value at 1.

2.4. LEMMA. *If* $x \leqq y$ *then*
(i) $xy^{-1}=(y/x)^{-1}yy^{-1}$,
(ii) $y^{-1}x=y^{-1}y(y/x)^{-1}$,
(iii) $yx^{-1}=(y/x)xx^{-1}$,
(iv) $x^{-1}y=x^{-1}x(y/x)$.

Proof. To see (i), note that

$$\begin{aligned} xy^{-1} &= x((y/x)x)^{-1} = xx^{-1}(y/x)^{-1} = xx^{-1}(y/x)^{-1}(y/x)(y/x)^{-1} \\ &= (y/x)^{-1}(y/x)xx^{-1}(y/x)^{-1} = (y/x)^{-1}yy^{-1}. \end{aligned}$$

Part (ii) is proved similarly and (iii) and (iv) are trivial.

The next result is, in a sense, an analogue of a theorem of Gluskin [2, Lemma 1.2] and follows immediately from the above lemma.

2.5. LEMMA. *Let* $x, y, z \in P$. *Then the elements* $xy^{-1}z$ *and* $x^{-1}yz^{-1}$ *of* F_P *can also be written as follows*:

(i)
$$\begin{aligned} xy^{-1}z &= xz/y && \textit{if } y \leqq x, z, \\ &= (y/x)^{-1}z && \textit{if } x \leqq y \leqq z, \\ &= x(y/z)^{-1} && \textit{if } z \leqq y \leqq x, \\ &= (y/x)^{-1}y(y/z)^{-1} && \textit{if } x, z \leqq y. \end{aligned}$$

(ii)
$$\begin{aligned} x^{-1}yz^{-1} &= (xz/y)^{-1} && \textit{if } y \leqq x, z, \\ &= x^{-1}(y/z) && \textit{if } z \leqq y \leqq x, \\ &= (y/x)z^{-1} && \textit{if } x \leqq y \leqq z, \\ &= (y/x)y^{-1}(y/z) && \textit{if } x, z \leqq y. \end{aligned}$$

(iii) *There exist* a, b, c *in* P *such that* $b \geqq a, c$ *and* $x^{-1}yz^{-1}=ab^{-1}c$.

Proof. Parts (i) and (ii) follow immediately from Lemma 2.4. Using (ii) we can write $x^{-1}yz^{-1}$ as $ab^{-1}c$ if we choose a, b and c as follows: if $y \leqq x, z$ let $a=1$, $b=xz/y$, $c=1$; if $z \leqq y \leqq x$, let $a=1$, $b=x$, $c=y/z$; if $x \leqq y \leqq z$, let $a=y/x$, $b=z$, $c=1$; and if $x, z \leqq y$, let $a=y/x$, $b=y$, $c=y/z$. In each case $b \geqq a$, c, and a, b and c are in P.

2.6. THEOREM. $F_P=PP^{-1}P=P^{-1}PP^{-1}$ *and* $F_{P_0}=PP_0^{-1}P=P^{-1}P_0P^{-1}$.

Proof. It is an immediate consequence of 2.5(i) that $PP^{-1}P \subset P^{-1}PP^{-1}$. Hence $P^{-1}PP^{-1}=(PP^{-1}P)^{-1} \subset (P^{-1}PP^{-1})^{-1}=PP^{-1}P$ and so $PP^{-1}P=P^{-1}PP^{-1}$. Note also that $(PP^{-1}P)^2=(PP^{-1}P)(PP^{-1}P) \subset P(P^{-1}PP^{-1})P=P(PP^{-1}P)P \subset PP^{-1}P$. Hence $PP^{-1}P$ is an inverse subsemigroup of F_P. Since $P \subset PP^{-1}P$, we obtain $F_P =PP^{-1}P$. Now suppose $u \in F_{P_0}=F_P \backslash \{1\}$. Then there exist $x, y, z \in P$ such that $u=x^{-1}yz^{-1}$. Now it follows from 2.5(iii) that there exist $a, b, c \in P$ with $b \geqq a$, c so that $u=x^{-1}yz^{-1}=ab^{-1}c$. However, at least one of a, b, c is not 1, and so $b\neq 1$. This says that $u\in PP_0^{-1}P$. On the other hand, choose $xy^{-1}z$ in $PP_0^{-1}P$. Suppose $1=xy^{-1}z$. Note that $y\neq 1$. If $x=z=1$, then $y^{-1}=1$. So $y=1$ which is a contradiction. Thus, either $x\neq 1$ or $z\neq 1$. Without loss of generality, suppose $x\neq 1$. Now if $z=1$, then $1=xy^{-1}\in F_{P_0}$, which is a contradiction. So $z\neq 1$. Thus none of x, y, or z is 1. Therefore $1=xy^{-1}z\in F_{P_0}$, another contradiction. Thus $xy^{-1}z\neq 1$; i.e., $xy^{-1}z\in F_{P_0}$. Hence $PP_0^{-1}P=F_{P_0}$.

2.7. Theorem. *Each element of F_P can be written in one and only one way in the form $xy^{-1}z$ where $x, y, z\in P$ with $x, z\leqq y$. Refer to this as the canonical representation of elements of F_P. Then if $u, v\in F_P$ with canonical representations $u=xy^{-1}z$ and $v=rs^{-1}t$, then uv has as its canonical representation*

$$uv = (xzr/y\wedge zr)(yzrs/(y\wedge zr)(zr\wedge s))^{-1}(zrt/zr\wedge s).$$

Proof. Let $u\in F_P$. Then by 2.6 there are elements, $a, b, c\in P$ such that $u=a^{-1}bc^{-1}$. Now using 2.5(iii) we can write $u=xy^{-1}z$ where $x, z\leqq y$. To show that the representation is unique, we make use of the semigroup B_P defined earlier. Let $f, g\colon P\to B_P$ be the homomorphisms given by $f(x)=(x, 1)$ and $g(x)=(1, x)$. Let $\bar f$ and $\bar g$ be the extensions of f and g respectively to F_P. Now suppose that $u\in F_P$ has two representations $xy^{-1}z$ and $rs^{-1}t$ where $x, z\leqq y$ and $r, t\leqq s$. Then $\bar f(xy^{-1}z)=f(x)f(y)^{-1}f(z)=(x, 1)(1, y)(z, 1)=(x, y/z)$ and similarly $\bar f(rs^{-1}t)=(r, s/t)$, $\bar g(xy^{-1}z)=(y/x, z)=\bar g(rs^{-1}t)=(s/r, t)$. Hence $r=x$, $s=y$ and $z=t$ and thus the representation is unique.

To establish the rule for multiplication, let $u, v\in F_P$ with representations (not necessarily canonical) $u=xy^{-1}z$ and $v=rs^{-1}t$. It then follows from 3.4(ii) that

$$\begin{aligned} uv &= x(ys/zr)^{-1}t && \text{if } zr\leqq s, y,\\ &= xy^{-1}(zrt/s) && \text{if } s\leqq zr\leqq y,\\ &= (xzr/y)s^{-1}t && \text{if } y\leqq zr\leqq s,\\ &= (xzr/y)(zr)^{-1}(zrt/s) && \text{if } s, y\leqq zr.\end{aligned}$$

Now since $y\wedge zr\leqq xzr$ and $zr\wedge s\leqq zrt$ it follows that $xzr/(y\wedge zr)$, $zrt/(zr\wedge s)$, and $yzrt/((y\wedge zr)(zr\wedge s))$ are all in P. It is a simple matter to check using the four cases above that in fact,

$$uv = (xzr/y\wedge zr)[yzrs/(y\wedge zr)(zr\wedge s)]^{-1}(zrt/zr\wedge s).$$

Further, if $xy^{-1}z$ and $rs^{-1}t$ are canonical; i.e. if $x, z\leqq y$ and $r, t\leqq s$ then it is easily checked that $xzr/y\wedge zr$, $zrt/zr\wedge s\leqq yzrs/(y\wedge zr)(zr\wedge s)$ and so the representation for the product above is canonical. This completes the proof.

2.8. Corollary. *The elements of $F_{P_0}=F_P\backslash\{1\}$ consist precisely of those elements of F_P whose canonical representation $xy^{-1}z$ is such that $y\neq 1$.*

Proof. Let $u\in F_{P_0}$ and let $xy^{-1}z$ be its canonical representation. If $y=1$ then $x=z=1$ and so $u=1$. Hence $y\neq 1$. Conversely, if $xy^{-1}z\in F_P$ with $x, z\leqq y\neq 1$, then $xy^{-1}z\in PP_0^{-1}P=F_{P_0}$, by 2.6. Q.E.D.

Using 2.7 and 2.8 we immediately obtain the following parametrization theorem for F_P and F_{P_0}.

2.9. COROLLARY. *Let* $T_P=\{(x, y, z) \mid x, y, z \in P \text{ with } x, z \leqq y\}$. *Define an operation on* T_P *by*

$$(x, y, z)(r, s, t) = (xzr/y \wedge zr, yzrs/(y \wedge zr)(zr \wedge s), zrt/zr \wedge s).$$

Then the map $\phi: F_P \to T_P$ *defined by* $\phi(u)=(x, y, z)$ *for* $u \in F_P$ *with canonical representation* $u=xy^{-1}z$ *is an isomorphism from* F_P *onto* T_P. *Further if* $T_{P_0}=T_P\backslash\{(1, 1, 1)\}$, *then* $\phi|F_{P_0}$ *is an isomorphism from* F_{P_0} *onto* T_{P_0}.

2.10. REMARK. If T_P is given the subspace topology from the product space $P\times P\times P$, where P is given the subspace topology from R with the usual topology, then it is easily seen that the multiplication and inversion on T_P are continuous; that is, T_P is a topological inverse semigroup. This follows from the fact that multiplication and inversion on R and the $\wedge$ operation on P are all continuous operations. Hence there is a natural topology on F_P making F_P into a topological inverse semigroup. Indeed, F_P is freely generated by P even in the topological sense; that is, any continuous homomorphism from P into a topological inverse semigroup S extends to a unique continuous homomorphism from F_P into S.

The idempotent structure of F_P is determined next.

2.11. LEMMA. *Let* $u \in F_P$ *with canonical representation* $u=xy^{-1}z$. *Then the canonical representation of* u^{-1} *is* $(y/z)y^{-1}(y/x)$.

Proof. Note y/z, $y/x \in P$. *Also note* $u^{-1}=z^{-1}yx^{-1}$. Hence by 2.5(ii) $u^{-1}=(y/z)y^{-1}(y/x)$.

For $x \in P$, let $e_x=xx^{-1}$ and $f_x=x^{-1}x$, and let $E=\{e_x \mid x \in P\}$, $F=\{f_x \mid x \in P\}$. Note E, $F \subseteq E_P$, the set of idempotents of F_P.

2.12. THEOREM. *Let* $u \in F_P$ *with canonical representation* $xy^{-1}z$. *Then* $u \in E_P$ *if and only if* $y=xz$. *Furthermore, each element of E can be written in one and only one way in the form* e_xf_x *for some* $x, z \in P$. *Thus* E_P *is the direct sum of the two subsemilattices E and F. Also* $e_xf_y \leqq e_uf_v$ *if and only if* $u \leqq x$ *and* $v \leqq y$.

Proof. Suppose $u \in E_P$ and $xy^{-1}z$ is the canonical representation of u. Then by 2.9, $u=u^{-1}=(y/z)y^{-1}(y/x)$. Hence $(y/z)=x$, that is, $y=xz$. On the other hand, if $y=zx$ then $xy^{-1}z=(xx^{-1})(z^{-1}z)=e_xf_z \in E_P$. Hence to establish the last statement we need only show the uniqueness of the representation. So suppose $x, z, r, t \in P$ with $xx^{-1}z^{-1}z=e_xf_z=e_rf_t=rr^{-1}t^{-1}t$. Then, using the homomorphisms $\bar{f}$ and $\bar{g}$ of 2.7 we see that $f(xx^{-1}z^{-1}z)=f(x)f(x)^{-1}f(z)^{-1}f(z)=(x, 1)(1, x)(1, z)(z, 1)=(x, x)=\bar{f}(rr^{-1}tt^{-1})=(r, r)$ and similarly $\bar{g}(xx^{-1}z^{-1}z)=(z, z)=\bar{g}(rr^{-1}t^{-1}t)=(t, t)$. Hence $x=r$ and $z=t$. The last assertion follows easily upon noting that $e_xe_u=e_{x\vee u}$. 2.13 follows immediately from 2.12 and the fact that $F_{P_0}=F_P\backslash\{1\}$.

2.13. COROLLARY. *The idempotents of F_{P_0} are precisely those elements of F_P which can be written (uniquely) in the form $e_x f_z$ where $\{x, z\} \cap P_0 \neq \varnothing$.*

Next we determine Green's relations (confer with [1]) on F_P.

2.14. THEOREM. *Let $u, v \in F_P$ with canonical representations $u = xy^{-1}z$ and $v = rs^{-1}t$. Then*

(i) $u \mathcal{R} v$ *if and only if* $x=r$ *and* $y=s$,

(ii) $u \mathcal{L} v$ *if and only if* $y=s$ *and* $z=t$,

(iii) $u \mathcal{H} v$ *if and only if* $x=r$, $y=s$ *and* $z=t$,

(iv) $u \mathcal{D} v$ *if and only if* $y=s$.

Proof. (i) We know $u \mathcal{R} v$ if and only if $uu^{-1}=vv^{-1}$. But

$$uu^{-1} = (xy^{-1}z)((y/z)y^{-1}(y/x)) = xy^{-1}(y/x)$$

and similarly $vv^{-1}=rs^{-1}(s/t)$. Hence by 2.7 $uu^{-1}=vv^{-1}$ if and only if $x=r$ and $y=s$.

(ii) Analogous to (i).

(iii) Follows immediately from (i) and (ii).

(iv) Suppose $u \mathcal{D} v$. Then there is an element w of F with $u \mathcal{R} w$ and $w \mathcal{L} v$. Let $ab^{-1}c$ be the canonical representation of w. Then by (i) $y=b$ and by (ii) $b=s$. Hence $y=s$. On the other hand, if $y=s$ let $w=xy^{-1}t$. Then $u \mathcal{R} w$ by (i) and $w \mathcal{L} v$ by (ii). Hence $u \mathcal{D} v$. This completes the proof of 2.12.

From 2.14 we get that there is a $\mathcal{D}$-class D_y for each element y of D: $D_y = \{xy^{-1}z \mid x, z \in P \text{ with } x, z \leqq y\}$. Note also that $E_P \cap D_y = \{e_x f_z \mid xz = y\}$. Hence the $\mathcal{D}$-class D_y can be pictured as in Figure 1.

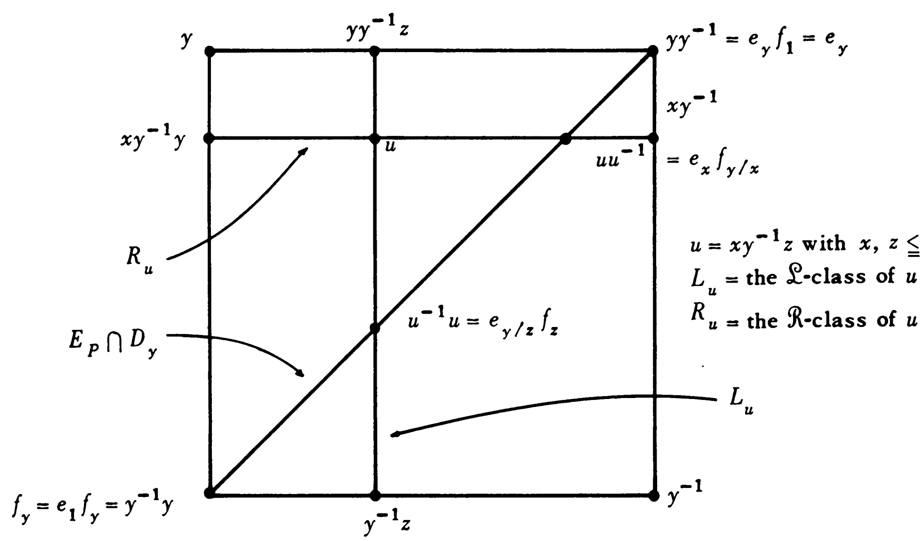


FIGURE 1

It may be helpful to visualize F_P as in Figure 2.

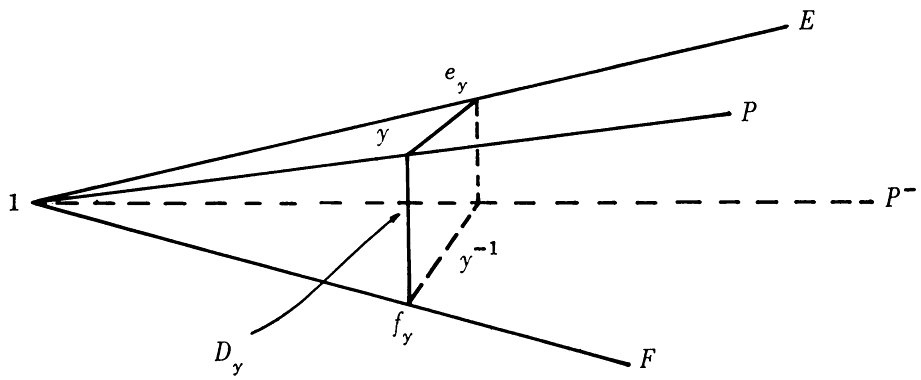


FIGURE 2

Note that the idempotents of F_P lie in a plane which cuts F_P into two pieces.

Next we determine the ideal structure of F_P. For $y \in R$, let

$$I_y = \bigcup \{D_t \mid t \geqq y \text{ and } t \in P\}$$

and let

$$I_y^\circ = \bigcup \{D_t \mid t > y \text{ and } t \in P\}.$$

2.15. THEOREM. *For each $y \in P$, I_y and I_y° are ideals of F_P. Conversely, if I is an ideal of F_P, then there is an element $y \geqq 1$ of R such that $I = I_y$ or $I = I_y^\circ$. Consequently the ideals of F_P are totally ordered with respect to set inclusion.*

Proof. The fact that I_y and I_y° are ideals of F_P follows readily from the rule for multiplication expressed in 2.7. On the other hand, if I is an ideal of F_P, then let y denote the greatest lower bound of the set of all $t \in P$ such that $D_t \cap I \neq \varnothing$. It is not difficult to show that if $D_t \cap I \neq \varnothing$, then $D_{tt_1} \subset I$ for all $t_1 \in P$, and hence $I = I_y$ if $D_y \cap I \neq \varnothing$ or $I = I_y^\circ$ if $D_y \cap I = \varnothing$. Q.E.D.

2.16. REMARK. If we give F_P the natural topology described in 2.10 then the closed ideals are the ones which can be written in the form I_y.

3. **The lattice of congruences on F_P.** In this section as in the last, G is an arbitrary subgroup of R, the multiplicative group of positive reals, and $P = \{x \in G \mid x \geqq 1\}$. We shall describe here the structure of the lattice of congruences on the free one-parameter inverse semigroup F_P, and hence obtain a description of every one-parameter inverse semigroup.

The set $\Lambda(S)$ of congruences on a semigroup S is well known to be a complete lattice with respect to the operations

$$\sigma \wedge \rho = \sigma \cap \rho \quad \text{and} \quad \sigma \vee \rho = \bigcap \{\delta \in \Lambda(S) \mid \cup \rho \sigma \subset \delta\}.$$

The largest (resp. smallest) congruence on S, which is $S^2 = S \times S$ (resp. $\Delta S^2 = \{(x, x) \mid x \in S\}$), is denoted by 1 (resp. 0). The θ relation on $\Lambda(S)$, first defined and studied on regular semigroups S by Reilly and Scheiblich [4] provides a useful aid in visualizing $\Lambda(S)$. The relation is defined by $\sigma \, \theta \, \rho$ if and only if $\sigma \cap E^2 =$

$\rho \cap E^2$, where E is the set of idempotents on S. It is shown in [4] that if S is an inverse semigroup, then θ is a lattice congruence on $\Lambda(S)$. The θ-class of 1 is the set of group congruences on S; the θ-class of 0 is the set of idempotent-separating congruences; in general, each θ-class is a complete lattice of commuting congruences on S.

A congruence ω on E, the idempotents of an inverse semigroup S, is *normal* provided whenever $e\,\omega\,f$, then $xex^{-1}\,\omega\,xfx^{-1}$ for all $x \in S$. The normal congruences on E are precisely those congruences ω on E such that $\omega = \sigma \cap E^2$ for some $\sigma \in \Lambda(S)$. In fact one sees that $\Lambda(S)/\theta$ is isomorphic with the lattice of normal congruences on E, under the map induced by the map from $\Lambda(S)$ to the normal congruences on E given by $\sigma \to \sigma \cap E^2$.

As a first step in describing $\Lambda(F_P)$, we shall determine the normal congruences on E_P, the set of idempotents of F_P. Recall 2.12, which says that E_P is the direct sum of $E = \{xx^{-1} \mid x \in P\}$ and $F = \{x^{-1}x \mid x \in P\}$.

3.1. Lemma. *Let $x, y, t \in P$. Then*

(i) $$te_xf_yt^{-1} = \begin{cases} e_{tx}f_{y/t} & \text{if } t \leqq y \\ e_{tx} & \text{if } y \leqq t \end{cases} = e_{xt}f_{y/y\wedge t},$$

(ii) $$t^{-1}e_xf_yt = \begin{cases} e_{x/t}f_{ty} & \text{if } t \leqq x \\ f_{ty} & \text{if } x \leqq t \end{cases} = e_{x/x\wedge t}f_{ty}.$$

Proof. This follows from the rule for multiplication expressed in 2.7.

Let A and B denote the relations on E_P defined by $e_xf_y\,A\,e_rf_s$ if and only if $x=r$ and $e_xf_y\,B\,e_rf_s$ if and only if $y=s$. These are clearly congruence relations on E_P. Furthermore, it is also clear that $A \vee B = E_P^2$ and $A \wedge B = \Delta E_P^2$. Let I be an ideal of F_P, and let $IA = (A \cap I^2) \cup \Delta E_P^2$, $IB = (B \cap I^2) \cup \Delta E_P^2$, and $IE_P^2 = (E_P^2 \cap I^2) \cup \Delta E_P^2$. We see immediately that IA, IB, and IE_P^2 are all congruences on E_P also.

3.2. Theorem. *Each of the above congruences on E_P is normal. As a set of normal congruences, they form a lattice with the structure as indicated in the diagram below: (I, J ideals of F_P with $J \subseteq I$)*

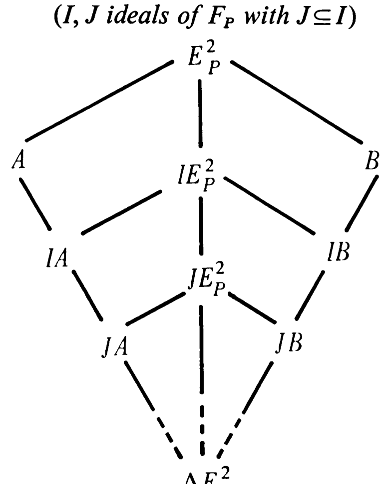


Figure 3

Proof. If I is an ideal of F_P and ω is a normal congruence on E_P, then $I\omega=(\omega \cap I^2) \cup \Delta E_P^2$ is clearly a normal congruence on E_P, since it is the intersection of the two normal congruences ω and $(I^2 \cap E_P^2) \cup \Delta E_P^2$. Hence the only assertion requiring proof is that A and B are normal. To see this, let $u=ab^{-1}c \in F_P$ and note that by 3.1

$$ue_xf_yu^{-1} = e_{acx/b\wedge cx}f_{(by/y\wedge c)/(a\wedge[by/(y\wedge c)])}.$$

From this we see that A and B are normal. Q.E.D.

3.3. Lemma. *Suppose ω is a normal congruence on E_P, and suppose $x_0, y_0, t_0 \in P$ with $t_0 \neq 1$. Let I denote the ideal $I_{x_0y_0}=\bigcup \{D_t \mid t \geqq x_0y_0\}$ of F_P. Then*

(i) *if $e_{x_0}f_{y_0}\ \omega\ e_{x_0}f_{y_0t_0}$, then $IA \subseteq \omega$,*

(ii) *if $e_{x_0}f_{y_0}\ \omega\ e_{x_0t_0}f_{y_0}$, then $IB \subseteq \omega$.*

Proof. (i) Suppose $x, y, t \in P$ with $xy \geqq x_0y_0$. We wish to show that $e_xf_y\ \omega\ e_xf_{yt}$. Note that $e_xf_y=xf_{xy}x^{-1}$ and $e_xf_{yt}=xf_{xyt}x^{-1}$; hence the result follows if $f_{xy}\ \omega\ f_{xyt}$. To see this, first note that $f_{x_0y_0}=x_0^{-1}e_{x_0}f_{y_0}x_0\ \omega\ x_0^{-1}e_{x_0}f_{y_0t_0}x_0=f_{x_0y_0t_0}$. Hence $f_{x_0y_0t_0}=t_0^{-1}f_{x_0y_0}t_0\ \omega\ t_0^{-1}f_{x_0y_0t_0}t_0=f_{x_0y_0t_0^2}$, and so $f_{x_0y_0}\ \omega\ f_{x_0y_0t_0^2}$. Inductively, we have that $f_{x_0y_0}\ \omega\ f_{x_0y_0t_0^n}$ for each positive integer n. Now choose n so large that $x_0y_0t_0^n \geqq xyt \geqq xy$. Then since ω is a congruence on E_P,

$$f_{xy} = f_{xy}\cdot f_{x_0y_0}\ \omega\ f_{xy}\cdot f_{x_0y_0t_0^n} = f_{x_0y_0t_0^n}$$

and

$$f_{xyt} = f_{xyt}\cdot f_{x_0y_0}\ \omega\ f_{xyt}\cdot f_{x_0y_0t_0^n} = f_{x_0y_0t_0^n}.$$

Hence $f_{xy}\ \omega\ f_{xyt}$ and the proof of (i) is complete. The proof of (ii) is analogous.

3.4. Theorem. *Let ω be a nonzero normal congruence on E_P. Then there is an ideal I of F_P such that ω is one of the congruences IA, IB, or IE_P^2. Consequently the lattice shown in 3.2 is the lattice of all normal congruences on E_P.*

Proof. Since $\omega \neq \Delta E_P^2$, there exist $x, y, r, s \in P$ with $x \neq r$ or $y \neq s$ such that $e_xf_y\ \omega\ e_rf_s$. Suppose $x \neq r$; say $x<r$. Then since $e_xf_{y\vee s}=e_xf_y(f_{y\vee s})\ \omega\ e_rf_s(f_{y\vee s})=e_rf_{y\vee s}$, we have by 3.3 that $I_{x(s\vee y)}B \subseteq \omega$. Similarly, if $y<s$, then $I_{(x\vee r)y}A \subseteq \omega$. In any event, at least one of the sets $L=\{t \in P : I_tA \subseteq \omega\}$ and $R=\{T \in P : I_tB \subseteq \omega\}$ is nonvoid.

Suppose $R=\varnothing$ and $L \neq \varnothing$. Let $I_L=\bigcup \{I_t : t \in L\}$ and note that $I_LA=\bigcup \{I_tA : t \in L\} \subseteq \omega$. So let $e_xf_y\ \omega\ e_rf_s$; $x=r$, otherwise $R \neq \varnothing$. Assume $y<s$. Then $(e_xf_y, e_rf_s) \in I_{xy}A$. But by 3.3, $I_{xy}A \subseteq \omega$ so $xy \in L$; hence $I_{xy}A \subseteq I_LA$. Therefore $\omega=I_LA$. By an analogous argument we conclude that if $L=\varnothing$, then $R \neq \varnothing$, so $I_RB=\omega$ where $I_R=\bigcup \{I_t : t \in R\}$.

If neither L nor R is void, then we claim $L=R$ and $\omega=I_LE_P^2$. To see that $L=R$, let $t \in L$. Choose any $t_0 \in R$. Then $(e_tf_1, e_tf_{t_0}) \in I_tA \subseteq \omega$ as $t \in L$; also $(e_tf_{t_0}, e_{tt_0}f_{t_0}) \in I_{t_0}B \subseteq \omega$ and $(e_{tt_0}f_{t_0}, e_{tt_0}f_1) \in I_tA \subseteq \omega$. So $(e_tf_1, e_{tt_0}f_1) \in \omega$. By 3.3 we conclude that $I_tB \subseteq \omega$; i.e. $t \in R$. Thus $L \subseteq R$. Similarly $R \subseteq L$. So $L=R$.

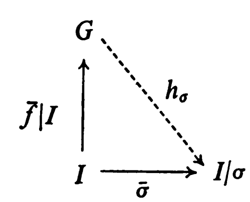
Note that $I_L E_P^2 \subseteq \omega$ since $I_L A \subseteq \omega$ and $I_R B \subseteq \omega$, and $I_L A \vee I_L B = I_L E_P^2$. Now suppose $e_x f_y \, \omega \, e_r f_s$. If $x=r$ and $y=s$, then $(e_x f_y, e_r f_s) \in \Delta E_P^2 \subseteq I_L E_P^2$. Without loss of generality assume $x \neq r$, say $x > r$. If $y=s$, then $(e_x f_y, e_r f_y) \in \omega$, so $I_{ry} B \subseteq \omega$. Thus $I_{ry} B \subseteq I_L E_P^2$, so $(e_x f_y, e_r f_s) = (e_x f_y, e_r f_y) \in I_L E_P^2$. Similarly for the case $x < r$. A similar argument shows if $x=r$ and $y \neq s$, then $(e_x f_y, e_r f_s) \in I_L E_P^2$. Now if $x \neq r$ and $y \neq s$, w.l.o.g. assume $x > r$. Then $e_x f_y \, \omega \, e_x f_s$, and hence $e_x f_s \, \omega \, e_r f_s$. By 3.3, this implies $I_{x(y \wedge s)} A \subseteq \omega$ and $I_{rs} B \subseteq \omega$, so $I_x(y \wedge s) A$ and $I_{rs} B \subseteq I_L E_P^2$. Therefore, $e_x f_y (I_L E_P^2) e_x f_s (I_L E_P^2) e_r f_s$, so $(e_x f_y, e_r f_s) \in I_L E_P^2$, and $\omega \subseteq I_L E_P^2$. This completes the proof.

Now that we have determined the lattice of normal congruences on E_P (and hence the lattice $\Lambda(F_P)/\theta$), we concentrate on determining each θ-class of $\Lambda(F_P)$. If ω is a normal congruence on E_P then the θ-class belonging to ω is the set of all congruences $\sigma \in \Lambda(F_P)$ such that $\sigma \cap E_P^2 = \omega$.

Let I be an arbitrary ideal of F_P. In the next three theorems we shall determine the θ-class belonging to IE_P^2. Let f denote the inclusion map of P into G and let $\bar{f}$ denote the extension of f to F_P. Note that $\bar{f}(xy^{-1}z) = xz/y$, and that $\bar{f}|I$ is onto G.

3.5. THEOREM. *A congruence σ on I is a group congruence if and only if there is a subgroup N of G such that for each $u, v \in I$ (with canonical representations $u = xy^{-1}z$, $v = rs^{-1}t$), $u \, \sigma \, v$ if and only if $xzs/rty \in N$.*

Proof. Let σ be a group congruence on I, and consider the following diagram:



$h_\sigma \bar{f} = \bar{\sigma}$

In order to check that the homomorphism h_σ exists, we note that if $\bar{f}|I(xy^{-1}z) = \bar{f}|I(rs^{-1}t)$, then $xzs = rty$. Hence $\bar{\sigma}(xy^{-1}z) = \bar{\sigma}(rs^{-1}t)$. Since $\bar{f}|I$ is onto, there is a unique homomorphism induced which we call h_σ. Now let $N = \ker h_\sigma$ and note that $xy^{-1}z \, \sigma \, rs^{-1}t$ if and only if $\bar{\sigma}(xy^{-1}z) = \bar{\sigma}(rs^{-1}t)$ if and only if $h_\sigma \bar{f}(xy^{-1}z) = h_\sigma \bar{f}(rs^{-1}t)$ if and only if $h_\sigma(xz/y) = h_\sigma(rt/s)$ if and only if $xz/y \div rt/s = xzs/rty \in \ker h_\sigma = N$.

Conversely suppose N is a subgroup of G. Let σ_N be the relation on I defined by $xy^{-1}z \, \sigma_N \, rs^{-1}t$ if and only if $xzs/rty \in N$, where $y \geqq x, z$ and $s \geqq r, t$ and $xy^{-1}z$, $rs^{-1}t \in I$. It is readily checked that σ_N is a congruence on I using the fact that N is a group.

To see that σ_N is a group congruence we need only show I/σ_N has only one idempotent. So let e, f be idempotents in I. Then by 2.10, $e = x(xz)^{-1}z$ and $f = r(rt)^{-1}t$ for some x, z, r, and t in P. Since $xz(rt)/rt(xz) = 1 \in N$ we have that $e \, \sigma_N \, f$. Thus I/σ_N is a group.

3.6. THEOREM. *The correspondences* $\sigma \to \ker h_\sigma$ *and* $N \to \sigma_N$ *described in* 3.1 *between the lattice of group congruences on I and the lattice of subgroups of G are mutually inversive lattice isomorphisms.*

Proof. Let σ be a group congruence on I, and let $\delta = \sigma_{\ker h_\sigma}$. Now as in 3.5 $xy^{-1}z \;\sigma\; rs^{-1}t$ if and only if $xzs/rty \in \ker h_\sigma$. But from the definition of δ, $xy^{-1}z \;\delta\; rs^{-1}t$ if and only if $xzs/rty \in \ker h_\sigma$. Hence $\sigma_{\ker h_\sigma} = \sigma$. On the other hand, let N be a subgroup of G. Let $u, v \in I$ with canonical representations $u = xy^{-1}z$ and $v = rs^{-1}t$. Now $u \;\sigma_N\; v$ if and only if $xzs/rty \in N$. Also using the induced homomorphism h_{σ_N}, $u \;\sigma_N\; v$ if and only if $xzs/rty \in \ker h_{\sigma_N}$. Hence $N = \ker h_{\sigma_N}$. Hence the correspondences are mutually inversive functions. To complete the proof we need only show that the correspondence $N \to \sigma_N$ is a lattice homomorphism.

Let N and M be subgroups of G. It will suffice to show that $N \subseteq M$ if and only if $\sigma_N \subseteq \sigma_M$. Now it is clear that $N \subseteq M$ implies $\sigma_N \subseteq \sigma_M$. Conversely if $\sigma_N \subseteq \sigma_M$ let x be in N with $x = y/z$ such that $y, z \in P$. Then $(1, y, 1) \;\sigma_N\; (1, z, 1)$ implies $(1, y, 1) \;\sigma_M\; (1, z, 1)$. Thus $x \in M$ and $N \subseteq M$. This completes the proof of 3.6.

3.7. THEOREM. *The* θ*-class belonging to the normal congruence* IE_P^2 *is isomorphic with the lattice of subgroups of G under the correspondence* $N \to \sigma_N \cup \Delta F_P^2$.

Proof. Let Γ denote the θ-class belonging to IE_P^2, Ω the lattice of subgroups of G, and Δ the lattice of group congruences on I. By 3.6 the function from Ω onto Δ taking N to σ_N is a lattice isomorphism. Hence we only need show that the function from Δ to Γ taking δ to $\delta \cup \Delta F_P^2$ is a 1-1 onto lattice isomorphism.

To see that this function is 1-1 and onto, let $\delta \cup \Delta F_P^2 = \delta'$ for $\delta \in \Delta$ and $\rho \cap I^2 = \rho^*$ for $\rho \in \Gamma$. Clearly $\delta' \in \Gamma$ and $\rho^* \in \Delta$. Also one sees without difficulty that $(\delta')^* = \delta$, for $\delta \in \Delta$. On the other hand if $\rho \in \Gamma$, then to show that $(\rho^*)' = \rho$ we need only show that whenever $u, v \in F_P$ with $u \neq v$ and $u \;\rho\; v$ then $u, v \in I$. We consider two cases: (1) If $u \notin I$, $v \in I$, then $uu^{-1} \notin I$ and $vv^{-1} \in I$. Also $uu^{-1} \;\rho\; vv^{-1}$. However this is impossible since $\rho \cap E_P^2 = IE_P^2$. (2) If $u \notin I$, $v \notin I$, then $uu^{-1}, vv^{-1}, u^{-1}u, v^{-1}v \notin I$; but $uu^{-1} \;\rho\; vv^{-1}$, so $uu^{-1} = vv^{-1}$ since $\rho \cap E_P^2 = IE_P^2$. Similarly $u^{-1}u = v^{-1}v$. However this implies that u and v are $\mathscr{H}$ related and so by 2.14 we conclude that $u = v$, a contradiction. This shows that $(\rho^*)' = \rho$. Hence the functions $\delta \to \delta'$ and $\rho \to \rho^*$ are mutually inversive functions; and thus $\sigma_N \to \sigma_N \cup \Delta F_P^2$ is a 1-1 onto function.

To see that it is a lattice isomorphism, let $\delta, \sigma \in \Delta$. Then $\delta \vee \sigma = \delta \circ \sigma$, since $\delta \circ \sigma = \sigma \circ \delta$. Also $\delta' \vee \sigma' = \delta' \circ \sigma'$ according to [4]. So $(\delta \vee \sigma)' = (\delta \circ \sigma) \cup \Delta F_P^2$, and $\delta' \vee \sigma' = (\delta \cup \Delta F_P^2) \circ (\sigma \cup \Delta F_P^2)$. From this it follows that $(\delta \vee \sigma)' = \delta' \vee \sigma'$; hence $\sigma_N \to \sigma_N \cup \Delta F_P^2$ preserves $\vee$. Since the inverse of this function clearly preserves $\wedge$, we conclude that $\sigma_N \to \sigma_N \cup \Delta F_P^2$ is a lattice isomorphism.

3.8. COROLLARY. *For each subgroup N of G, let* σ^N *denote the relation on* F_P *defined by* $u \;\sigma^N\; v$ *if and only if* $u = v$, *or* $u, v \in I$ *and* $xzs/rty \in N$, *where* $xy^{-1}z$ *and* $rs^{-1}t$ *are the canonical representations of u and v respectively. Then* σ^N *is a member* *of the θ-class belonging to IE_P^2. Furthermore if M is a subgroup of G then $\sigma^N \vee \sigma^M = \sigma^{NM}$ and $\sigma^N \cap \sigma^M = \sigma^{N \cap M}$.*

Now we shall determine the θ-class belonging to IA and IB. It turns out that they are both degenerate. Let $g, h\colon P \to B_P$ be the homomorphisms given by $g(x) = (x, 1)$ and $h(x) = (1, x)$. Let $\bar{g}, \bar{h}\colon F_P \to B_P$ denote the extensions of g and h, and let α, β be the congruences on F_P determined by $\bar{g}, \bar{h}$ respectively. Note that $u \,\alpha\, v (u \,\beta\, v)$ if and only if $x=r$ and $yt=sz$ ($z=t$ and $yr=sx$) where $xy^{-1}z$ and $rs^{-1}t$ are the canonical representations of u and v. Let $I\alpha = (\alpha \cap I^2) \cup \Delta F_P^2$ ($I\beta = (\beta \cap I^2) \cup \Delta F_P^2$). It is readily checked that $I\alpha$ ($I\beta$) is a congruence on F_P lying in the θ-class belonging to IA (IB).

3.9. Theorem. *The θ-class belonging to IA (IB) has $I\alpha$ ($I\beta$) as its only member.*

Proof. Let Γ denote the θ-class belonging to IA, and let ρ and σ denote the largest and smallest elements of Γ respectively. It follows from Theorem 4.2 of [4] that for $u, v \in F_P$ with canonical representations $xy^{-1}z$ and $rs^{-1}t$ respectively that $u \,\sigma\, v$ if and only if $uu^{-1} \,(IA)\, vv^{-1}$ and $eu = ev$ for some $e \in E_P$ such that $e \,IA\, uu^{-1}$. To prove the theorem we need only show that $u \,\rho\, v$ implies $u \,\sigma\, v$. So suppose $u \,\rho\, v$. Then $u^{-1} \,\rho\, v^{-1}$ so $uu^{-1} \,\rho\, vv^{-1}$. Thus $e_x f_{y/x} = uu^{-1} \,(IA)\, vv^{-1} = e_r f_{s/r}$ and so $x = r$. Also $e_{y/z} f_z = u^{-1}u \,(IA)\, v^{-1}v = e_{s/t} f_t$ and so $yt = sz$. Now let $e = e_x f_{sy}$ and note that $eu = ev$ and $e \,IA\, uu^{-1}$. Hence $u \,\sigma\, v$, and we conclude that $\sigma = \rho = I\alpha$. The proof that the θ-class belonging to IB contains only $I\beta$ is analogous.

The following corollary sums up the information contained in 3.7 and 3.9. For an arbitrary ideal I of F_P and an arbitrary congruence σ on F_P, let $I\sigma$ denote the congruence $(\sigma \cap I^2) \cup \Delta F_P^2$ on F_P. The *top* of $\Lambda(F_P)$, T, is the set of group congruences on F_P together with the two congruences α and β.

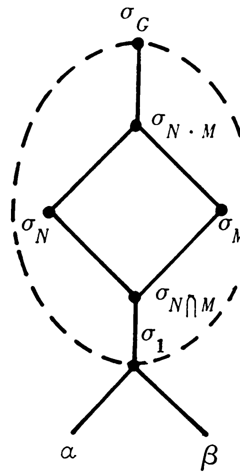


Figure 4

3.10. Corollary. *Every nonzero congruence* σ *on* F_P *can be written uniquely in the form* $I\delta$ *for some* $\delta \in T$ *and some ideal* I *of* F_P. *Furthermore for ideals* I *and* J *of* F_P *and* γ *and* δ *in* T, $I\gamma \subset J\delta$ *if and only if* $I \subset J$ *and* $\gamma \subset \delta$.

3.11. Remark. If we consider F_P with the topology described in 2.10, then it is natural to ask what the closed congruences on F_P are. It is not hard to see that 1, 0, α and β are closed. Also the group congruence σ_N is closed if and only if N is cyclic, and if I is an ideal of F_P and $\sigma \in T$ then $I\sigma$ is closed if and only if I is closed and σ is closed.

Several additional pieces of information can be obtained from the preceding theorems. We state them below.

3.12. Corollary. $\Lambda(F_P)$ *is a nonmodular lattice.*

3.13. Corollary. *All one-parameter inverse semigroups except those of the form* F_P *have a kernel* (*i.e. minimal ideal*). *In particular, if* I *is an ideal of* F_P *then* $F_P/I\alpha$ *and* $F_P/I\beta$ *have a kernel isomorphic with* B_P *and* $F_P/I\sigma_N$ *has a kernel isomorphic with* G/N.

3.14. Corollary. *The lattice of congruences on* F_{P_0} *is isomorphic with the complement of the top of* $\Lambda(F_P)$ *under the mapping* $\sigma \to \sigma \cup \{(1, 1)\}$.

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University of Kentucky, Lexington, Kentucky 40506