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COVERING RELATIONS IN THE LATTICE OF T_1-TOPOLOGIES

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Abstract. A topology $\mathscr{T}_1$ is said to cover another topology $\mathscr{T}_2$ if $\mathscr{T}_2 \subsetneq \mathscr{T}_1$ and no other topology may be included between the two. In this paper, we characterize the relationship between a T_1-topology and its covers. This characterization is used to prove that the lattice of T_1-topologies is both upper and lower semimodular. We also prove that the sublattice generated by the covers of a T_1-topology is isomorphic to the Boolean lattice of all subsets of the set of covers.

1. **Introduction.** Since 1936 when Garrett Birkhoff introduced both the lattice of topologies and the lattice of T_1-topologies [4], there appear to have been few results concerning the modularity of either lattice. In 1947, R. Vaidyanathaswamy remarked that the lattice of topologies is not distributive [10]. In 1954, Bagley proved that the lattice of T_1-topologies is not modular, and hence not distributive [2]. In 1966, Anne Steiner noted that the lattice of topologies on three or more elements is not modular [8]. O. Ore discussed some of these same properties in 1943 [7] in connection with the lattice of closure relations. Among his results is the discovery that the lattice of closure relations is lower semimodular.

In this paper, we investigate when a given T_1-topology has covers and under what conditions it covers other topologies. The results will be used to prove that the lattice of T_1-topologies is both upper and lower semimodular. We also give an example to show that the lattice of topologies is, in general, neither upper nor lower semimodular. In addition to this, we will extend a result of Bagley's [1] to show that the sublattice generated by covers of a T_1-topology is isomorphic to the Boolean lattice of all subsets of the set of covers.

Before proceeding, we note a few definitions and notations.

If $(L, \leqq)$ is a lattice and a and b are elements of L such that $a \neq b$, then b is said to *cover* a iff $a \leqq b$ and $a \leqq c \leqq b$ implies that $a = c$ or $c = b$.

Recalling that a lattice is modular iff it does not contain a sublattice of the form

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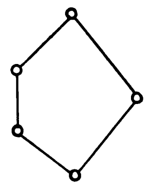


we say that a lattice $(L, \leqq)$ is *upper semimodular* iff for any two distinct elements a and b in L, which both cover a third element c, it follows that both a and b are covered by $a \vee b$.

A lattice L is called *lower semimodular* iff for any two distinct elements a and b in L, which are both covered by a third element c, it follows that a and b both cover $a \wedge b$.

If a lattice L has a least element 0, then any element of L which covers 0 is called an *atom*. If every element of L, except 0, can be written as the least upper bound of atoms, then L is called an *atomic lattice*. If L has a greatest element 1, then any element of L which is covered by 1 is called an *anti-atom* of the lattice. If L is such that every element in it, except 1, can be written as the greatest lower bound of anti-atoms, then it is called an *anti-atomic lattice*.

The lattice of all topologies on a set X we shall denote by $(\Sigma(X), \subseteq)$ or, if there is no danger of confusion, simply by $\Sigma(X)$. Similarly, $(\Lambda(X), \subseteq)$ or $\Lambda(X)$ shall be the lattice of all T_1-topologies on X.

The lattice $\Sigma(X)$ has as least element the indiscrete topology $[\phi, X]$ and as largest element the discrete topology $\mathscr{P}(X)$. The lattice $\Sigma(X)$ is atomic. The atoms are $[\phi, A, X]$, $\phi \neq A \subsetneq X$. That the lattice is also anti-atomic was shown by Fröhlich [6]. The anti-atoms, which in $\Sigma(X)$ will be called *ultratopologies*, are all topologies of the form $\mathscr{T}(x, \mathscr{U}) = \mathscr{P}(X \sim [x]) \cup \mathscr{U}$, where $\mathscr{U}$ is an ultrafilter, $\mathscr{U} \neq [A \mid x \in A \subseteq X]$.

In this paper, we will be concerned mainly with T_1-topologies. However, when a result can easily be established for arbitrary topologies, we shall do so.

2. **Covers.** If $\mathscr{T}$ is a topology on X and $G \subseteq X$, then we let $\mathscr{T}(G) = \mathscr{T} \vee [\phi, G, X]$. We can see that $\mathscr{T}(G) = [(A \cap G) \cup B \mid A, B \in \mathscr{T}]$. If $\mathscr{T}$ is a T_1-topology on X, we would like to know when $\mathscr{T}(G)$ covers $\mathscr{T}$ and when $\mathscr{T}$ covers $\mathscr{T} \cap \mathscr{T}(x, \mathscr{U})$ where $G \subseteq X$ and $\mathscr{T}(x, \mathscr{U})$ is an ultratopology on X. Our first two theorems give such characterizations.

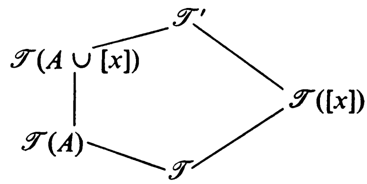
The proofs of the first two lemmas follow easily from the atomic and anti-atomic properties of $\Sigma(X)$ and are omitted.

LEMMA 1. *If $\mathscr{T}$ and $\mathscr{T}'$ are topologies on X, then $\mathscr{T}'$ covers $\mathscr{T}$ iff $\mathscr{T}' = \mathscr{T}(G)$ for every $G \in \mathscr{T}' \sim \mathscr{T}$.*

LEMMA 2. *If $\mathscr{T}'$ and $\mathscr{T}$ are topologies on X such that $\mathscr{T}'$ covers $\mathscr{T}$, then there exists an ultratopology $\mathscr{T}(x, \mathscr{U})$ such that $\mathscr{T} = \mathscr{T}' \cap \mathscr{T}(x, \mathscr{U})$.*

In Example 1, we show that $\mathscr{T}(G)$ need not cover $\mathscr{T}$ and that $\mathscr{T}$ need not cover $\mathscr{T} \cap \mathscr{T}(x, \mathscr{U})$ even if $\mathscr{T}$ is T_1.

EXAMPLE 1. Let X be an infinite set and let $A \subseteq X$ be such that A and $X \sim A$ are both infinite. Let $\mathscr{T}$ be the minimum T_1-topology on X, choose $x \notin A$, and let $\mathscr{T}' = \mathscr{T} \vee [\phi, [x], A, A \cup [x], X]$. The following diagram is valid:



We see that $\mathscr{T}(A \cup [x])$ does not cover $\mathscr{T}$. Further, it is easily verified that, if we take $\mathscr{U}$ to be a nonprincipal ultrafilter which contains $X \sim A$, then $\mathscr{T}' \cap \mathscr{T}(x, \mathscr{U}) = \mathscr{T}(A)$, which is not covered by $\mathscr{T}'$. Note that $\mathscr{T}([x])$ does cover $\mathscr{T}$ and that the topologies of the form $\mathscr{T}([x])$ are the atoms in $\Lambda(X)$ [1].

We also have here a simple example to show that $\Lambda(X)$ is not modular. Even though the lattice in Example 1 is neither upper nor lower semimodular, this does not show that $\Lambda(X)$ is neither upper nor lower semimodular since neither of these properties is hereditary. For an example of a lattice which is upper and lower semimodular, but not modular, see Birkhoff [3].

DEFINITION 1. Let $\mathscr{T}$ and $\mathscr{T}'$ be topologies on X such that $\mathscr{T}'$ covers $\mathscr{T}$. Then we will say *$\mathscr{T}'$ covers $\mathscr{T}$ by x* iff there exists an ultratopology $\mathscr{T}(x, \mathscr{U})$ such that $\mathscr{T} = \mathscr{T}' \cap \mathscr{T}(x, \mathscr{U})$.

It is possible for a non-T_1-topology to cover another topology by x and y where $x \neq y$. For example, the atom $[\phi, [x, y], X]$ covers the indiscrete topology by both x and y. However, this is not possible for T_1-topologies.

LEMMA 3. *If $\mathscr{T}$ and $\mathscr{T}'$ are T_1-topologies on X such that $\mathscr{T}'$ covers $\mathscr{T}$, then there exists a unique $x \in X$ such that $\mathscr{T}'$ covers $\mathscr{T}$ by x.*

Proof. That at least one such x exists follows from Lemma 2. To see that this x is unique, suppose that $\mathscr{T} = \mathscr{T}' \cap \mathscr{T}(x, \mathscr{U})$ and that there exists an ultratopology $\mathscr{T}(y, \mathscr{V})$ with $y \in X$ and $x \neq y$ and $\mathscr{T} = \mathscr{T}' \cap \mathscr{T}(y, \mathscr{V})$. For any $G \in \mathscr{T}'$, $G \sim [x]$ and $G \sim [y]$ are both open in $\mathscr{T}'$, since $\mathscr{T}'$ is T_1. Therefore, $G \sim [x] \in \mathscr{T}' \cap \mathscr{T}(x, \mathscr{U})$ and $G \sim [y] \in \mathscr{T}' \cap \mathscr{T}(y, \mathscr{V})$, which implies that G is open in $\mathscr{T}$ since $G = (G \sim [x]) \cup (G \sim [y])$.

This implies $\mathscr{T} = \mathscr{T}'$, which is a contradiction and the proof is complete.

Since it may easily happen that $\mathscr{T}(G) = \mathscr{T}(H)$ even though $G \neq H$, we define the following equivalence relation.

DEFINITION 2. Let $\mathscr{T}$ be a topology on X, G and H be open in $\mathscr{T}$, and $x \in X$, then we will say that G is *locally equal* to H at x (with respect to $\mathscr{T}$) iff there exists an open neighborhood of x in $\mathscr{T}$, N, such that $G \cap N = H \cap N$. We will denote this as $G =_x H$.

It is shown in Bourbaki [5, p. 65] that local equality is an equivalence relation on the open subsets of $(X, \mathscr{T})$. Bourbaki calls the resulting equivalence classes "germs." We denote the equivalence class containing A as A_x, that is, $A_x = [B \mid B \in \mathscr{T}, A =_x B]$.

DEFINITION 3. If G and H are open sets in a space $(X, \mathscr{T})$ and $x \in X$, we will say $G \leqq_x H$ iff there exists a neighborhood of x in $\mathscr{T}$, N, such that $N \cap H \subseteq N \cap G$. We say $G_x \leqq H_x$ iff $G \leqq_x H$. (It is easily verified that the second statement is well defined.)

LEMMA 4. *If $\mathscr{T}$ is a topology on X, and $x \in X$, and G and H are open sets in $\mathscr{T}$ which do not contain x, then $\mathscr{T}(G \cup [x]) \subseteq \mathscr{T}(H \cup [x])$ iff $G \leqq_x H$.*

Proof. Suppose that $G \leqq_x H$. Choose $N \in \mathscr{T}$ such that $x \in N$ and $N \cap H \subseteq N \cap G$. Then $G \cup [x] = (N \cap (H \cup [x])) \cup G$ which implies that $G \cup [x]$ is open in $\mathscr{T}(H \cup [x])$. Therefore, $\mathscr{T}(G \cup [x]) \subseteq \mathscr{T}(H \cup [x])$.

Now, assume that $\mathscr{T}(G \cup [x]) \subseteq \mathscr{T}(H \cup [x])$.

Case 1. If $\mathscr{T} = \mathscr{T}(H \cup [x])$, then let $N = (H \cup [x]) \cap (G \cup [x])$. It follows that $N \cap H \subseteq N \cap G$, and we have that $G \leqq_x H$.

Case 2. If $\mathscr{T} \neq \mathscr{T}(H \cup [x])$, then choose $A, B \in \mathscr{T}$ such that

$$G \cup [x] = ((H \cup [x]) \cap A) \cup B.$$

If $x \in B$, then $G \cup [x] = G \cup B$, which would imply that $G \cup [x]$ is open in $\mathscr{T}$, and then $H \cap (G \cup [x]) \subseteq G \cap (G \cup [x])$. Finally, if $x \in A$, and if we let $N = A$, we have $N \cap H \subseteq N \cap G$, and the proof is complete.

LEMMA 5. *If $\mathscr{T}$ is a topology on X, and $x \in X$, and G and H are open sets in $\mathscr{T}$ which do not contain x, then $\mathscr{T}(G \cup [x]) = \mathscr{T}(H \cup [x])$ iff $G =_x H$.*

Proof. The proof of this lemma follows immediately from Lemma 4 once we see that $G =_x H$ iff $G \leqq_x H$ and $H \leqq_x G$.

LEMMA 6. *If $\mathscr{T}$ is a topology on X, and $G \in \mathscr{T}$, then $\mathscr{T}(G \cup [x])$ covers $\mathscr{T}$ by x iff G_x is a minimal element in the set $[A_x \mid A \in \mathscr{T}, A \cup [x] \notin \mathscr{T}]$.*

Proof. Suppose that $\mathscr{T}(G \cup [x])$ covers $\mathscr{T}$ by x. For any $H \in \mathscr{T}$, such that $H \cup [x] \notin \mathscr{T}$, if $H \leqq_x G$, we know by Lemma 4 that $\mathscr{T} \subseteq \mathscr{T}(H \cup [x]) \subseteq \mathscr{T}(G \cup [x])$. But, since $\mathscr{T}(G \cup [x])$ covers $\mathscr{T}$, this implies that $\mathscr{T}(G \cup [x]) = \mathscr{T}(H \cup [x])$, and by Lemma 5, $H =_x G$, and $H_x = G_x$. Therefore, G_x is a minimal element in $[A_x \mid A \in \mathscr{T}, A \cup [x] \notin \mathscr{T}]$.

If we assume that G_x is a minimal element in $[A_x \mid A \in \mathscr{T}, A \cup [x] \notin \mathscr{T}]$, then assume that there exists a topology $\mathscr{T}^*$ such that $\mathscr{T} \subseteq \mathscr{T}^* \subseteq \mathscr{T}(G \cup [x])$. If there exists an $H^* \in \mathscr{T}^*$ such that $H^* \notin \mathscr{T}$, it follows that $x \in H^*$. Let $H = H^* \sim [x]$. Then $H \in \mathscr{T}$, and $H \cup [x] \notin \mathscr{T}$. Therefore, $\mathscr{T} \subseteq \mathscr{T}(H \cup [x]) \subseteq \mathscr{T}^* \subseteq \mathscr{T}(G \cup [x])$, which implies that $H \leqq_x G$ and $H_x \leqq G_x$. Since G_x is minimal in

$$[A_x \mid A \in \mathscr{T}, A \cup [x] \notin \mathscr{T}]$$

it follows that $G_x = H_x$ and $G =_x H$. Therefore, $\mathscr{T}(H \cup [x]) = \mathscr{T}(G \cup [x])$ and $\mathscr{T}^* = \mathscr{T}(G \cup [x])$, and it follows that $\mathscr{T}(G \cup [x])$ covers $\mathscr{T}$.

Notice that if $\mathscr{T}$ is not a T_1-topology, then it is possible for $\mathscr{T}(G \cup [x])$ to cover $\mathscr{T}$, where $x \notin G$, and $G \notin \mathscr{T}$. If $\mathscr{T}$ is a T_1-topology, we will see that this is not possible.

LEMMA 7. *If $\mathscr{T}$ is a T_1-topology on X and $\mathscr{T}(G \cup [x])$ covers $\mathscr{T}$ by x, where $x \notin G$, then $G \in \mathscr{T}$.*

Proof. Since $\mathscr{T}(G \cup [x])$ is T_1, G is open in $\mathscr{T}(G \cup [x])$. By Lemma 2, let $\mathscr{T}(x, \mathscr{U})$ be an ultratopology such that $\mathscr{T} = \mathscr{T}(G \cup [x]) \cap \mathscr{T}(x, \mathscr{U})$. Then $G \in \mathscr{T}(x, \mathscr{U})$ since $x \notin G$, and $G \in \mathscr{T}$.

THEOREM 1. *If $\mathscr{T}$ is a T_1-topology on X, and $x \notin G$, then $\mathscr{T}(G \cup [x])$ covers $\mathscr{T}$ by x iff G_x is a minimal element in $[A_x \mid A \in \mathscr{T}, A \cup [x] \notin \mathscr{T}]$.*

The proof follows from Lemma 6 once we notice that Lemma 7 implies $G \in \mathscr{T}$.

From Theorem 1, we can see that many topologies do not have covers. In Example 2, we will show that this is the case with the usual topology on the real numbers.

EXAMPLE 2. Let $(X, \mathscr{T})$ be the space of real numbers with the usual topology. If there exist $G \in \mathscr{T}$, $x \in X$, such that $\mathscr{T}(G \cup [x])$ covers $\mathscr{T}$, then, since $X \sim G$ is closed in $(X, \mathscr{T})$ and $X \sim (G \cup [x])$ is not closed in $(X, \mathscr{T})$, we may choose a sequence in $X \sim (G \cup [x])$ which converges to x. Let $[x_n \mid n \in N]$ be such a sequence. We may, and do, assume that this sequence is monotone increasing. Now let $S = [(x_n, x_{n+2}) \mid n = 1, 3, 5, \ldots]$, where (x_n, x_{n+2}) is the open interval between x_n and x_{n+2}, and let $H = G \cup S$. H is open in $\mathscr{T}$, but $H \cup [x]$ is not open in $\mathscr{T}$, since the sequence $[x_n \mid n = 1, 3, 5, \ldots]$ converges to x and lies outside of H. Since $G \subseteq H$, it is clear that $H \leqq_x G$, but there does not exist an $N_x \in \mathscr{T}$ such that $N_x \cap H \subseteq N_x \cap G$, since the sequence $[x_n \mid n = 2, 4, 6, \ldots]$ is contained in H, but not in G. Therefore, G_x is not a minimal element in $[A_x \mid A \in \mathscr{T}, A \cup [x] \notin \mathscr{T}]$ and, by Theorem 1, $\mathscr{T}(G \cup [x])$ does not cover $\mathscr{T}$.

LEMMA 8. *If $\mathscr{T}$ is a T_1-topology on X, and $\mathscr{T}(x, \mathscr{U})$ is an ultratopology on X (which is not stronger than $\mathscr{T}$) then $\mathscr{T}$ covers $\mathscr{T} \cap \mathscr{T}(x, \mathscr{U})$ iff for any two neighborhoods of x in $\mathscr{T}$, G and H, such that $H \notin \mathscr{U}$, there exists another neighborhood of x in $\mathscr{T}$, N, such that $N \in \mathscr{U}$, and $N \cap H \subseteq G$.*

Proof. Let $\mathscr{T}' = \mathscr{T} \cap \mathscr{T}(x, \mathscr{U})$. Let G and H be any two neighborhoods of x in $\mathscr{T}$ such that $H \notin \mathscr{U}$. If we assume that $\mathscr{T}$ covers $\mathscr{T}'$, then by Lemma 1, we know that there exists $A, B \in \mathscr{T}'$ such that $G = (H \cap B) \cup A$. If $x \in A$, then $A \in \mathscr{U}$, so let $A = N$, and we have $N \subseteq G$, which implies that $N \cap H \subseteq G$. If $x \in B$, then $B \in \mathscr{U}$, and we may let $B = N$; then, again, $N \cap H \subseteq G$.

Now, if we assume that there exists $N \in \mathscr{U}$ such that $N \cap H \subseteq G$, then $G \sim [x]$ is open in $\mathscr{T}'$, since $\mathscr{T}$ is T_1, and N is open in $\mathscr{T}'$ since $N \in \mathscr{U}$. Therefore, $G = (H \cap N) \cup (G \sim [x])$, which implies that $G \in \mathscr{T}'(H)$. But, then since G was an arbitrary neighborhood of x in $\mathscr{T}$, we know that $\mathscr{T} = \mathscr{T}'(H)$, and by Lemma 1, $\mathscr{T}$ covers $\mathscr{T}'$.

THEOREM 2. *If $\mathscr{T}$ is a T_1-topology on X, and $\mathscr{T}(x, \mathscr{U})$ is an ultratopology on X which is not stronger than $\mathscr{T}$, then $\mathscr{T}$ covers $\mathscr{T} \cap \mathscr{T}(x, \mathscr{U})$ iff for any two neighborhoods of x in $\mathscr{T}$, G and H, such that $H \notin \mathscr{U}$, it follows that $(G \cup (X \sim H))^i \in \mathscr{U}$, where A^i denotes the interior of A with respect to $\mathscr{T}$.*

Proof. Assume that $\mathscr{T}$ covers $\mathscr{T} \cap \mathscr{T}(x, \mathscr{U})$, and let G and H be two neighborhoods of x in $\mathscr{T}$ such that $H \notin \mathscr{U}$. Then by Lemma 8, we know there exist some $N \in \mathscr{U}$ such that $N \cap H \subseteq G$. But then $N \subseteq (G \cup (X \sim H))$, and since N is open in $\mathscr{T}$, $N \subseteq (G \cup (X \sim H))^i$, which implies that $(G \cup (X \sim H))^i$ is an element of $\mathscr{U}$.

Conversely, assume that for any two neighborhoods of x in $\mathscr{T}$, G and H, such that $H \notin \mathscr{U}$, $(G \cup (X \sim H))^i \in \mathscr{U}$. Then let $N = (G \cup (X \sim H))^i$, and we have $N \cap H \subseteq G$, so again by Lemma 8, we know that $\mathscr{T}$ covers $\mathscr{T} \cap \mathscr{T}(x, \mathscr{U})$, and the proof is complete.

To see that Theorem 2 is not valid in non-T_1-spaces, we give the following example.

EXAMPLE 3. Let $X = [a, b, c]$ and let $\mathscr{U}(a)$ be the principal ultrafilter of a. If $\mathscr{T} = [\phi, [a], [b], [a, b], [b, c], X]$, then $\mathscr{T}(b, \mathscr{U}(a))$ is an ultratopology on X which is not finer than $\mathscr{T}$. Furthermore, $\mathscr{T}$ does not cover $\mathscr{T} \cap \mathscr{T}(b, \mathscr{U}(a))$ as can be seen in the following diagram:

$$\begin{tikzcd}
 & \mathscr{P}(X) \arrow[ddl, dash] \arrow[dr, dash] & \\
 & & \mathscr{T}(b, \mathscr{U}(a)) \arrow[dd, dash] \\
{[\phi, [a], [b], [a, b], X]} \arrow[ddr, dash] & & \\
 & & {[\phi, [a], [a, c], [a, b], X]} \arrow[dl, dash] \\
 & {[\phi, [a], [a, b], X]} &
\end{tikzcd}$$

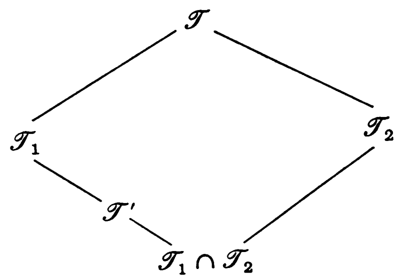
Now let H be any neighborhood of b in $\mathscr{T}$ such that $H \notin \mathscr{U}(a)$. There are only two possibilities for H: $H = [b]$ or $H = [b, c]$, and in either case $a \in X \sim H$. But, for any choice of G, $[a] \subseteq (G \cup (X \sim H))^i$ and $(G \cup (X \sim H))^i \in \mathscr{U}(a)$. Hence, the conditions of Theorem 2 are satisfied, and yet $\mathscr{T}$ does not cover $\mathscr{T} \cap \mathscr{T}(b, \mathscr{U}(a))$.

Example 3 may also be used to conclude that $\Sigma(X)$ is neither upper nor lower semimodular if X contains three elements. A slight generalization of this example would show that $\Sigma(X)$ is neither upper nor lower semimodular if X contains more than three elements (i.e. let $\mathscr{T} = [\phi, [a], [b], [a, b], [b, c], [a, b, c], X]$ and let $\mathscr{T}(b, \mathscr{U}(a))$ be replaced by $[\phi, [a], [c], [a, b], [a, c], [a, b, c], X]$).

3. **Semimodularity.** We shall now prove that $\Lambda(X)$, the lattice of T_1-topologies, is upper as well as lower semimodular.

THEOREM 3. *$\Lambda(X)$ is lower semimodular.*

Proof.



Assume that $\mathscr{T}$ covers $\mathscr{T}_1$ by x_1, and that $\mathscr{T}$ covers $\mathscr{T}_2$ by x_2. If $\mathscr{T}_1$ does not cover $\mathscr{T}_1 \cap \mathscr{T}_2$, let $\mathscr{T}'$ be a topology lying strictly between them, as in the diagram. Now choose $G' \in \mathscr{T}' \sim \mathscr{T}_2$ and $G \in \mathscr{T}_1 \sim \mathscr{T}'$. By Lemma 1, we know that $\mathscr{T} = [(G' \cap B) \cup A \mid A, B \in \mathscr{T}_2]$, so choose $A, B \in \mathscr{T}_2$ such that $G = (G' \cap B) \cup A$.

Case 1. Assume $x_1 \neq x_2$. The proofs of the four cases given in this and the next theorem are not symmetrical, but they do involve similar arguments. Therefore, we will try to be explicit in Case 1 of this proof and omit some of the details in the remaining cases.

(1) G was chosen to be open in $\mathscr{T}_1$, but not in $\mathscr{T}'$. If $x_2 \notin G$, then $G \sim [x_2] = G \in \mathscr{T}_2$. But then G would also be open in $\mathscr{T}_1 \cap \mathscr{T}_2$, and hence open in $\mathscr{T}'$, which contradicts the choice of G. Therefore, $x_2 \in G$.

(2) We know that $x_2 \in G = (G' \cap B) \cup A$. If $x_2 \in A$, then $G = (G \sim [x_2]) \cup A$, which implies that $G \in \mathscr{T}_2$, and hence, $G \in \mathscr{T}_1 \cap \mathscr{T}_2 \subseteq \mathscr{T}'$. This again is a contradiction, so $x_2 \in G' \cap B$.

(3) B was chosen to be open in $\mathscr{T}_2$, and since $\mathscr{T}_2$ is a T_1-topology, $B \sim [x_1] \in \mathscr{T}_2$. However, $B \sim [x_1]$ is open in $\mathscr{T}_1$ also. Therefore, $B \sim [x_1]$ is open in $\mathscr{T}_1 \cap \mathscr{T}_2$, and hence, it is open in $\mathscr{T}'$.

(4) $G \sim [x_2] \in \mathscr{T}'$. This argument is similar to that used in showing that $B \sim [x_1]$ is open in $\mathscr{T}'$.

(5) $G = (G' \cap (B \sim [x_1])) \cup (G \sim [x_2])$. This is true since we have shown that $x_2 \in G' \cap B \subseteq G$.

(6) $G \in \mathscr{T}'$, since we have shown that all three of the sets in the above representation G are open in $\mathscr{T}'$.

Statement number six contradicts the choice of G, and hence, contradicts the existence of the topology $\mathscr{T}'$. Therefore, $\mathscr{T}_1$ covers $\mathscr{T}_1 \cap \mathscr{T}_2$.

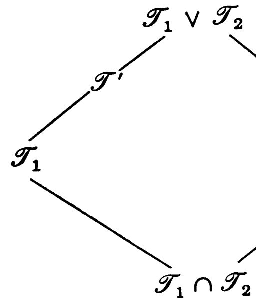
Case 2. Assume $x_1 = x_2 = x$. Again, $x \in B \cap G'$, which implies that

$$G = (G' \cap (B \cup (G \sim [x]))) \cup (G \sim [x]),$$

where G', $B \cup (G \cup [x])$ and $G \sim [x]$ are open in $\mathscr{T}'$. (Note that $B \cup (G \sim [x]) = B \cup (G \cup [x]) = B \cup G$.) But again, this places G in $\mathscr{T}'$, which contradicts our assumption. Therefore, $\mathscr{T}_1$ covers $\mathscr{T}_1 \cap \mathscr{T}_2$, and by a similar argument, we could show that $\mathscr{T}_2$ covers $\mathscr{T}_1 \cap \mathscr{T}_2$.

THEOREM 4. *$\Lambda(X)$ is upper semimodular.*

Proof. Note that $\Lambda(X)$ is not self-dual, and a separate proof must be given for this theorem.



Assume that $\mathscr{T}_1$ covers $\mathscr{T}_1 \cap \mathscr{T}_2$ by x_1, and that $\mathscr{T}_2$ covers $\mathscr{T}_1 \cap \mathscr{T}_2$ by x_2. If $\mathscr{T}_1 \vee \mathscr{T}_2$ does not cover $\mathscr{T}_1$, assume $\mathscr{T}'$ is a topology lying strictly between them, as in the diagram. Now choose $G' \in (\mathscr{T}' \sim \mathscr{T}_1)$ and $G \in (\mathscr{T}_2 \sim \mathscr{T}')$.

Case 1. Assume $x_1 \neq x_2$. It can be shown that $G' \sim [x_1]$ is open in $\mathscr{T}_2$ but not open in $\mathscr{T}_1$; therefore, by Lemma 1, it follows that there exist $A, B \in \mathscr{T}_1 \cap \mathscr{T}_2$ such that $G = ((G' \sim [x_1]) \cap B) \cup A$. But, this places G in $\mathscr{T}'$, which is a contradiction.

Case 2. Assume $x_1 = x_2 = x$.

(1) For each $y \in G'$, choose $A_y \in \mathscr{T}_1$, $B_y \in \mathscr{T}_2$ such that $y \in A_y \cap B_y \subseteq G'$. Then $G' = \bigcup [A_y \cap B_y \mid y \in G']$.

(2) By Lemma 7, $A_y \sim [x] \in \mathscr{T}_1 \cap \mathscr{T}_2$ and $B_y \sim [x] \in \mathscr{T}_1 \cap \mathscr{T}_2$ for each $y \in G'$. Therefore, $G' \sim [x] \in \mathscr{T}_1 \cap \mathscr{T}_2$ since $G' \sim [x] = \bigcup [(A_y \sim [x]) \cap (B_y \sim [x]) \mid y \in G']$.

(3) $B_x \cup (G' \sim [x]) \notin \mathscr{T}_1$ since $G' = (G' \sim [x]) \cup (((G' \sim [x]) \cup B_x) \cap A_x)$, and $G' \notin \mathscr{T}_1$.

(4) By Lemma 1, there exist $A, B \in \mathscr{T}_1 \cap \mathscr{T}_2$ such that

$$G = ((B_x \cup (G' \sim [x])) \cap B) \cup A.$$

(5) $G' \cup (B_x \sim [x]) = (G' \sim [x]) \cup B_x$ and since $B_x \sim [x] \in \mathscr{T}_1 \cap \mathscr{T}_2$, we know $(G' \sim [x]) \cup B_x$ is open in $\mathscr{T}'$ and, by (4), $G \in \mathscr{T}'$ which is a contradiction. Therefore, $\mathscr{T}_1 \vee \mathscr{T}_2$ covers $\mathscr{T}_1$ and could similarly be shown to cover $\mathscr{T}_2$.

4. **The sublattice** $\Gamma(\mathscr{T})$. Let $\mathscr{T}$ be a topology on X and let $[\mathscr{T}_a \mid a \in A]$ be the set of all topologies on X which cover $\mathscr{T}$. Then let $\Gamma(\mathscr{T}) = [\bigvee [\mathscr{T}_a \mid a \in B] \mid B \subseteq A] \cup [\mathscr{T}]$. $\Gamma(\mathscr{T})$ is clearly closed under least upper bounds in the lattice of topologies. It is also clear that $\Gamma(\mathscr{T})$ contains a least element, $\mathscr{T}$, and a greatest element, $\bigvee [\mathscr{T}_a \mid a \in A]$. In order to conclude that $\Gamma(\mathscr{T})$ is a sublattice of the lattice of topologies, we will show that $\Gamma(\mathscr{T})$ is also closed under intersections.

LEMMA 9. *If $\mathscr{T}$, $\mathscr{T}_1$, and $\mathscr{T}_2$ are T_1-topologies on X such that $\mathscr{T}_1$ and $\mathscr{T}_2$ both cover $\mathscr{T}$ and if there exists an ultratopology on X, $\mathscr{T}(x, \mathscr{U})$ such that $\mathscr{T} = \mathscr{T}_1 \cap \mathscr{T}(x, \mathscr{U}) = \mathscr{T}_2 \cap \mathscr{T}(x, \mathscr{U})$, it follows that $\mathscr{T}_1 = \mathscr{T}_2$.*

Proof. (1) Choose $G_1 \in \mathscr{T}_1 \sim \mathscr{T}$ and $G_2 \in \mathscr{T}_2 \sim \mathscr{T}$. It follows that $\mathscr{T}_1 = \mathscr{T}(G_1)$ and $\mathscr{T}_2 = \mathscr{T}(G_2)$. Note that $x \in (G_1 \cap G_2)$, so $G_1 \cap G_2 \neq \varnothing$. We now have the following diagram:

$$\begin{tikzcd}
& \mathscr{T}(G_1) \vee \mathscr{T}(G_2) & \\
\mathscr{T}_1 = \mathscr{T}(G_1) \arrow[ur, dash] & & \mathscr{T}(G_2) = \mathscr{T}_2 \arrow[ul, dash] \\
& \mathscr{T} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

(2) We claim that $\mathscr{T}_1 \vee \mathscr{T}_2 = \mathscr{T}(G_1 \cap G_2)$. $\mathscr{T}(G_1) \subseteq \mathscr{T}(G_1 \cap G_2)$ since $G_1 = (G_1 \cap G_2) \cup (G_1 \sim [x])$ implies that G_1 is open in $\mathscr{T}(G_1 \cap G_2)$. Similarly, $\mathscr{T}(G_2) \subseteq \mathscr{T}(G_1 \cap G_2)$, and therefore, $\mathscr{T}_1 \vee \mathscr{T}_2$ is contained in $\mathscr{T}(G_1 \cap G_2)$. To see that $\mathscr{T}(G_1 \cap G_2) \subseteq \mathscr{T}_1 \vee \mathscr{T}_2$, we observe that $\mathscr{T} \subseteq \mathscr{T}_1 \vee \mathscr{T}_2$ and $(G_1 \cap G_2)$ is open in $\mathscr{T}_1 \vee \mathscr{T}_2$.

(3) Now we claim that $\mathscr{T}(G_1 \cap G_2) \cap \mathscr{T}(x, \mathscr{U}) = \mathscr{T}$. For any $G \in \mathscr{T}(x, \mathscr{U}) \cap \mathscr{T}(G_1 \cap G_2)$, $G = (G_1 \cap G_2 \cap B) \cup A$ for some $A, B \in \mathscr{T}$. If $x \notin G$, then $G = (((G_1 \cap G_2) \sim [x]) \cap B) \cup A$ which implies that $G \in \mathscr{T}$. If $x \in G$, then $G \in \mathscr{U}$ which implies that $G \cup G_1 \in \mathscr{T}$ since $G \cup G_1 = (G \sim [x]) \cup G_1 \in \mathscr{T}_1$ and $\mathscr{T} = \mathscr{T}_1 \cap \mathscr{T}(x, \mathscr{U})$. Similarly, $G \cup G_2 \in \mathscr{T}$. But then $((G \cup G_1) \cap (G \cup G_2) \cap B) \cup A = G$ and $G \in \mathscr{T}$. Therefore, for any G in $\mathscr{T}(x, \mathscr{U}) \cap \mathscr{T}(G_1 \cap G_2)$, we have shown that $G \in \mathscr{T}$. Since it is clear that $\mathscr{T} \subseteq \mathscr{T}(x, \mathscr{U}) \cap \mathscr{T}(G_1 \cap G_2)$, it follows that $\mathscr{T} = \mathscr{T}(x, \mathscr{U}) \cap \mathscr{T}(G_1 \cap G_2)$.

(4) Note that neither G_1 nor G_2 are elements of $\mathscr{U}$, and therefore, $G_1 \cup G_2 \notin \mathscr{U}$. This implies $G_1 \cup G_2 \notin \mathscr{T}$. However, $G_1 \cup G_2 = G_1 \cup (G_2 \sim [x]) = G_2 \cup (G_1 \sim [x])$ which implies $G_1 \cup G_2$ is open in both $\mathscr{T}_1$ and $\mathscr{T}_2$. Since $G_1 \cup G_2 \notin \mathscr{T}$, we must conclude that $\mathscr{T}_1 \cap \mathscr{T}_2 \neq \mathscr{T}$, and that $\mathscr{T}_1 = \mathscr{T}_2$.

Lemma 9 indicates that the covers of a T_1-topology induce an equivalence relation on all ultratopologies which are greater than $\mathscr{T}$. That is, for any ultratopology, $\mathscr{T}(x, \mathscr{U})$, greater than $\mathscr{T}$, $\mathscr{T}(x, \mathscr{U})$ is either greater than every cover of $\mathscr{T}$ or it is greater than all but one cover of $\mathscr{T}$. We will use this idea in the proof of the following lemma.

LEMMA 10. *If $\mathscr{T}$ is a T_1-topology on X, and $\mathscr{T}_1, \mathscr{T}_2 \in \Gamma(\mathscr{T})$, then $\mathscr{T}_1 \cap \mathscr{T}_2 \in \Gamma(\mathscr{T})$.*

Proof. Let $[\mathscr{T}_a \mid a \in A]$ be the set of all covers of $\mathscr{T}$, and let

$$\mathscr{T}_1 = \bigvee [\mathscr{T}_a \mid a \in A_1 \subseteq A], \quad \mathscr{T}_2 = \bigvee [\mathscr{T}_a \mid a \in A_2 \subseteq A],$$

and $\mathscr{T}' = \bigvee [\mathscr{T}_a \mid a \in A]$.

$$\begin{aligned}
\mathscr{T}_1 &= \bigcap [\mathscr{T}(x, \mathscr{U}) \mid \mathscr{T}_1 \subseteq \mathscr{T}(x, \mathscr{U})] \\
&= (\bigcap [\mathscr{T}(x, \mathscr{U}) \mid \mathscr{T}' \subseteq \mathscr{T}(x, \mathscr{U})]) \\
&\quad \cap (\bigcap [\mathscr{T}(x, \mathscr{U}) \mid \mathscr{T}_1 \subseteq \mathscr{T}(x, \mathscr{U}) \text{ and } \mathscr{T}' \nsubseteq \mathscr{T}(x, \mathscr{U})]) \\
&= \mathscr{T}' \cap (\bigcap [\mathscr{T}(x, \mathscr{U}) \mid \mathscr{T}(x, \mathscr{U}) \cap \mathscr{T}_a = \mathscr{T} \text{ for some } a \in A \sim A_1]).
\end{aligned}$$

To justify the last equality, assume $\mathcal{T}(x, \mathcal{U})$ is an ultratopology on X such that $\mathcal{T}_1 \subseteq \mathcal{T}(x, \mathcal{U})$ and $\mathcal{T}' \nsubseteq \mathcal{T}(x, \mathcal{U})$. If $\mathcal{T}_a \subseteq \mathcal{T}(x, \mathcal{U})$ for every $a \in A \sim A_1$, then $\mathcal{T}' \subseteq \mathcal{T}(x, \mathcal{U})$, which contradicts our assumption. Therefore, since each $\mathcal{T}_a$ covers $\mathcal{T}$ and since $\mathcal{T} \subseteq \mathcal{T}_1 \subseteq \mathcal{T}(x, \mathcal{U})$, it follows that $\mathcal{T}_a \cap \mathcal{T}(x, \mathcal{U}) = \mathcal{T}$ for some $a \in A \sim A_1$.

Conversely, if $\mathcal{T}(x, \mathcal{U}) \cap \mathcal{T}_a = \mathcal{T}$ for some $a \in A \sim A_1$, by Lemma 9, $\mathcal{T}(x, \mathcal{U}) \cap \mathcal{T}_{a_1} \neq \mathcal{T}$ for any $a_1 \in A_1$. This implies $\mathcal{T}_{a_1} \subseteq \mathcal{T}(x, \mathcal{U})$ for each $a_1 \in A_1$ and $\mathcal{T}_1 \subseteq \mathcal{T}(x, \mathcal{U})$. Since $\mathcal{T}(x, \mathcal{U}) \cap \mathcal{T}_a = \mathcal{T}$, it is clear that $\mathcal{T}' \nsubseteq \mathcal{T}(x, \mathcal{U})$.

Now

$$\mathcal{T}_1 \cap \mathcal{T}_2 = \mathcal{T}' \cap (\bigcap [\mathcal{T}(x, \mathcal{U}) \mid \mathcal{T}(x, \mathcal{U}) \cap \mathcal{T}_a = \mathcal{T} \text{ for some } a \in A \sim A_1]) \\ \cap (\bigcap [\mathcal{T}(x, \mathcal{U}) \mid \mathcal{T}(x, \mathcal{U}) \cap \mathcal{T}_a = \mathcal{T} \text{ for some } a \in A \sim A_2]).$$

This may be written as

$$\mathcal{T}' \cap (\bigcap [\mathcal{T}(x, \mathcal{U}) \mid \mathcal{T}(x, \mathcal{U}) \cap \mathcal{T}_a = \mathcal{T} \text{ for some } a \in A \sim (A_1 \cap A_2)]).$$

If $A_1 \cap A_2 = \varnothing$, then

$$\mathcal{T}' \cap (\bigcap [\mathcal{T}(x, \mathcal{U}) \mid \mathcal{T}(x, \mathcal{U}) \cap \mathcal{T}_a = \mathcal{T} \text{ for some } a \in A]) = \mathcal{T}' \cap \mathcal{T} = \mathcal{T}.$$

If $A_1 \cap A_2 \neq \varnothing$, then

$$\mathcal{T}' \cap (\bigcap [\mathcal{T}(x, \mathcal{U}) \mid \mathcal{T}(x, \mathcal{U}) \cap \mathcal{T}_a = \mathcal{T} \text{ for some } a \in A \sim (A_1 \cap A_2)]) \\ = \bigvee [\mathcal{T}_a \mid a \in A_1 \cap A_2]$$

by the same argument used above in the representation of $\mathcal{T}_1$. Therefore, $\mathcal{T}_1 \cap \mathcal{T}_2 \in \Gamma(\mathcal{T})$.

Bagley has shown that $\Gamma(\mathcal{T})$ is isomorphic to the Boolean lattice of all subsets of X, when $\mathcal{T}$ is the minimum T_1-topology on X [1]. We recall that any complete, uniquely complemented, atomic lattice is isomorphic with the Boolean lattice of all subsets of its atoms [3].

We have now shown that $\Gamma(\mathcal{T})$ is a sublattice of the lattice of topologies. The covers of $\mathcal{T}$ form the atoms in $\Gamma(\mathcal{T})$ and $\Gamma(\mathcal{T})$ is atomic by definition. The completeness of $\Gamma(\mathcal{T})$ follows from the completeness of the lattice of topologies, and this brings us to our final lemma.

LEMMA 11. *If $\mathcal{T}$ and $\mathcal{T}'$ are T_1-topologies on X such that $\mathcal{T}'$ covers $\mathcal{T}$, and $[\mathcal{T}_a \mid a \in A]$ is a collection of topologies on X, each of which covers $\mathcal{T}$, then $\mathcal{T}' \subseteq \bigvee [\mathcal{T}_a \mid a \in A]$ iff $\mathcal{T}' = \mathcal{T}_a$ for some $a \in A$.*

Proof. One direction of the proof is clear: that is, if $\mathcal{T}' = \mathcal{T}_a$ for some $a \in A$, then of course $\mathcal{T}' \subseteq \bigvee [\mathcal{T}_a \mid a \in A]$. The other direction follows from the preceding lemma. Assume $\mathcal{T}' \subseteq \bigvee [\mathcal{T}_a \mid a \in A]$ and $\mathcal{T}' \neq \mathcal{T}_a$ for any $a \in A$. Let $\mathcal{T} = \mathcal{T}' \cap \mathcal{T}(x, \mathcal{U})$. By Lemma 9, $\mathcal{T}_a \cap \mathcal{T}(x, \mathcal{U}) \neq \mathcal{T}$ for any $\mathcal{T}_a$. Since each $\mathcal{T}_a$ covers $\mathcal{T}$ and since $\mathcal{T} \subseteq \mathcal{T}(x, \mathcal{U})$, it follows that $\mathcal{T}_a \subseteq \mathcal{T}(x, \mathcal{U})$ for each $\mathcal{T}_a$. This implies that $\bigvee [\mathcal{T}_a \mid a \in A] \subseteq \mathcal{T}(x, \mathcal{U})$, which in turn implies that $\mathcal{T}' \subseteq \mathcal{T}(x, \mathcal{U})$. This is a contradiction, since $\mathcal{T} = \mathcal{T}' \cap \mathcal{T}(x, \mathcal{U})$. Therefore, $\mathcal{T}' = \mathcal{T}_a$ for some $a \in A$.

THEOREM 5. *If $\mathscr{T}$ is any T_1-topology on X, then $\Gamma(\mathscr{T})$ is isomorphic to the Boolean lattice of all subsets of the set of covers of $\mathscr{T}$.*

Proof. Since $\Gamma(\mathscr{T})$ is complete and atomic, it suffices to show that $\Gamma(\mathscr{T})$ is uniquely complemented. Let $[\mathscr{T}_a \mid a \in A]$ be the set of all covers of $\mathscr{T}$. Let $\mathscr{T}'$ be any element in $\Gamma(\mathscr{T})$, where $\mathscr{T}' = \bigvee [\mathscr{T}_a \mid a \in A' \subseteq A]$ and let

$$\mathscr{T}'' = \bigvee [\mathscr{T}_a \mid a \in A \sim A'].$$

It is clear that $\mathscr{T}' \vee \mathscr{T}''$ is the greatest element of $\Gamma(\mathscr{T})$ and since $A' \cap (A \sim A') = \varnothing$, it follows from the proof of Lemma 10 that $\mathscr{T}' \cap \mathscr{T}'' = \mathscr{T}$.

If there exists another $\mathscr{T}^* \in \Gamma(\mathscr{T})$ such that $\mathscr{T}^*$ is a complement of $\mathscr{T}'$, let $\mathscr{T}^* = \bigvee [\mathscr{T}_a \mid a \in A^* \subseteq A]$. Since $\mathscr{T}^* \vee \mathscr{T}' = \bigvee [\mathscr{T}_a \mid a \in A]$, it follows from Lemma 11 that, for every $a \in A$, $a \in A' \cup A^*$. Therefore, $A \subseteq A' \cup A^*$ and $A \sim A' \subseteq A^*$. If there exists $a \in A^* \cap A'$, then $\mathscr{T}_a \subseteq \mathscr{T}' \cap \mathscr{T}^*$, which is a contradiction; therefore, $(A \sim A') = A^*$ and $\mathscr{T}'' = \mathscr{T}^*$. The proof is complete.

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