SEMIGROUPS SATISFYING VARIABLE IDENTITIES. II

BY

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Abstract. The concept of a semigroup satisfying an identity $xy=f(x,y)$ is generalized by considering identities in $n$-variables and letting the identity depend on the variables. The property of satisfying a "variable identity" is studied. Semigroups satisfying certain types of identities are characterized in terms of unions and semi-lattices of groups.

Introduction. Semigroups satisfying an identity of the form $xy=f(x,y)$ have been studied by Tully [5] and Tamura [4]. In [2], we generalized Tamura's result. We considered semigroups $S$ satisfying: for every $a, b \in S$ there exists a positive integer $m$ such that $ab = b^{\lambda_1}a^{\mu_1} \cdots b^{\lambda_m}a^{\mu_m}$ where $\lambda_i, \mu_i$ are integers greater than one, $i=1, \ldots, m$, and $\sum_{i=1}^{m} \mu_i \geq 2$. We proved that a semigroup $S$ satisfies this condition if and only if $S$ is an inflation of a semi-lattice of periodic groups. The purpose of this article is to consider semigroups satisfying the analogous condition for $n$ variables.

1. Preliminaries. Throughout $S$ will denote a semigroup and $E=E(S)$ the set of idempotents of $S$ and $n$ will be an integer, $n \geq 2$. Let $F_n$ denote the free (non-commutative) semigroup generated by the distinct letters $x_1, \ldots, x_n$. Denote by $C_n$ the subsemigroup of $F_n$ consisting of all elements $x \in F_n$ each of which is the product of all of the $x_1, \ldots, x_n$, allowing repeated use. Let $R_n$ denote the semigroup ring of $C_n$ over the integers $\mathbb{Z}$. Thus $R_n$ is the set of functions of finite support from $C_n$ to $\mathbb{Z}$. It is well known that $R_n$ can be considered as the set of finite formal sums of elements in $C_n$ and coefficients in $\mathbb{Z}$.

Definition. We define a subset $\mathcal{V}_n$ of $R_n$ by $\mathcal{V}_n = \{ f \mid f \in R_n, f = f_1 - f_2, \text{there exists } c_1, c_2 \in C_n, c_1 \neq c_2 \text{ such that } f_i(c) = 0 \text{ unless } c = c_i, f_i(c_i) = 1 \}$.

Thus $\mathcal{V}_n$ is the set of all $f \in R_n$ which are the differences of two different monomials. That is, $f \in \mathcal{V}_n$ if and only if $f = x_{k_1} \cdots x_{k_r} - x_{i_1} \cdots x_{i_s} \neq 0$, where each $x_i (i=1, \ldots, n)$ appears at least once in $x_{k_1} \cdots x_{k_r}$ and at least once in $x_{i_1} \cdots x_{i_s}$.

If $a_1, \ldots, a_n \in S$ there exists a homomorphism $\varphi: C_n \to S$ such that $\varphi(x_i) = a_i$. This homomorphism extends to a homomorphism $\varphi'$ from $R_n$ into the semigroup ring of $S$ over $\mathbb{Z}$. If $f = f_1 - f_2 \in \mathcal{V}_n$ and $a_1, \ldots, a_n \in S$ we say that $f_1(a_1, \ldots, a_n) = f_2(a_1, \ldots, a_n)$ if $f \in \ker \varphi'$. Thus $f_1(a_1, \ldots, a_n) = f_2(a_1, \ldots, a_n)$ if and only if considering $f_1$ and $f_2$ as monomials in $x_1, \ldots, x_n$ when $a_i$ is substituted for $x_i$ and

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multiplication performed in \( S \), we have \( f_1(a_1, \ldots, a_n) = f_2(a_1, \ldots, a_n) \). We will use without further comment this characterization of \( f_1(a_1, \ldots, a_n) = f_2(a_1, \ldots, a_n) \).

If \( \mathcal{X} \) is a subset of \( \mathcal{Y}_n \), we say that \( S \) is an \( \mathcal{X} \)-semigroup if for every \( a_1, \ldots, a_n \in S \) there exists \( f = f_1 - f_2 \in \mathcal{X} \) (depending on \( a_1, \ldots, a_n \)) such that

\[
f_1(a_1, \ldots, a_n) = f_2(a_1, \ldots, a_n).
\]

We prove a preliminary result for arbitrary subsets \( \mathcal{X} \subseteq \mathcal{Y}_n \).

**Theorem 1.1.** Suppose \( T \) is a semilattice \( \Omega \) of semigroups \( T_\alpha, \alpha \in \Omega \), such that each \( T_\alpha \) has an identity element. Then \( T \) is an \( \mathcal{X} \)-semigroup if and only if each \( T_\alpha \) is an \( \mathcal{X}_\alpha \)-semigroup.

**Proof.** If \( T \) is an \( \mathcal{X} \)-semigroup then each \( T_\alpha \) is an \( \mathcal{X}_\alpha \)-semigroup since subsemigroups of \( \mathcal{X} \)-semigroups are \( \mathcal{X}_\alpha \)-semigroups. To prove the converse let \( a_1, \ldots, a_n \in T \). Suppose \( a_\alpha \in T_\alpha \) and let \( \alpha = \alpha_1 \cdots \alpha_n \) (the product in the semilattice \( \Omega \)). Then \( a = a_1 \cdots a_n \in T_\alpha \). Let \( e \) be the identity of \( T_\alpha \). Since \( \alpha \alpha = \alpha \) we have that \( a \in T_\alpha \) and \( a_\alpha e \) are in \( T_\alpha \), for \( i = 1, \ldots, n \). Furthermore \( e a_i = (e a_i) e = e (a_i e) = a_i e \). Now applying the hypothesis to \( T_\alpha \), there exists \( f = f_1 - f_2 \in \mathcal{X} \) such that \( f_1(a_\alpha e, \ldots, a_n e) = f_2(a_\alpha e, \ldots, a_n e) \). Since each \( x_i \) appears at least once in \( f_1 \) and \( f_2 \) and since \( \alpha \alpha = \alpha \), it follows that \( f_1(a_\alpha e, \ldots, a_n e) \) and \( f_2(a_\alpha e, \ldots, a_n e) \) are in \( T_\alpha \). Hence \( f_1(a_\alpha e, \ldots, a_n e) = f_2(a_\alpha e, \ldots, a_n e) = f_2(a_\alpha e, \ldots, a_n e) = f_2(a_\alpha e, \ldots, a_n e) = f_2(a_\alpha e, \ldots, a_n e) \). Therefore \( T \) is an \( \mathcal{X} \)-semigroup.

**Definition.** Let \( S \) be a semigroup, \( T \) a subsemigroup of \( S \). Then \( S \) is an \( n \)th inflation of \( T \) if there exists a homomorphism \( \theta: S \to T \) such that \( \theta \) is the identity map on \( T \) and for each \( a_1, \ldots, a_n \in S \), \( (a_1 \theta) \cdots (a_n \theta) = a_1 \cdots a_n \).

**Remark.** The 2nd inflation corresponds to the usual concept of inflation (cf. [1, p. 98]). If \( S \) is an \( n \)th inflation of \( T \) then \( (a_1 \theta) \cdots (a_m \theta) = a_1 \cdots a_m \) for all \( m \geq n \).

**Corollary 1.2.** Let \( \mathcal{X} \subseteq \mathcal{Y}_n \). Suppose \( S \) is an \( n \)th inflation of \( T \), that \( T \) is a semilattice of semigroups \( T_\alpha \), and that each \( T_\alpha \) is an \( \mathcal{X}_\alpha \)-semigroup with identity. Then \( S \) is an \( \mathcal{X}_n \)-semigroup.

**Proof.** Let \( a_1, \ldots, a_n \in S \). Then \( a_1, \ldots, a_n \in T \) so by Theorem 1.1 there exists \( f = f_1 - f_2 \in \mathcal{X} \) such that \( f_1(a_1 \theta, \ldots, a_n \theta) = f_2(a_1 \theta, \ldots, a_n \theta) \). By the above remark, for \( j = 1, 2, f_j(a_1, \ldots, a_n) = f_j(a_1 \theta, \ldots, a_n \theta) \) since each \( x_i \) appears at least once in \( f_j \). Thus \( f_1(a_1, \ldots, a_n) = f_1(a_1 \theta, \ldots, a_n \theta) = f_2(a_1 \theta, \ldots, a_n \theta) = f_2(a_1, \ldots, a_n) \). Hence \( S \) is an \( \mathcal{X}_n \)-semigroup.

**Remark.** The proof of Corollary 1.2 can be modified to prove that if \( S \) is an ideal extension of \( T \) by an \( \mathcal{X}_n \)-semigroup, then \( S \) is an \( \mathcal{X}_n \)-semigroup.

**Definition.** Let \( n, i, \alpha \) be positive integers \( n \geq 2 \), \( 1 \leq i \leq n \). Certain subsets of \( \mathcal{Y}_n \) are defined by:

1. \( \mathcal{L}_n = \{ f \mid f = f_1 - f_2 \in \mathcal{Y}_n; \ f_1 = x_1 \cdots x_n; \ x_j \) appears at least twice in \( f_2, j = 1, \ldots, n \} \).
2. \( \mathcal{M}_{n, \alpha} = \{ f \mid f = f_1 - f_2 \in \mathcal{Y}_n, f_1 = x_1 \cdots x_n, x_i \) appears at least \( \alpha \) times in \( f_2 \} \).
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(3) If \( \mathcal{X} \subseteq \mathcal{Y}_n \), then \( \overline{\mathcal{X}} = \{ f \mid f = f_1 - f_2 \in \mathcal{X}, f_2 \) starts with \( x_j, f \neq 1 \) and ends with \( x_k, k \neq n \}. \)

(4) \( S \) is an \( M_{\alpha}^{\infty} \)-semigroup if \( S \) is an \( M_{\alpha}^{\infty} \)-semigroup for all \( \alpha \geq 2 \).

Remark. Let \( \mathcal{X}, \mathcal{Y} \subseteq \mathcal{Y}_n \). Then \( \mathcal{X} \subseteq \mathcal{Y} \) implies that: (i) every \( \mathcal{X} \)-semigroup is a \( \mathcal{Y} \)-semigroup and (ii) \( \overline{\mathcal{X}} \subseteq \overline{\mathcal{Y}} \). Also subsemigroups and homomorphic images of \( \mathcal{X} \)-semigroups are \( \mathcal{X} \)-semigroups.

We will now prove several lemmas which are needed for the main theorems.

Let \( a_1, \ldots, a_n \in S \) and suppose \( x = a_1^{\mu_1} \cdots a_n^{\mu_n} \). We say that the length of \( x \) in the \( a_i \)'s is \( \sum_{i=1}^{n} \mu_i \).

Lemma 1.3. (i) Every \( M_n \)-semigroup is an \( M_{n,0} \)-semigroup.

(ii) Every \( M_{n,2} \)-semigroup is an \( M_{n,0} \)-semigroup.

Proof. (i) The proof follows by repeated application of the equation

\[ a_1 \cdots a_n = f_2(a_1, \ldots, a_n). \]

(ii) Since \( M_{n,\alpha} \subseteq M_n \) it suffices to show that for \( \alpha \geq 2 \), every \( M_{n,\alpha} \)-semigroup is an \( M_{n,\alpha + 1} \)-semigroup. Let \( a_1, \ldots, a_n \in S \). Then by (2) \( a_1 \cdots a_n = f_2(a_1, \ldots, a_n) \), where each \( a_i \) appears at least once on the right-hand side and \( a_i \) appears at least \( \alpha \) times. Since we can apply the hypothesis repeatedly, we may assume, without loss of generality, that the length of \( f_2 \) in the \( a_i \)'s is greater than \( 2n^2 \). There are two possibilities:

(i) \( f_2(a_1, \ldots, a_n) = u a_1 h_1 \cdots h_n \) with \( u, h_j \in \langle a_1, \ldots, a_n \rangle, a_i \) appearing in at least one \( h_j \) and the length of \( h_j \) in the \( a_k \)'s greater than or equal to \( n \).

(ii) \( f_2(a_1, \ldots, a_n) = h_1 \cdots h_n a u, h_i, u \) as in (i).

We assume (i), the proof for (ii) being similar. Applying the \( M_{n,\alpha} \)-hypothesis to \( (u a_1 h_1) h_2 \cdots h_n \) we have

\[ (u a_1 h_1) h_2 \cdots h_n = f_2(u a_1 h_1, h_2, \ldots, h_n) = h_{k_1} \cdots h_{k_n} u a_1 h_1 h_{k_1+1} \cdots h_{k_n} = g_1 \cdots g_{n} \]

where each \( g_i \) is a product of the \( a_i \)'s and \( a_i \) appears in at least one \( g_j \). Again applying the \( M_{n,\alpha} \)-hypothesis we have \( g_1 \cdots g_{i-1} a g_{i+1} \cdots g_{n} = f_3(g_1, \ldots, a, \ldots, g_n) \).

Now since \( a_i \) appears in some \( g_j \), \( a_i \) appears at least \( \alpha + 1 \) times on the right-hand side. Hence \( a_1 \cdots a_n = f_2(a_1, \ldots, a_n) = u a_1 h_1 \cdots h_n = f_3(g_1, \ldots, a, \ldots, g_n) \) and the proof is complete.

Lemma 1.4. Suppose that \( S \) is a semigroup with zero and that \( S \) is an \( M_{n,0}^{\infty} \)-semigroup with no nonzero idempotents. Then \( S^n = \{0\} \).

Proof. Let \( s \in S \). Letting \( a_1 = a_2 = \cdots = a_n \in S \), we have from (2) that \( s^n = s^k \), with \( k > n \). Hence \( s^n \in E(S) \) for some \( m, n \leq m \leq k \). Thus \( s^n = 0 \) and hence \( s^m = s^k = 0 \).

Let \( l \) be the least positive integer for which \( S^{l-1} s^l S^{n-l} = \{0\} \). We prove that the assumption \( l > 1 \) leads to a contradiction. Let \( t_1, \ldots, t_n \in S \). Let \( \alpha = n + 1 \) and define

\[ a_j = s^{l-1} t_j \quad \text{if} \quad 1 \leq j < i, \]

\[ a_j = s^{l-1} t_i s^{l-1} \quad \text{if} \quad j = i, \]

\[ a_j = t_i s^{l-1} \quad \text{if} \quad i < j \leq n. \]
Since $S$ is an $\mathcal{M}_n^\alpha$-semigroup we have $a_1 \cdots a_n = f_2(a_1, \ldots, a_n)$ with $a_i$ appearing at least $\alpha = n+1$ times on the right. It follows that $s^l$ appears at least $n$ times on the right. Isolating the $i$th $s^l$ we see that $f_2(a_1, \ldots, a_n) \in S^{n-i}S^n = \{0\}$. Thus $s^l = t_1 \cdots t_{l-1} t_n s^{l-1} \cdots t_n s^{l-1} = f_2(a_1, \ldots, a_n) = 0$. But the $t_i$'s are arbitrary in $S$. Hence $(s^{n-i}S^n)^{n+1} = 0$. Now apply the $\mathcal{M}_n^\alpha$ condition with $\alpha = 2n+2$ to the elements $a_j = t_j$, $j \neq i$ and $a_i = s^{l-1}$ where the $t_j$ are arbitrary elements of $S$. We have

$$t_1 \cdots t_{l-1} s^{l-1} t_{l+1} \cdots t_n = f_2(t_1, \ldots, t_{l-1}, s^{l-1}, t_{l+1}, \ldots, t_n) \in S^{l}(s^{l-1}S^n)^{n+1} = \{0\}.$$  

Since the $t_j$ are arbitrary, we conclude that $S^{l-1}S^{n-i} = \{0\}$, a contradiction. Hence $l = 1$ and $S^{l-1}S^{n-i} = \{0\}$, for every $s \in S$. Thus $S^n = \{0\}$.

**Corollary 1.5.** Let $S$ be a semigroup with zero and no nonzero idempotents. If $S$ is either an $\mathcal{L}_n$-semigroup or an $\mathcal{M}_n^{1,2}$-semigroup then $S^n = \{0\}$.

**Definition.** Let $S$ be a semigroup. $I = \mathcal{I}(S) = \{x \mid x \in S, x^l = x \text{ for some positive integer } l \geq 2\}$. $E = \mathcal{E}(S) = \{x \mid x \in S, x^2 = x\}$.

**Lemma 1.6.** If $S$ is an $\mathcal{L}_n$-semigroup or an $\mathcal{M}_n^{1,2}$-semigroup then $SE \cup ES \subseteq I$.

**Proof.** (i) Let $S$ be an $\mathcal{L}_n$-semigroup. We will show that $SE \subseteq I$; the proof that $ES \subseteq I$ is similar. Let $a \in S$, $e \in E$. Then by (1), there exists $f_k$ such that $ae = ae \cdots e = f_2(ae, e, \ldots, e) = e^k(ae)e^l = e^k(ae)^l$ where $l \geq 2$ and $k = 0$ or $k = 1$. If $k = 1$, $ae = e(ae)^l$ so that $iae = ae$. Hence $ae = e(ae)^l = (e(ae))e^r = (ae)^l$. Hence, for either $k = 0$ or $k = 1$, we obtain $ae = (ae)^l$, with $l \geq 2$. Consequently $ae \in I$.

(ii) Let $S$ be an $\mathcal{M}_n^{1,2}$-semigroup. Let $a \in S$, $e \in E$. From (2) and (3) it follows that $ae = ae \cdots e = es$ for some $s \in S$. Hence $ae = eae$. Now letting $a_i = e$ for $j \neq i$ and $a_i = ae$ we have, again applying (2) and (3), that

$$e \cdots e(ae)e \cdots e = f_2(e, \ldots, ae, \ldots, e) = e^k(ae)^l$$

where $k = 0$ or $l$ and $l \geq 2$. Hence $ae = eae = e^k(ae)^l = (e^r(ae)(ae)^l = (ae)^l$ and $ae \in I$, and so $SE \subseteq I$. Similarly $ES \subseteq I$.

**Lemma 1.7.** Let $S$ be an $\mathcal{L}_n$-semigroup or an $\mathcal{M}_n^{1,2}$-semigroup. Then $E \subseteq I$ and $I$ is an ideal. In particular, the Rees factor semigroup $S/I$ has no nonzero idempotents.

**Proof.** Either hypothesis implies that $S$ is periodic and hence $E \neq \emptyset$. Clearly $E \subseteq I$. Let $a \in I$, $x \in S$. Then there exists $l \geq 2$ such that $a = a^l$. Hence $a^{-1} \in E$. Consequently $ax = a^l x = a^l x = a^{-1}(ax) \in I$, by Lemma 1.6.

2. Main theorems.

**Theorem 2.1.** Let $S$ be either an $\mathcal{L}_n$-semigroup or an $\mathcal{M}_n^{1,2}$-semigroup. Then $S^n$ is a disjoint union of periodic groups.

**Proof.** By Lemma 1.7, $I$ is an ideal and $S/I$ has no nonzero idempotents. Since $S/I$ is a homomorphic image of $S$, $S/I$ is either an $\mathcal{L}_n$-semigroup or an $\mathcal{M}_n^{1,2}$-semigroup. By Corollary 1.5, $S^n \subseteq I$. Hence for all $x \in S^n$, there exists $l \geq 2$ such
that $x^1 = x$. It is well known (cf. [1, p. 23, Exercise 6a]), that this condition implies that $S^n$ is a disjoint union of periodic groups.

**Remark.** A semigroup $S$ is an $\mathcal{L}_n$-semigroup if and only if $S^n$ is a disjoint union of periodic groups. Theorem 2.4 shows that a union of periodic groups is not necessarily an $\mathcal{M}_n^{1,2}$-semigroup.

**Definition.** A semigroup $S$ is viable if $ab, ba \in E$ implies $ab = ba$. Idempotents are central in $S$ if $ae = ea$, for every $e \in E, a \in S$.

**Lemma 2.2.** (i) If $S$ is an $\mathcal{M}_n^{1,2}$-semigroup then idempotents are central in $S$.

(ii) For any semigroup $S$, if idempotents are central in $S$, then $S$ is viable.

(iii) Let $S$ be an $\mathcal{L}_n$-semigroup. Then $S$ is viable if and only if idempotents are central in $S$ if and only if idempotents commute in $S$.

**Proof.** (i) Let $e \in E, a \in S$. By (2) and (3), there exists $f_2$ such that $ea = e \cdots ea = se$ for some $s$ in $S$. Hence $ea = eae$. Similarly $ae = eae$. Thus $ea = ae$.

(ii) Suppose $ab, ba \in E$. Then $ab = (ab)(ab) = a(ba)b = (ba)(ab) = b(ab)a = (ba)(ba) = ba$.

(iii) By Theorem 2.1, $S^n$ is a union of groups. If $S$ is viable then $S^n$ is viable. By [3, Theorem 13], idempotents of $S^n$ are central in $S^n$. Let $e \in E, a \in S$. Then $e = e^n$; $ea = e^{n-1}a \in S^n$. Hence $ea = e(ea) = (ea)e$. Similarly $ea = ae$. Therefore $ea = ae$, showing idempotents are central in $S$. The converse follows from (ii). If idempotents commute in $S$ then (cf. [1, pp. 126, 127]), idempotents are central in $S^n$. The above argument shows idempotents are central in $S$. The following lemma is Lemma 3 of [2].

**Lemma 2.3.** Let $S$ be a semigroup with central idempotents. If $a_1, a_2 \in S$ and $e_1 \in \langle a_1 \rangle, e_2 \in \langle a_2 \rangle, e \in \langle a_1a_2 \rangle$ where $e, e_1, e_2 \in E$, then $e_1e_2 = e$.

**Theorem 2.4.** Let $S$ be a semigroup. The following are equivalent.

(i) $S$ is an $\mathcal{M}_n^{1,2}$-semigroup.

(ii) $S$ is an $\mathcal{M}_n^{1,2}$-semigroup.

(iii) $S$ is an $\mathcal{L}_n$-semigroup and idempotents commute.

(iv) $S$ is an $\mathcal{L}_n$-semigroup with central idempotents.

(v) $S$ is a viable $\mathcal{L}_n$-semigroup.

(vi) $S$ is an $n$th inflation of a semilattice of periodic groups.

**Proof.** (i) $\Rightarrow$ (ii) a fortiori.

(ii) $\Rightarrow$ (iii). By Theorem 2.1, $S^n$ is a disjoint union of periodic groups. Let $a_1, \ldots, a_n \in S$. Then $a_1 \cdots a_n \in S^n$, so there exists $k \geq 2$ such that $a_1 \cdots a_n = (a_1 \cdots a_n)^k$. Consequently $S$ is an $\mathcal{L}_n$-semigroup. By Lemma 2.2(i), idempotents are central in $S$ so surely they commute.

(iii) $\Rightarrow$ (iv) follows by Lemma 2.2(iii).

(iv) $\Rightarrow$ (v). By Lemma 2.2(ii). $S$ is viable.

(v) $\Rightarrow$ (vi). By Theorem 2.1, $S^n$ is a disjoint union of periodic groups. By Lemma 2.2(iii), idempotents are central in $S$, so certainly idempotents are central in $S^n$. 

Hence (cf. [1, Theorem 4.11]), $S^n$ is a semilattice of periodic groups. We show that $S$ is an $n$th inflation of $S^n$. Define $\theta: S \to S^n$ by: $a\theta = ae$ where $e \in \langle a \rangle$. Clearly $\theta$ is the identity on $S^n$. Also if $a, b \in S$, $e_1 \in \langle a \rangle$, $e_2 \in \langle b \rangle$, then by Lemma 2.3, $e_1 e_2 \in \langle ab \rangle$. Hence $(ab)\theta = abe_1 e_2 = (ae_1)(be_2) = (a\theta)(b\theta)$. Hence $\theta$ is a homomorphism. If $a_1, \ldots, a_n \in S$, then $(a_1 \theta) \cdots (a_n \theta) = (a_1 \cdots a_n)\theta = a_1 \cdots a_n$ since $\theta$ is the identity on $S^n$.

(vi) $\Rightarrow$ (i). Let $G$ be a periodic group with identity $e$ and let $a_1, \ldots, a_n \in G$. There exist integers $i > 1$, $j > 1$, $k > 1$ such that $a_1^i = e$, $a_2^j = e$, $(a_1 \cdots a_n)^k = (a_1 \cdots a_n)$. Thus $a_1 \cdots a_n = a_2^k (a_1 \cdots a_n)^k a_1^i$. Consequently $G$ is an $\mathcal{P}_n$-semigroup. Thus by Corollary 1.2, $S$ is an $\mathcal{P}_n$-semigroup.

**Corollary 2.5.** Let $\mathcal{X} \subseteq \mathcal{M}_n^{1, \infty}$. Then $S$ is an $\mathcal{X}$-semigroup if and only if $S$ is an $n$th inflation of a semilattice of $\mathcal{X}$-groups.

**Proof.** If $S$ is an $\mathcal{X}$-semigroup, then it is an $\mathcal{M}_n^{1, \infty}$-semigroup. Hence $S$ is an $n$th inflation of a semilattice of groups. Each of these groups, being a subgroup of $S$, is an $\mathcal{X}$-group. The converse follows from Corollary 1.2.

**Corollary 2.6.** Let $\mathcal{X} \subseteq \mathcal{M}_n^{1, \infty}$. Then $S^{n-1}$ is in the center of $S$ for every $\mathcal{X}$-semigroup $S$ if and only if

(i) $\mathcal{X} \subseteq \mathcal{M}_n^{1, \infty}$, and

(ii) every $\mathcal{X}$-group is abelian.

**Proof.** The necessity of (ii) is clear. If (i) does not hold then there exists $f = x_1 \cdots x_n - f_2 \in \mathcal{X}$ such that $f_2$ starts with $x_1$ or ends with $x_n$. Let $S_1$ (respectively $S_2$) be a nontrivial right-(left-)zero semigroup. Then either $S_1$ or $S_2$ is an $(f)$-semigroup and hence an $\mathcal{X}$-semigroup. But $S_1^{n-1} = S_1$ ($S_2^{n-1} = S_2$) which is not in the center of $S_1$ ($S_2$).

Conversely assume (i) and (ii) hold. By Corollary 2.5, $S$ is an $n$th inflation of a semilattice of $\mathcal{X}$-groups $G_a$. By (ii) each $G_a$ is abelian and hence satisfies every permutation identity in $n$-variables. By Corollary 1.2, letting $\mathcal{X}$ be any single permutation identity in $n$-variables, $S$ itself satisfies every permutation identity in $n$-variables. In particular $S^{n-1}$ is in the center.

**Remark.** In Corollary 2.6, we can replace the words “$S^{n-1}$ is in the center of $S$” with “$S$ satisfies every permutation identity in $n$-variables.”

Theorem 2.4 and Corollary 2.6 yield the main theorem and corollary of Tamura [4], when $n = 2$ and $\mathcal{X} = \{x_1 x_2 - f(x_1, x_2)\}$, $f$ a fixed monomial in $x_1$ and $x_2$. In addition Theorem 2.4 generalizes the main theorem of [2].

### 3. Examples and problems.

**Example 1.** Theorem 2.1 is not true for $\mathcal{M}_n^{1, \infty}$-semigroups. Let $S$ be the multiplicative semigroup of $2 \times 2$ matrices over $GF(2)$ consisting of $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then $S$ satisfies $x_1 x_2 = x_1^2 x_2$ for all $\alpha \geq 2$ and hence $S$ is an $\mathcal{M}_2^{1, \infty}$-semigroup. However $S^2 = S$ is not a union of groups.
Example 2. Theorem 1.1 is not true if the condition that every \( T_a \) has an identity is deleted. Let \( n = 2 \), \( \mathcal{X} = \{x_1x_2 - x_2x_1\} \). Let \( S \) be the semigroup given by:

\[
\begin{array}{ccc}
0 & a & b \\
0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 \\
b & 0 & a & b \\
\end{array}
\]

Let \( T_1 = \{0, a\} \), \( T_2 = \{b\} \). Then \( T_1 \) and \( T_2 \) are \( \mathcal{X} \)-semigroups and \( S \) is a semilattice of \( T_1 \) and \( T_2 \). But \( S \) is not an \( \mathcal{X} \)-semigroup.

Problem 1. Theorems 2.1 and 2.4 characterize \( \mathcal{L}_n \)-semigroups and \( \mathcal{M}_n^{1,2} \)-semigroups. Characterize other \( \mathcal{X} \)-semigroups. Study semigroups which are \( \{f\} \)-semigroups for some \( f \in \mathcal{X} \).

Problem 2. Is there a "nice" subset \( \mathcal{X} \) of \( \mathcal{Y}_n \) such that \( S \) is an \( \mathcal{X} \)-semigroup if and only if \( S^n \) is a band of groups?

References


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