TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 168, June 1972

THE CONNECTEDNESS OF THE COLLECTION OF ARC CLUSTER SETS

BY
PETER LAPPAN[1]

Abstract. Let f be a continuous complex-valued function defined on the unit disk and let p be a boundary point of the disk. A very natural topology on the collection of all arc cluster sets of f at the point p has been investigated by Belna and Lappan [1] who proved that this collection is a compact set under certain suitable conditions. It is proved here that this collection is an arcwise connected set under the topology in question, but is not in general locally arcwise connected or even locally connected. It is also shown by example that it is generally not possible to map the real line onto the collection of arc cluster sets at p in a continuous manner.

1. Let D denote the open unit disk, let Γ denote the unit circle, and for $p \in \Gamma$, let $\mathfrak{T}(p)$ denote the collection of all Jordan arcs in $D \cup \{p\}$ with one endpoint in D and the other at p. If Q is a subset of $\bar{D}$ and if f is a complex-valued function defined on D, we define the *cluster set of f at p relative to Q* by

$$C_Q(f, p) = \bigcap_{r>0} \mathrm{Cl}\,(f(Q \cap N(r, p))),$$

where $N(r, p) = \{z \in D : |z-p| < r\}$ and the closure is taken relative to the Riemann sphere W. Thus each arc $t \in \mathfrak{T}(p)$ determines a nonempty closed set $C_t(f, p)$. Let $\mathfrak{C}_f(p) = \{C_t(f, p) : t \in \mathfrak{T}(p)\}$ *be called the collection of all arc cluster sets of f at p*. Thus $\mathfrak{C}_f(p)$ is a set for which each element $C_t(f, p)$ is itself a closed subset of W. If f is a continuous function it is well known that $C_t(f, p)$ is a connected subset of W for each $t \in \mathfrak{T}(p)$.

Let z and z^1 be two points of the Riemann sphere W and let $d(z, z^1)$ denote the chordal distance between z and z^1. If A is a subset of W, let

$$d(z, A) = \inf\{d(z, a); a \in A\}$$

denote the distance between the point z and the set A. If A and B are subsets of W, define the M-distance between A and B by

$$M(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}.$$

The M-distance defines a Hausdorff topology on the set of all closed subsets of W [2], and hence defines a Hausdorff topology on $\mathfrak{C}_f(p)$ for each $p \in \Gamma$ and each complex-valued function f.

Received by the editors April 15, 1971 and, in revised form, November 17, 1971.

AMS 1969 *subject classifications.* Primary 3062; Secondary 5455.

Key words and phrases. Cluster set, arc cluster set, arcwise connectedness.

[1] The author acknowledges partial support from the National Science Foundation under grant GP 11825.

Lappan [3] has proved that if f is a continuous function in D then, for each $p \in \Gamma$, $\mathfrak{C}_f(p)$ is either a singleton or else has continuum many elements. Belna and Lappan [1] have proved that if f is a continuous function in D and if $p \in C$ is not an ambiguous point in the sense of Bagemihl then $\mathfrak{C}_f(p)$ is a compact set in the M-topology. In particular, this means that if f is continuous then $\mathfrak{C}_f(p)$ is compact for all but a countable set of points $p \in \Gamma$. In this paper we will consider connectedness properties for the set $\mathfrak{C}_f(p)$, where f is a continuous function.

2. We begin by defining a concept especially designed for the proofs to follow.

DEFINITION. *If $p \in \Gamma$ and if $t \in \mathfrak{T}(p)$, a set E is said to be a tree with trunk t if $E = t \cup \bigcup_{n=1}^{\infty} s_n$, where the sets s_n satisfy all of the following conditions*:

(1) *For each n, either $s_n = \varnothing$ or else s_n is a Jordan arc in D having its initial endpoint p_n on t, but no other point of t lies in s_n. (The Jordan arc s_n need not include its other endpoint, although, of course, it may include this endpoint.)*

(2) *$s_n \cap s_k = \varnothing$ for $k \neq n$, and $\bar{s}_n \cap \{p\} = \varnothing$ for each n.*

(3) *For each n, there exists a real number r_n with $0 < r_n < 1$ and $\lim_{n\to\infty} r_n = 1$ such that $s_n \subset \{z \in D : |z| > r_n\}$.*

(4) *$\bar{E} \cap \Gamma = \{p\}$.*

We remark that the arc t itself is a tree under this definition.

We shall have occasion to refer to what we call a double tree. Let $p \in \Gamma$ and let t_0 and t_1 be elements of $\mathfrak{T}(p)$ such that $t_0 \cap t_1 \cap D = \varnothing$. It is no loss of generality to assume that both t_0 and t_1 have initial points on the circle $\{z : |z| = \frac{1}{2}\}$, and that all other points of $t_0 \cup t_1$ are contained in $\{z : |z| > \frac{1}{2}\}$. Let s_1' be a subarc of the circle $\{z : |z| = \frac{1}{2}\}$ joining the endpoints of t_0 and t_1 such that $t_0 \cup t_1 \cup s_1'$ is the boundary of a bounded region Δ. For each $n > 1$, let s_n' be a subarc of the circle $\{z : |z| = n/(n+1)\}$ which meets both t_0 and t_1 and also disconnects the region Δ in such a way that the component of $\Delta - s_n'$ having p as a boundary point does not contain s_{n-1}'. Let p_n be the endpoint of s_n' on t_0, let q_n be the endpoint of s_n' on t_1, let u_n be the portion of t_0 between p_n and p_{n+1}, and let v_n be the portion of t_1 between q_n and q_{n+1}, where u_n includes the point p_n but not the point p_{n+1} and v_n includes the point q_n but not the point q_{n+1}. Setting $s_n = s_n' \cup v_n$, we have that $E = t_0 \cup t_1 \cup \bigcup_{n=1}^{\infty} s_n' = t_0 \cup \bigcup_{n=1}^{\infty} s_n$ is a tree with trunk t_0. However, if we set $s_n'' = s_n' \cup u_n$, we have that the same point set E can be expressed as $E = t_1 \cup \bigcup_{n=1}^{\infty} s_n''$, so that E may also be considered as a tree with trunk t_1. We will refer to this point set E as the *double tree on t_0 and t_1*.

Each tree E with trunk t in $\mathfrak{T}(p)$ determines a cluster set $C_E(f, p)$ for a given function f. To simplify notation, we will use CE in place of $C_E(f, p)$ whenever no ambiguity will result.

LEMMA 1. *If $p \in \Gamma$, $t \in \mathfrak{T}(p)$, if E is a tree with trunk t, and if f is a continuous function in D, then $CE \in \mathfrak{C}_f(p)$.*

Proof. Let $E = t \cup \bigcup_{n=1}^{\infty} s_n$, where the nonempty sets s_n are listed in the order in which their initial points p_n appear on t_0. Let μ_n denote the portion of t_0 between

p_n and p_{n+1}, where μ_n includes the point p_n but not the point p_{n+1}. Then $s_n \cup \mu_n$ is a Jordan arc (possibly not including either of its endpoints). There exists a Jordan arc t_n which has its initial point at p_n, its terminal point at p_{n+1}, and approximates $s_n \cup \mu_n$ in such a manner that both

$$(5) \qquad \sup\{d(z, s_n \cup \mu_n) : z \in t_n\} + \sup\{d(z, t_n) : z \in s_n \cup \mu_n\} < 1/n$$

and

$$(6) \quad \sup\{d(f(z), f(s_n \cup \mu_n)) : z \in t_n\} + \sup\{d(f(z), f(t_n)) : z \in s_n \cup \mu_n\} < 1/n.$$

Furthermore, the arcs t_n can be chosen to satisfy (5) and (6) and such that $t_n \cap t_{n+1} = \{p_{n+1}\}$ and $t_n \cap t_j = \varnothing$ for $j \geqq n+2$. Setting $t = \bigcup_{n=1}^{\infty} t_n \cup \{p\}$, we have that $t \in T(p)$ and $C_t(f, p) = CE \in \mathfrak{C}_f(p)$.

LEMMA 2. *Let $p \in \Gamma$, $t \in \mathfrak{T}(p)$, and let E and E' be two trees with the same trunks t such that $E' \subset E$. Let f be a continuous function in D such that $CE' \neq CE$. Given r, $0 < r < 1$, there exists a tree E'' with trunk t satisfying $E' \subset E'' \subset E$ and $M(CE', CE'') = rM(CE', CE)$.*

Proof. We first note that $CE' \subset CE$, and if $E' \subset E'' \subset E$, then $CE' \subset CE'' \subset CE$. Let r be given, $0 < r < 1$, and let $G = \{z \in D : d(f(z), CE') < rM(CE', CE)\}$. If $E' = t \cup \bigcup_{n=1}^{\infty} s'_n$ and $E = t \cup \bigcup_{n=1}^{\infty} s_n$, then we may suppose that $s'_n \subset s_n$ for each n, since the points of E can be redistributed among the arcs s_n, if necessary, and the arcs s'_n can be re-indexed as necessary without changing either tree as a point set. There exists a component G' of G and an integer N such that $s'_n \subset G'$ for $n \geqq N$. For $n \geqq N$, let s''_n be the component of $s_n \cap G'$ containing s'_n, and for $n < N$ set $s''_n = s'_n$. Then, setting $E'' = t \cup \bigcup_{n=1}^{\infty} s''_n$, we have $E' \subset E'' \subset E$ and also $M(CE', CE'') = rM(CE', CE)$, so that E'' is the desired tree.

LEMMA 3. *Let f be a continuous function in D, let $p \in \Gamma$, let $t \in \mathfrak{T}(p)$, and let E be a tree with trunk t. For each $\xi > 0$ there exists a finite number of trees E_n, $n = 0, 1, 2, \ldots, m(\xi)$, each with trunk t such that*

$$t = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{m(\xi)} = E$$

and $M(CE_n, CE_{n+1}) < \xi$ for $0 \leqq n < m(\xi)$.

Proof. Let $\xi > 0$ be given and let j be a positive integer such that $j > 1/\xi$. Set $E_0 = t$. If $M(CE_0, CE) \leqq 1/j$, then set $E = E_1$ and we are finished. If $M(CE_0, CE) > 1/j$, then by a repeated application of Lemma 2 we can construct a sequence $\{E_n\}$ of trees, each with trunk t, such that $E_n \subset E_{n+1}$ and $M(CE_n, CE_{n+1}) = 1/j$ for each n, where we continue the construction as far as possible. Since CE_n is a subset of the Riemann sphere W, we have that CE_{n+1} contains a point of W a distance $1/j$ from the set CE_n. Since W is compact, the construction of the trees E_n must terminate after a finite number of steps.

This completes the proof of the lemma.

LEMMA 4. *Let f be a continuous function in D, let $p \in \Gamma$, let $t \in \mathfrak{T}(p)$ and let E be a tree with trunk t. There exists a countable collection of trees $\{E_n : n=0, 1, 2, \ldots\}$ each with trunk t, with $E_0=t$ and $E_1=E$ and such that there exists a sequence $\{a_n\}$ of real numbers dense in the interval* $[0, 1]$ *with $a_0=0$ and $a_1=1$ and such that*

(7) *if $a_n < a_m$ then E_n is a subset of E_m;*

(8) *for each $\xi > 0$ there exist integers $j > 0$, $k > 1$ such that $M(CE_0, CE_j) < \xi$ and $M(CE_k, CE_1) < \xi$; and*

(9) *for each real number x, $0 < x < 1$, and each $\xi > 0$ there exist integers q and s such that $a_q < x < a_s$ and $M(CE_q, CE_s) < \xi$.*

Proof. By Lemma 3, we can choose a finite collection $\mathfrak{S}_1$ of trees, each with trunk t, such that $\mathfrak{S}_1 = \{E_{1,j} : 0 \leqq j \leqq m_1\}$ with $t = E_{1,0}$, $E = E_{1,m_1}$, and for each j, $0 \leqq j < m_1$, both $E_{1,j} \subset E_{1,j+1}$ and $M(CE_{1,j}, CE_{1,j+1}) \leqq \frac{1}{2}$ are satisfied, where the equality holds for all j with the possible exception of $j = m_1 - 1$. By repeated application of Lemma 3, we can find a sequence $\{\mathfrak{S}_n\}$ of finite collections of trees each with trunk t satisfying $\mathfrak{S}_n \subset \mathfrak{S}_{n+1}$, $\mathfrak{S}_n = \{E_{n,j} : 0 \leqq j \leqq m_n\}$, $E_{n,0} = t$, $E_{n,m_n} = E$, $E_{n,j} \subset E_{n,j+1}$, and $M(CE_{n,j}, CE_{n,j+1}) \leqq 1/(n+1)$ for each n and each j, $0 \leqq j < m_n$, where the equality holds for all n and j with the possible exceptions of those n and j for which $E_{n,j+1} \in \mathfrak{S}_{n-1}$.

We now construct the sequence $\{a_n\}$. Let $a_0 = 0$, $a_1 = 1$, and choose $a_2, a_3, \ldots, a_{m_1}$ to be the points $a_{n+1} = n/m_1$. For $2 \leqq j \leqq m_1$, assign a_j to the tree $E_{1,j-1}$. Proceeding inductively, we assume that to each tree in $\mathfrak{S}_n$ we have assigned a unique number a_j with the property that each interval in $[0, 1]$ of length $1/n$ contains at least one of the numbers $a_0, a_1, \ldots, a_{m_n}$, and that the order of size of the numbers a_j corresponds to the order under set inclusion of the corresponding trees of $\mathfrak{S}_n$. We can then assign to each of the $(m_{n+1} - m_n)$ trees in $\mathfrak{S}_{n+1} - \mathfrak{S}_n$ a number a_j, where the number a_j is assigned in such a way that the order of size of the numbers a_j, $0 \leqq j \leqq m_{n+1}$, corresponds to the order under set inclusion of the corresponding trees in $\mathfrak{S}_{n+1}$, and such that if k new numbers are assigned to an interval I determined by adjacent numbers corresponding to trees in $\mathfrak{S}_n$, then the new numbers will divide the interval I into $(k+1)$ intervals of equal length. By this construction, the set $\{a_n : n=0, 1, 2, \ldots\}$ is a dense subset of the interval $[0, 1]$. We may now list the trees in $\bigcup \mathfrak{S}_n$ as $E_0, E_1, E_2, \ldots$ in such a way that (7), (8), and (9) are satisfied.

3. We are now in a position to state and prove some results concerning the connectedness of $\mathfrak{C}_f(p)$.

THEOREM 1. *Let f be a continuous function in D, let $p \in \Gamma$, let $t \in \mathfrak{T}(p)$, and let E be a tree with trunk t. There exists a continuous function π_E from* $[0, 1]$ *into $\mathfrak{C}_f(p)$ such that $\pi_E(0) = C_t(f, p)$ and $\pi_E(1) = CE$.*

Proof. Let $\{a_n\}$ and $\{E_n\}$ be as described in Lemma 4. Define $\pi_E(x) = \bigcap_{a_n \geqq x} CE_n$

for each $x \in [0, 1]$, and let $E(x)=\bigcap_{a_n \geqq x} E_n$. For $0<x<1$ and $\xi>0$, by (7) and (9) there exist numbers a_q and a_s such that $a_q<x<a_s$, $E_q \subset E(x) \subset E_s$, $CE_q \subset CE(x) \subset CE_s$, and $M(CE_q, CE_s)<\xi$. It follows that $CE(x)=\pi_E(x)$ so that $\pi_E(x) \in \mathfrak{C}_f(p)$ by Lemma 1 and π_E is continuous on the open interval (0, 1). By using (8) in place of (9), we obtain in a similar manner that π_E is continuous at $x=0$ and at $x=1$. Thus π_E is the desired function.

THEOREM 2. *If f is a continuous function in D and if $p \in \Gamma$, then $\mathfrak{C}_f(p)$ is arcwise connected in the M-topology.*

Proof. We will show that for each pair of arcs $t_1, t_2 \in \mathfrak{T}(p)$ there is a continuous function π from [0, 1] into $\mathfrak{C}_f(p)$ such that $\pi(0)=C_{t_1}(f, p)$ and $\pi(1)=C_{t_2}(f, p)$.

If $C_{t_1}(f, p)=C_{t_2}(f, p)$, then we can define $\pi(x)=C_{t_1}(f, p)$ for $0 \leqq x \leqq 1$ and we are finished.

If $C_{t_1}(f, p) \neq C_{t_2}(f, p)$ and if $t_1 \cap t_2=\{p\}$, then we may assume that t_1 and t_2 both originate on the circle $\{z : |z|=\frac{1}{2}\}$. Let E be the double tree on t_1 and t_2 and consider E first as a tree with trunk t_1. By Theorem 1 there exists a continuous function π_E from [0, 1] into $\mathfrak{C}_f(p)$ such that $\pi_E(0)=C_{t_1}(f, p)$ and $\pi_E(1)=CE$. Now let $E'=E$, where E' is considered as a tree with trunk t_2. Again, by Theorem 1, there exists a continuous function $\pi_{E'}$ from [0, 1] into $\mathfrak{C}_f(p)$ such that $\pi_{E'}(0)=C_{t_2}(f, p)$ and $\pi_{E'}(1)=CE'$. The function $\pi(x)$, defined such that $\pi(x)=\pi_E(2x)$ for $0 \leqq x \leqq \frac{1}{2}$ and $\pi(x)=\pi_{E'}(2-2x)$ for $\frac{1}{2} \leqq x \leqq 1$, is a continuous function from [0, 1] into $\mathfrak{C}_f(p)$ with $\pi(0)=C_{t_1}(f, p)$ and $\pi(1)=C_{t_2}(f, p)$, so that π is the desired function in this case.

Finally, if $t_1 \cap t_2 \cap D \neq \varnothing$ then there exists an arc $t_3 \in \mathfrak{T}(p)$ such that $t_1 \cap t_3=\{p\}=t_2 \cap t_3$. By the argument just completed there exist continuous functions π_1 and π_2 from [0, 1] into $\mathfrak{C}_f(p)$ such that $\pi_1(0)=C_{t_1}(f, p)$, $\pi_1(1)=C_{t_3}(f, p)$, $\pi_2(0)=C_{t_3}(f, p)$, and $\pi_2(1)=C_{t_2}(f, p)$. We obtain the desired function π by taking $\pi(x)=\pi_1(2x)$ for $0 \leqq x \leqq \frac{1}{2}$, and $\pi(x)=\pi_2(2x-1)$ for $\frac{1}{2} \leqq x \leqq 1$. This completes the proof of the theorem.

REMARK. The proof of Theorem 2 involves a reasonably smooth shift of trees with a fixed trunk with one "discontinuity" by considering the same point set as different trees with different trunks. The elements of $\mathfrak{T}(p)$ corresponding to these two trees are not at all close together in any intuitive sense. It would seem reasonable that an arc in $\mathfrak{C}_f(p)$, i.e. the continuous function π in Theorem 2, would describe a smooth shift in elements of $\mathfrak{T}(p)$, without a "discontinuity" of the sort mentioned. A proof of Theorem 2 involving such a smooth shift, without any "discontinuity" of the type mentioned, would be highly desirable.

Theorem 2 shows that $\mathfrak{C}_f(p)$ is an arcwise connected set, and hence a connected set whenever f is continuous. However, it is not true that $\mathfrak{C}_f(p)$ need be locally connected, much less locally arcwise connected, where f is continuous.

THEOREM 3. *There exists a function f continuous in D for which $\mathfrak{C}_f(1)$ is not locally connected.*

Proof. Let A_n be the chord of D given by

$$A_n = \{z \in D : \arg(z-1) = \pi/2+1/n\},$$

let $z_n=(1/n)e^{i/n}$ and let $w_n=e^{i/n}$, $n=1, 2, 3, \ldots$. Define f on the set $\{z : \arg(z-1) \geqq \pi\}$ by $f(z)=0$ there. For all odd n, define $f(z)=z_n$ for $z \in A_n$, while for all even n define $f(z)=w_n$ for $z \in A_n$. For these z satisfying $\pi/2+1/(n+1)<\arg(z-1)<\pi/2+1/n$, define $f(z)=ta_n+(1-t)a_{n+1}$ for $\arg(z-1)=\pi/2+1/(n+1)+t/n(n+1)$, $0<t<1$, $n=1, 2, \ldots$, where a_n and a_{n+1} are the values assumed by $f(z)$ on A_n and A_{n+1}, respectively. Finally, for z such that $\pi/2+1<\arg(z-1)<\pi$, define

$$f(z) = (2e^i/(\pi-2))(\pi-\arg(z-1)).$$

The function f is now defined as continuous in D.

The value 0 is an asymptotic value of f at 1 so $\{0\} \in \mathfrak{C}_f(1)$. If ξ is given such that $0<\xi<\frac{1}{2}$, then there exist points z_{n_1} and z_{n_2}, n_1 and n_2 both odd, such that $0<d(0, z_{n_1})<\xi$, $0<d(0, z_{n_2})<\xi$. But the points z_{n_1} and z_{n_2} are located in different components of $\{z : |z|<\xi\} \cap C_D(f, 1)$. It follows that $\mathfrak{C}_f(1)$ is not locally connected.

4. In light of Theorem 2, it would be interesting to be able to list elements of $\mathfrak{C}_f(p)$ in a systematic (i.e., continuous) manner. The following result shows that this is not possible.

THEOREM 4. *There exists a continuous function f in D for which there is no continuous function from the real line onto $\mathfrak{C}_f(1)$.*

Proof. Let P denote the Cantor ternary set on $[0, 1]$, let $\{I_n\}$ be the open intervals removed from $[0, 1]$ to obtain P, let y_n be the midpoint of I_n. For each θ, $-\pi/2<\theta<\pi/2$, let $L(\theta)$ be the chord

$$L(\theta) = \{z \in D: \arg(z-1) = \pi+\theta\}.$$

For each $n=1, 2, 3, \ldots$, let

$A_n=\{z \in D : |z-1| \leqq 1/n, \arg(z-1)-\pi \in I_n\}$,

$B_n=\{z \in D : |z-1| \geqq 1/(n-1), \arg(z-1)-\pi \in I_n\}$, and

$C_n=\{z \in D : 1/n<|z-1|<1/(n-1), \arg(z-1)-\pi \in I_n\}$.

(Interpret $B_1=\varnothing$ and $C_1=\{z \in D : |z-1|>1, \arg(z-1) \in I_1\}$.) For each $x \in P$, let S_x be the line segment from x to the point $\frac{1}{2}+i$ and let $K=\bigcup \{S_x : x \in P\}$.

We begin to define a continuous function f on the unit disk by setting $f(z)=0$ for $z \in L(\theta)$, $\theta<0$, setting $f(z)=1$ for $z \in L(\theta)$, $\theta \geqq 1$, and setting $f(z)=\theta$ for $z \in L(\theta)$, $\theta \in P$. Then f is defined at each z except for those z on some $L(\theta)$, $\theta \in \bigcup I_n$. For $\theta \in I_n$ and $z \in L(\theta)$, we have $z \in A_n \cup B_n \cup C_n$. If $z \in B_n$, set $f(z)=\theta$ and if $z \in L(y_n) \cap A_n$, set $f(z)=\frac{1}{2}+i$. Let $x_{n,1}$ and $x_{n,2}$ be the right- and left-hand endpoints of I_n. If $\theta \in (x_{n,1}, y_n)$ and $z \in L(\theta) \cap A_n$, and if $\theta=tx_{n,1}+(1-t)y_n$, $0<t<1$, set $f(z)=tx_{n,1}+(1-t)(\frac{1}{2}+i)$, while if $\theta \in (y_n, x_{n,2})$ and $z \in L(\theta) \cap A_n$, and if $\theta=sx_{n,2}+(1-s)y_n$, $0<s<1$, set $f(z)=sx_{n,2}+(1-s)(\frac{1}{2}+i)$. Finally if $\theta \in I_n$, and $z \in L(\theta) \cap C_n$, let $a=L(\theta) \cap \{z : |z-1|=1/n\}$ and $b=L(\theta) \cap \{z : |z-1|=1/(n-1)\}$ and letting $z=ta+(1-t)b$, $0<t<1$, define $f(z)=tf(a)+(1-t)f(b)$. (For $z \in C_1$, take $a=L(\theta)=\{z : |z-1|=1\}$ and set $f(z)=f(a)$.) The function f is now defined on all of D and continuous there, $C_D(f, 1)=K$ and for each $x \in P$ we have $\{x\} \in \mathfrak{C}_f(1)$.

Suppose that π is a continuous function for the real line onto $\mathfrak{C}_f(1)$. For each $x \in P$, let t_x be an element of $\pi^{-1}(x)$. Since the set $P^1=\{t_x : x \in P\}$ is uncountable, there exists a point t^1 which is a limit point of the set P^1. Let $\{t_n\}$ be a monotone sequence of points in P^1 converging to t^1. For each s there must exist a point t_n^1 between t_n and t_{n+1} such that $\frac{1}{2}+i \in \pi(t_n^1)$ by the continuity of π. But $\pi(t^1)=\lim_{n\to\infty} \pi(t_n)$ is a subset of P, while $\pi(t^1)=\lim_{n\to\infty} \pi(t_n^1)$ is a continuum containing $\frac{1}{2}+i$. It follows that the continuity of the function π is untenable, so the theorem is proved.

Theorem 4 may be interpreted as saying that the topological space $\mathfrak{C}_f(1)$ is not a Peano continuum.

REMARK. Neither of the examples constructed in the proofs of Theorems 3 and 4 are meromorphic functions. It would be interesting if meromorphic functions satisfying these theorems could be found. The construction of meromorphic functions with the desired properties appears to the author to be very difficult.

5. In what has preceded, we have dealt only with the local situation at a single point p. We now make a brief consideration of the global situation.

Let f be a function mapping D conformally onto $D-S$, where S is the spiral $\{z=re^{i\theta} : r=\theta/(\theta+\pi),\ \theta \geqq \pi\}$. There exists a point $p_0 \in \Gamma$ such that for each $t \in T(p_0)$ we have $C_t(f, p_0)=\Gamma$, while for each point $p \in \Gamma-\{p_0\}$ the cluster set $C_D(f, p)$ is a singleton set. It follows that $\bigcup \{\mathfrak{C}_f(p) : p \in \Gamma\}$ is not a connected set, so that the most obvious attempt at an analogue to Theorem 2 in a global setting is not valid.

However, if we broaden the family of cluster sets under consideration we can find a valid global version of Theorem 2. Let $\mathfrak{T}$ be the family of all Jordan arcs $z(t)$, $0 \leqq t < 1$, in D for which $|z(t)| \to 1$ as $t \to 1$. Then $\mathfrak{T}$ includes arcs other than those with an endpoint on Γ. If f is a continuous function in D and $t \in \mathfrak{T}$, let

$$C_t(f) = \bigcap_{0<r<1} \mathrm{Cl}\,(f(t) \cap \{z : r<|z|<1\})$$

and let $\mathfrak{C}^*(f)=\{C_t(f) : t \in \mathfrak{T}\}$. Then Theorem 2 is valid with $\mathfrak{T}$ and $\mathfrak{C}^*(f)$ used in place of $\mathfrak{T}(p)$ and $\mathfrak{C}_f(p)$, respectively. To prove this we need only modify the definition of a tree by allowing the trunk to be in $\mathfrak{T}$ rather than $\mathfrak{T}(p)$ and by eliminating condition (4) from the definition. It is then easy to modify all of the proofs of the lemmas and Theorems 1 and 2 to obtain the result. It should be noted that Theorems 3 and 4 remain valid with $\mathfrak{T}$ and $\mathfrak{C}^*(f)$ in place of $\mathfrak{T}(p)$ and $\mathfrak{C}_f(p)$, respectively, because it is easy to modify each of the functions constructed so that $\mathfrak{C}_f(1)=\mathfrak{C}^*(f)$ for each function.

References

1. C. L. Belna and P. Lappan, *The compactness of the set of arc cluster sets*, Michigan Math. J. **16** (1969), 211–214. MR **9**, 7111.

2. C. Kuratowski, *Topologie*. I. *Espaces métrisables, espaces complets*, 2ième éd., Monografie Mat., Tom 20, PWN, Warsaw, 1948. MR **10**, 389.

3. P. Lappan, *Continua which are curvilinear cluster sets*, Nagoya Math. J. **34** (1969), 25–34. MR **39** #1660.

Department of Mathematics, Michigan State University, East Lansing, Michigan 48823