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SEMIGROUPS SATISFYING VARIABLE IDENTITIES. II

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Abstract. The concept of a semigroup satisfying an identity $xy=f(x,y)$ is generalized by considering identities in n-variables and letting the identity depend on the variables. The property of satisfying a "variable identity" is studied. Semigroups satisfying certain types of identities are characterized in terms of unions and semilattices of groups.

Introduction. Semigroups satisfying an identity of the form $xy=f(x,y)$ have been studied by Tully [5] and Tamura [4]. In [2], we generalized Tamura's result. We considered semigroups S satisfying: for every $a, b \in S$ there exists a positive integer m such that $ab=b^{\lambda_1}a^{\mu_1}\cdots b^{\lambda_m}a^{\mu_m}$ where λ_i, μ_i are integers greater than one, $i=1,\ldots,m$, and $\sum_{i=1}^{m}\mu_i \geqq 2$. We proved that a semigroup S satisfies this condition if and only if S is an inflation of a semilattice of periodic groups. The purpose of this article is to consider semigroups satisfying the analogous condition for n variables.

1. **Preliminaries.** Throughout S will denote a semigroup and $E=E(S)$ the set of idempotents of S and n will be an integer, $n\geqq 2$. Let F_n denote the free (noncommutative) semigroup generated by the distinct letters $x_1,\ldots,x_n$. Denote by C_n the subsemigroup of F_n consisting of all elements $x\in F_n$ each of which is the product of *all* of the $x_1,\ldots,x_n$, allowing repeated use. Let R_n denote the semigroup ring of C_n over the integers Z. Thus R_n is the set of functions of finite support from C_n to Z. It is well known that R_n can be considered as the set of finite formal sums of elements in C_n and coefficients in Z.

Definition. We define a subset $\mathscr{V}_n$ of R_n by $\mathscr{V}_n=\{f \mid f\in R_n, f=f_1-f_2$, there exists $c_1, c_2\in C_n$, $c_1\neq c_2$ such that $f_i(c)=0$ unless $c=c_i, f_i(c_i)=1\}$.

Thus $\mathscr{V}_n$ is the set of all $f\in R_n$ which are the differences of two different monomials. That is, $f\in\mathscr{V}_n$ if and only if $f=x_{k_1}\cdots x_{k_r}-x_{l_1}\cdots x_{l_s}\neq 0$, where each x_i $(i=1,\ldots,n)$ appears at least once in $x_{k_1}\cdots x_{k_r}$ and at least once in $x_{l_1}\cdots x_{l_s}$.

If $a_1,\ldots,a_n\in S$ there exists a homomorphism $\varphi\colon C_n\to S$ such that $\varphi(x_i)=a_i$. This homomorphism extends to a homomorphism φ' from R_n into the semigroup ring of S over Z. If $f=f_1-f_2\in\mathscr{V}_n$ and $a_1,\ldots,a_n\in S$ we say that $f_1(a_1,\ldots,a_n)=f_2(a_1,\ldots,a_n)$ if $f\in\ker\varphi'$. Thus $f_1(a_1,\ldots,a_n)=f_2(a_1,\ldots,a_n)$ if and only if considering f_1 and f_2 as monomials in $x_1,\ldots,x_n$ when a_i is substituted for x_i and

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multiplication performed in S, we have $f_1(a_1, \ldots, a_n)=f_2(a_1, \ldots, a_n)$. We will use without further comment this characterization of $f_1(a_1, \ldots, a_n)=f_2(a_1, \ldots, a_n)$.

If $\mathscr{X}$ is a subset of $\mathscr{V}_n$, we say that S is an $\mathscr{X}$-semigroup if for every $a_1, \ldots, a_n \in S$ there exists $f=f_1-f_2 \in \mathscr{X}$ (depending on $a_1, \ldots, a_n$) such that

$$f_1(a_1, \ldots, a_n) = f_2(a_1, \ldots, a_n).$$

We prove a preliminary result for arbitrary subsets $\mathscr{X} \subseteq \mathscr{V}_n$.

THEOREM 1.1. *Suppose T is a semilattice Ω of semigroups T_α, $\alpha \in \Omega$, such that each T_α has an identity element. Then T is an $\mathscr{X}$-semigroup if and only if each T_α is an $\mathscr{X}$-semigroup.*

Proof. If T is an $\mathscr{X}$-semigroup then each T_α is an $\mathscr{X}$-semigroup since subsemigroups of $\mathscr{X}$-semigroups are $\mathscr{X}$-semigroups. To prove the converse let $a_1, \ldots, a_n \in T$. Suppose $a_i \in T_{\alpha_i}$ and let $\alpha=\alpha_1 \cdots \alpha_n$ (the product in the semilattice Ω). Then $a=a_1 \cdots a_n \in T_\alpha$. Let e be the identity of T_α. Since $\alpha_i\alpha=\alpha\alpha_i=\alpha$ we have that a_ie and ea_i are in T_α, for $i=1, \ldots, n$. Furthermore $ea_i=(ea_i)e=e(a_ie)=a_ie$. Now applying the hypothesis to T_α, there exists $f=f_1-f_2 \in \mathscr{X}$ such that $f_1(a_1e, \ldots, a_ne)=f_2(a_1e, \ldots, a_ne)$. Since each x_i appears at least once in f_1 and f_2 and since $\alpha_i\alpha=\alpha\alpha_i=\alpha$, it follows that $f_1(a_1, \ldots, a_n)$ and $f_2(a_1, \ldots, a_n)$ are in T_α. Hence $f_1(a_1, \ldots, a_n)=f_1(a_1, \ldots, a_n)e=f_1(a_1e, \ldots, a_ne)=f_2(a_1e, \ldots, a_ne)=f_2(a_1, \ldots, a_n)e=f_2(a_1, \ldots, a_n)$. Therefore T is an $\mathscr{X}$-semigroup.

DEFINITION. Let S be a semigroup, T a subsemigroup of S. Then S is an nth *inflation of* T if there exists a homomorphism $\theta: S \to T$ such that θ is the identity map on T and for each $a_1, \ldots, a_n \in S$, $(a_1\theta)\cdots(a_n\theta)=a_1 \cdots a_n$.

REMARK. The 2nd inflation corresponds to the usual concept of inflation (cf. [1, p. 98]). If S is an nth inflation of T then $(a_1\theta)\cdots(a_m\theta)=a_1 \cdots a_m$ for all $m \geqq n$.

COROLLARY 1.2. *Let $\mathscr{X} \subseteq \mathscr{V}_n$. Suppose S is an nth inflation of T, that T is a semilattice of semigroups T_α, and that each T_α is an $\mathscr{X}$-semigroup with identity. Then S is an $\mathscr{X}$-semigroup.*

Proof. Let $a_1, \ldots, a_n \in S$. Then $a_1\theta, \ldots, a_n\theta \in T$ so by Theorem 1.1 there exists $f=f_1-f_2 \in \mathscr{X}$ such that $f_1(a_1\theta, \ldots, a_n\theta)=f_2(a_1\theta, \ldots, a_n\theta)$. By the above remark, for $j=1, 2$, $f_j(a_1, \ldots, a_n)=f_j(a_1\theta, \ldots, a_n\theta)$ since each x_i appears at least once in f_j. Thus $f_1(a_1, \ldots, a_n)=f_1(a_1\theta, \ldots, a_n\theta)=f_2(a_1\theta, \ldots, a_n\theta)=f_2(a_1, \ldots, a_n)$. Hence S is an $\mathscr{X}$-semigroup.

REMARK. The proof of Corollary 1.2 can be modified to prove that if S is an ideal extension of T by an $\mathscr{X}$-semigroup, then S is an $\mathscr{X}$-semigroup.

DEFINITION. Let n, i, α be positive integers $n \geqq 2$, $1 \leqq i \leqq n$. Certain subsets of $\mathscr{V}_n$ are defined by:

(1) $\mathscr{L}_n=\{f \mid f=f_1-f_2 \in \mathscr{V}_n;\ f_1=x_1 \cdots x_n;\ x_j$ appears at least twice in f_2, $j=1, \ldots, n\}$.

(2) $\mathscr{M}_n^{i,\alpha}=\{f \mid f=f_1-f_2 \in \mathscr{V}_n, f_1=x_1 \cdots x_n, x_i$ appears at least α times in $f_2\}$.

(3) If $\mathscr{X} \subseteq \mathscr{V}_n$, then $\overline{\mathscr{X}} = \{f \mid f = f_1 - f_2 \in \mathscr{X}, f_2$ starts with $x_j; j \neq 1$ and ends with x_k, $k \neq n\}$.

(4) S is an $\mathscr{M}_n^{i,\infty}$-semigroup if S is an $\mathscr{M}_n^{i,\alpha}$-semigroup for all $\alpha \geqq 2$.

Remark. Let $\mathscr{X}, \mathscr{Y} \subseteq \mathscr{V}_n$. Then $\mathscr{X} \subseteq \mathscr{Y}$ implies that: (i) every $\mathscr{X}$-semigroup is a $\mathscr{Y}$-semigroup and (ii) $\overline{\mathscr{X}} \subseteq \overline{\mathscr{Y}}$. Also subsemigroups and homomorphic images of $\mathscr{X}$-semigroups are $\mathscr{X}$-semigroups.

We will now prove several lemmas which are needed for the main theorems.

Let $a_1, \dots, a_n \in S$ and suppose $x = a_{k_1}^{\mu_1} \cdots a_{k_t}^{\mu_t}$. We say that the length of x in the a_j's is $\sum_{i=1}^{t} \mu_i$.

Lemma 1.3. (i) *Every $\mathscr{L}_n$-semigroup is an $\mathscr{M}_n^{i,\infty}$-semigroup.*

(ii) *Every $\overline{\mathscr{M}}_n^{i,2}$-semigroup is an $\mathscr{M}_n^{i,\infty}$-semigroup.*

Proof. (i) The proof follows by repeated application of the equation

$$a_1 \cdots a_n = f_2(a_1, \dots, a_n).$$

(ii) Since $\overline{\mathscr{M}}_n^{i,\alpha} \subseteq \mathscr{M}_n^{i,\alpha}$ it suffices to show that for $\alpha \geqq 2$, every $\overline{\mathscr{M}}_n^{i,\alpha}$-semigroup is an $\overline{\mathscr{M}}_n^{i,\alpha+1}$-semigroup. Let $a_1, \dots, a_n \in S$. Then by (2) $a_1 \cdots a_n = f_2(a_1, \dots, a_n)$, where each a_j appears at least once on the right-hand side and a_i appears at least α times. Since we can apply the hypothesis repeatedly, we may assume, without loss of generality, that the length of f_2 in the a_j's is greater than $2n^2$. There are two possibilities:

(i) $f_2(a_1, \dots, a_n) = ua_ih_1 \cdots h_n$ with $u, h_j \in \langle a_1, \dots, a_n \rangle$, a_i appearing in at least one h_j and the length of h_j in the a_k's greater than or equal to n.

(ii) $f_2(a_1, \dots, a_n) = h_1 \cdots h_n a_i u$, h_j, u as in (i).

We assume (i), the proof for (ii) being similar. Applying the $\overline{\mathscr{M}}_n^{i,\alpha}$ hypothesis to $(ua_ih_1)h_2 \cdots h_n$ we have

$$\begin{aligned}(u_1a_ih_1)h_2 \cdots h_n &= f_3(ua_ih_1, h_2, \dots, h_n) = h_{k_1} \cdots h_{k_s} ua_ih_1h_{k_{s+1}} \cdots h_{k_n} \\ &= g_1 \cdots g_{i-1}a_ig_{i+1} \cdots g_n,\end{aligned}$$

where each g_i is a product of the a_j's and a_i appears in at least one g_j. Again applying the $\overline{\mathscr{M}}_n^{i,\alpha}$ hypothesis we have $g_1 \cdots g_{i-1}a_ig_{i+1} \cdots g_n = f_4(g_1, \dots, a_i, \dots, g_n)$. Now since a_i appears in some g_j, a_i appears at least $\alpha+1$ times on the right-hand side. Hence $a_1 \cdots a_n = f_2(a_1, \dots, a_n) = ua_ih_1 \cdots h_n = f_4(g_1, \dots, a_i, \dots, g_n)$ and the proof is complete.

Lemma 1.4. *Suppose that S is a semigroup with zero and that S is an $\mathscr{M}_n^{i,\infty}$-semigroup with no nonzero idempotents. Then $S^n = \{0\}$.*

Proof. Let $s \in S$. Letting $a_1 = a_2 = \cdots = a_n \in S$, we have from (2) that $s^n = s^k$, with $k > n$. Hence $s^m \in E(S)$ for some m, $n \leqq m \leqq k$. Thus $s^m = 0$ and hence $s^n = s^k = 0$. Let l be the least positive integer for which $S^{i-1}s^lS^{n-i} = \{0\}$. We prove that the assumption $l > 1$ leads to a contradiction. Let $t_1, \dots, t_n \in S$. Let $\alpha = n+1$ and define

$$\begin{aligned} a_j &= s^{l-1}t_j && \text{if } 1 \leqq j < i, \\ &= s^{l-1}t_is^{l-1} && \text{if } j = i, \\ &= t_js^{l-1} && \text{if } i < j \leqq n. \end{aligned}$$

Since S is an $\mathscr{M}_n^{i,\alpha}$-semigroup we have $a_1 \cdots a_n = f_2(a_1, \ldots, a_n)$ with a_i appearing at least $\alpha = n+1$ times on the right. It follows that s^l appears at least n times on the right. Isolating the ith s^l we see that $f_2(a_1, \ldots, a_n) \in S^{i-1}s^lS^{n-i} = \{0\}$. Thus $s^{l-1}t_1 \cdots s^{l-1}t_is^{l-1} \cdots t_ns^{l-1} = f_2(a_1, \ldots, a_n) = 0$. But the t_i's are arbitrary in S. Hence $(s^{l-1}S)^{n+1} = 0$. Now apply the $\mathscr{M}_n^{i,\alpha}$ condition with $\alpha = 2n+2$ to the elements $a_j = t_j$, $j \neq i$ and $a_i = s^{l-1}$ where the t_j are arbitrary elements of S. We have

$$t_1 \cdots t_{i-1}s^{l-1}t_{i+1} \cdots t_n = f_2(t_1, \ldots, t_{i-1}, s^{l-1}, t_{i+1}, \ldots, t_n) \in S^1(s^{l-1}S)^{n+1} = \{0\}.$$

Since the t_j are arbitrary, we conclude that $S^{i-1}s^{l-1}S^{n-i} = \{0\}$, a contradiction. Hence $l = 1$ and $S^{i-1}sS^{n-i} = \{0\}$, for every $s \in S$. Thus $S^n = \{0\}$.

COROLLARY 1.5. *Let S be a semigroup with zero and no nonzero idempotents. If S is either an $\mathscr{L}_n$-semigroup or an $\overline{\mathscr{M}}_n^{i,2}$-semigroup then $S^n = \{0\}$.*

DEFINITION. Let S be a semigroup. $I = I(S) = \{x \mid x \in S,\ x^l = x$ for some positive integer $l \geqq 2\}$. $E = E(S) = \{x \mid x \in S,\ x^2 = x\}$.

LEMMA 1.6. *If S is an $\mathscr{L}_n$-semigroup or an $\overline{\mathscr{M}}_n^{i,2}$-semigroup then $SE \cup ES \subseteq I$.*

Proof. (i) Let S be an $\mathscr{L}_n$-semigroup. We will show that $SE \subseteq I$; the proof that $ES \subseteq I$ is similar. Let $a \in S$, $e \in E$. Then by (1), there exists f_2 such that $ae = ae \cdots e = f_2(ae, e, \ldots, e) = e^k(ae)^le^i = e^k(ae)^l$ where $l \geqq 2$ and $k = 0$ or $k = 1$. If $k = 1$, $ae = e(ae)^l$ so that $eae = ae$. Hence $ae = e(ae)^l = (eae)(ae)^{l-1} = (ae)^l$. Hence, for either $k = 0$ or $k = 1$, we obtain $ae = (ae)^l$, with $l \geqq 2$. Consequently $ae \in I$.

(ii) Let S be an $\overline{\mathscr{M}}_n^{i,2}$-semigroup. Let $a \in S$, $e \in E$. From (2) and (3) it follows that $ae = ae \cdots e = es$ for some $s \in S$. Hence $ae = eae$. Now letting $a_j = e$ for $j \neq i$ and $a_i = ae$ we have, again applying (2) and (3), that

$$e \cdots e(ae)e \cdots e = f_2(e, \ldots, ae, \ldots, e) = e^k(ae)^l$$

where $k = 0$ or 1 and $l \geqq 2$. Hence $ae = eae = e^k(ae)^l = (e^kae)(ae)^{l-1} = (ae)(ae)^{l-1} = (ae^l)$. Thus $ae \in I$ and so $SE \subseteq I$. Similarly $ES \subseteq I$.

LEMMA 1.7. *Let S be an $\mathscr{L}_n$-semigroup or an $\overline{\mathscr{M}}_n^{i,2}$-semigroup. Then $E \subseteq I$ and I is an ideal. In particular, the Rees factor semigroup S/I has no nonzero idempotents.*

Proof. Either hypothesis implies that S is periodic and hence $E \neq \varnothing$. Clearly $E \subseteq I$. Let $a \in I$, $x \in S$. Then there exists $l \geqq 2$ such that $a = a^l$. Hence $a^{l-1} \in E$. Consequently $ax = a^lx = a^{l-1}(ax) \in I$, by Lemma 1.6.

2. **Main theorems.**

THEOREM 2.1. *Let S be either an $\mathscr{L}_n$-semigroup or an $\overline{\mathscr{M}}_n^{i,2}$-semigroup. Then S^n is a disjoint union of periodic groups.*

Proof. By Lemma 1.7, I is an ideal and S/I has no nonzero idempotents. Since S/I is a homomorphic image of S, S/I is either an $\mathscr{L}_n$-semigroup or an $\overline{\mathscr{M}}_n^{i,2}$-semigroup. By Corollary 1.5, $S^n \subseteq I$. Hence for all $x \in S^n$, there exists $l \geqq 2$ such that $x^l = x$. It is well known (cf. [1, p. 23, Exercise 6a]), that this condition implies that S^n is a disjoint union of periodic groups.

REMARK. A semigroup S is an $\mathscr{L}_n$-semigroup if and only if S^n is a disjoint union of periodic groups. Theorem 2.4 shows that a union of periodic groups is not necessarily an $\bar{\mathscr{M}}_n^{1,2}$-semigroup.

DEFINITION. A semigroup S is *viable* if $ab, ba \in E$ implies $ab = ba$. Idempotents are *central* in S if $ae = ea$, for every $e \in E$, $a \in S$.

LEMMA 2.2. (i) *If S is an $\bar{\mathscr{M}}_n^{1,2}$-semigroup then idempotents are central in S.*

(ii) *For any semigroup S, if idempotents are central in S, then S is viable.*

(iii) *Let S be an $\mathscr{L}_n$-semigroup. Then S is viable if and only if idempotents are central in S if and only if idempotents commute in S.*

Proof. (i) Let $e \in E$, $a \in S$. By (2) and (3), there exists f_2 such that $ea = e \cdots ea = se$ for some s in S. Hence $ea = eae$. Similarly $ae = eae$. Thus $ea = ae$.

(ii) Suppose $ab, ba \in E$. Then $ab = (ab)(ab) = a(ba)b = (ba)(ab) = b(ab)a = (ba)(ba) = ba$.

(iii) By Theorem 2.1, S^n is a union of groups. If S is viable then S^n is viable. By [3, Theorem 13], idempotents of S^n are central in S^n. Let $e \in E$, $a \in S$. Then $e = e^n$; $ea = e^{n-1}a \in S^n$. Hence $ea = e(ea) = (ea)e$. Similarly $eae = ae$. Therefore $ea = ae$, showing idempotents are central in S. The converse follows from (ii). If idempotents commute in S then (cf. [1, pp. 126, 127]), idempotents are central in S^n. The above argument shows idempotents are central in S. The following lemma is Lemma 3 of [2].

LEMMA 2.3. *Let S be a semigroup with central idempotents. If $a_1, a_2 \in S$ and $e_1 \in \langle a_1 \rangle$, $e_2 \in \langle a_2 \rangle$, $e \in \langle a_1 a_2 \rangle$ where $e, e_1, e_2 \in E$, then $e_1 e_2 = e$.*

THEOREM 2.4. *Let S be a semigroup. The following are equivalent.*

(i) *S is an $\bar{\mathscr{L}}_n$-semigroup.*

(ii) *S is an $\bar{\mathscr{M}}_n^{1,2}$-semigroup.*

(iii) *S is an $\mathscr{L}_n$-semigroup and idempotents commute.*

(iv) *S is an $\mathscr{L}_n$-semigroup with central idempotents.*

(v) *S is a viable $\mathscr{L}_n$-semigroup.*

(vi) *S is an nth inflation of a semilattice of periodic groups.*

Proof. (i) $\Rightarrow$ (ii) a fortiori.

(ii) $\Rightarrow$ (iii). By Theorem 2.1, S^n is a disjoint union of periodic groups. Let $a_1, \ldots, a_n \in S$. Then $a_1 \cdots a_n \in S^n$, so there exists $k \geqq 2$ such that $a_1 \cdots a_n = (a_1 \cdots a_n)^k$. Consequently S is an $\mathscr{L}_n$-semigroup. By Lemma 2.2(i), idempotents are central in S so surely they commute.

(iii) $\Rightarrow$ (iv) follows by Lemma 2.2(iii).

(iv) $\Rightarrow$ (v). By Lemma 2.2(ii). S is viable.

(v) $\Rightarrow$ (vi). By Theorem 2.1, S^n is a disjoint union of periodic groups. By Lemma 2.2(iii), idempotents are central in S, so certainly idempotents are central in S^n.

Hence (cf. [1, Theorem 4.11]), S^n is a semilattice of periodic groups. We show that S is an nth inflation of S^n. Define $\theta: S \to S^n$ by: $a\theta = ae$ where $e \in \langle a \rangle$. Clearly θ is the identity on S^n. Also if $a, b \in S$, $e_1 \in \langle a \rangle$, $e_2 \in \langle b \rangle$, then by Lemma 2.3, $e_1 e_2 \in \langle ab \rangle$. Hence $(ab)\theta = abe_1e_2 = (ae_1)(be_2) = (a\theta)(b\theta)$. Hence θ is a homomorphism. If $a_1, \ldots, a_n \in S$, then $(a_1\theta) \cdots (a_n\theta) = (a_1 \cdots a_n)\theta = a_1 \cdots a_n$ since θ is the identity on S^n.

(vi) $\Rightarrow$ (i). Let G be a periodic group with identity e and let $a_1, \ldots, a_n \in G$. There exist integers $i > 1, j > 1, k > 1$ such that $a_1^i = e$, $a_n^j = e$, $(a_1 \cdots a_n)^k = (a_1 \cdots a_n)$. Thus $a_1 \cdots a_n = a_n^j (a_1 \cdots a_n)^k a_1^i$. Consequently G is an $\overline{\mathscr{L}}_n$-semigroup. Thus by Corollary 1.2, S is an $\overline{\mathscr{L}}_n$-semigroup.

COROLLARY 2.5. *Let $\mathscr{X} \subseteq \overline{\mathscr{M}}_n^{1,2}$. Then S is an $\mathscr{X}$-semigroup if and only if S is an nth inflation of a semilattice of $\mathscr{X}$-groups.*

Proof. If S is an $\mathscr{X}$-semigroup, then it is an $\overline{\mathscr{M}}_n^{1,2}$-semigroup. Hence S is an nth inflation of a semilattice of groups. Each of these groups, being a subgroup of S, is an $\mathscr{X}$-group. The converse follows from Corollary 1.2.

COROLLARY 2.6. *Let $\mathscr{X} \subseteq \mathscr{M}_n^{1,2}$. Then S^{n-1} is in the center of S for every $\mathscr{X}$-semigroup S if and only if*

(i) *$\mathscr{X} \subseteq \overline{\mathscr{M}}_n^{1,2}$, and*

(ii) *every $\mathscr{X}$-group is abelian.*

Proof. The necessity of (ii) is clear. If (i) does not hold then there exists $f = x_1 \cdots x_n - f_2 \in \mathscr{X}$ such that f_2 starts with x_1 or ends with x_n. Let S_1 (respectively S_2) be a nontrivial right-(left-)zero semigroup. Then either S_1 or S_2 is an $\{f\}$-semigroup and hence an $\mathscr{X}$-semigroup. But $S_1^{n-1} = S_1$ ($S_2^{n-1} = S_2$) which is not in the center of S_1 (S_2).

Conversely assume (i) and (ii) hold. By Corollary 2.5, S is an nth inflation of a semilattice of $\mathscr{X}$-groups G_α. By (ii) each G_α is abelian and hence satisfies every permutation identity in n-variables. By Corollary 1.2, letting $\mathscr{X}$ be any single permutation identity in n-variables, S itself satisfies every permutation identity in n-variables. In particular S^{n-1} is in the center.

REMARK. In Corollary 2.6, we can replace the words "S^{n-1} is in the center of S" with "S satisfies every permutation identity in n-variables."

Theorem 2.4 and Corollary 2.6 yield the main theorem and corollary of Tamura [4], when $n = 2$ and $\mathscr{X} = \{x_1x_2 - f(x_1, x_2)\}$, f a fixed monomial in x_1 and x_2. In addition Theorem 2.4 generalizes the main theorem of [2].

3. Examples and problems.

EXAMPLE 1. Theorem 2.1 is not true for $\mathscr{M}_n^{1,\infty}$-semigroups. Let S be the multiplicative semigroup of 2×2 matrices over $GF(2)$ consisting of $\{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]\}$. Then S satisfies $x_1x_2 = x_1^\alpha x_2$ for all $\alpha \geqq 2$ and hence S is an $\mathscr{M}_2^{1,\infty}$-semigroup. However $S^2 = S$ is not a union of groups.

EXAMPLE 2. Theorem 1.1 is not true if the condition that every T_α has an identity is deleted. Let $n=2$, $\mathscr{X}=\{x_1x_2-x_2x_1\}$. Let S be the semigroup given by:

	0	a	b
0	0	0	0
a	0	0	0
b	0	a	b

Let $T_1=\{0, a\}$, $T_2=\{b\}$. Then T_1 and T_2 are $\mathscr{X}$-semigroups and S is a semilattice of T_1 and T_2. But S is not an $\mathscr{X}$-semigroup.

PROBLEM 1. Theorems 2.1 and 2.4 characterize $\mathscr{L}_n$-semigroups and $\overline{\mathscr{M}}_n^{1,2}$-semigroups. Characterize other $\mathscr{X}$-semigroups. Study semigroups which are $\{f\}$-semigroups for some $f\in\mathscr{V}_n$.

PROBLEM 2. Is there a "nice" subset $\mathscr{X}$ of $\mathscr{V}_n$ such that S is an $\mathscr{X}$-semigroup if and only if S^n is a band of groups?

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