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A GENERALIZED AREA THEOREM FOR HARMONIC FUNCTIONS ON HERMITIAN HYPERBOLIC SPACE

BY
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Abstract Let D be the noncompact realization of hermitian hyperbolic space. We consider functions on D which are harmonic with respect to the Laplace-Beltrami operator. The principal result is a generalized area theorem which gives a necessary and sufficient condition for the admissible convergence of harmonic functions.

1. **Introduction.** Let $D=\{z=(z_1,\dots,z_n)\in C^n : \operatorname{Im} z_1-\sum_2^n |z_k|^2>0\}$, which is the Cayley transform of the unit ball $\mathscr{D}$ in C^n. The basic machinery for studying harmonic analysis on D was developed in [3], [4], and [5]. In [6] Korányi defined the notion of admissible convergence in $\mathscr{D}$ and D and proved the following H^p result:

THEOREM (1.1). *If $f\in L^p(\partial D)$, $p\geqq 1$, and F is its Poisson integral, then F converges to f admissibly almost everywhere.*

The main object of this paper is to prove a generalized area theorem for harmonic functions in D and $\mathscr{D}$. For $n=1$, this is due to Marcinkiewicz and Zygmund [7] and Spencer [10]. This was extended to n real dimensions by Calderón [2] and Stein [11] for euclidean harmonic functions, and by Widman [13] to solutions of a large class of uniformly elliptic equations.

2. **Statement of the theorem.** Write $z\in C^n$ as $(z_1,\dots,z_n)$, $z_k=x_k+iy_k$,

$$\frac{\partial}{\partial z_k}=\frac{1}{2}\left(\frac{\partial}{\partial x_k}-i\frac{\partial}{\partial y_k}\right),\quad \frac{\partial}{\partial \bar z_k}=\frac{1}{2}\left(\frac{\partial}{\partial x_k}+i\frac{\partial}{\partial y_k}\right)\quad\text{and}\quad h(z)=y_1-\sum_2^n|z_k|^2.$$

Then $D=\{z\in C^n : h(z)>0\}$, $B=\partial D=\{z\in C^n : h(z)=0\}$, and the Laplace-Beltrami operator on D is

$$L=h(z)\left\{4y_1\frac{\partial^2}{\partial z_1\,\partial\bar z_1}+\sum_2^n\frac{\partial^2}{\partial z_k\,\partial\bar z_k}+2i\sum_2^n\bar z_k\frac{\partial^2}{\partial z_1\,\partial\bar z_k}-2i\sum_2^n z_k\frac{\partial^2}{\partial\bar z_1\,\partial z_k}\right\}.$$

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DEFINITION (KORÁNYI). If u is in B,

$$A_\alpha(u) = \left\{ z \in D : \left| i(\bar{z}_1 - u_1) - 2\sum_2^n u_k \bar{z}_k \right| < (1+\alpha)h(z),\, h(z) < 1 \right\}$$

is a (truncated) admissible domain of aperture $\alpha > 0$.

A function f defined on D converges admissibly at u to l if, for some $\alpha > 0$, $\lim_{z \to u; z \in A_\alpha(u)} f(z) = l$.

There are two groups of holomorphic automorphisms of D which will be used: $N = \{(a, c) : a \in R, c \in C^{n-1}\}$ acting by

$$(a, c)\cdot z = \left(z_1 + a + 2i \sum_2^n z_k \bar{c}_k + i \sum_2^n |c_k|^2, z_2 + c_2, \ldots, z_n + c_n \right)$$

and $S = \{t : t > 0\}$ acting by

$$t \cdot z = (tz_1, t^{1/2}z_2, \ldots, t^{1/2}z_n).$$

N acts simply transitively on B and $N \cdot S$ acts simply transitively on D. Thus if u is in B and z is in D, $u = (\operatorname{Re} u_1, \tilde{u})\cdot 0$ and $z = (x_1, \tilde{z})\cdot h(z) \cdot (i, 0, \ldots, 0)$ where $\tilde{w} = (w_2, \ldots, w_n)$.

Let

$$|\nabla f|^2 = 4h \left| \frac{\partial f}{\partial z_1} \right|^2 + \sum_2^n \left| 2i\bar{z}_k \frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_k} \right|^2.$$

THEOREM (2.1). *Let E be a measurable set in B and suppose f is a harmonic function on D.*

(a) *If f is admissibly bounded for each point of E then*

$$\int_{A_\alpha(u)} h(z)^{-n} |\nabla f|^2 \, d\mu(z) \tag{2.2}$$

is finite for almost every u in E and $\alpha > 0$, where μ is Lebesgue measure.

(b) *If, for each point u of E, there exists an $\alpha > 0$ such that the integral* (2.2) *is finite, then f converges admissibly at almost every point of E.*

The proof of (2.1)(b) which is outlined in [8] is incomplete. The original method shows that f is admissibly bounded at almost every point of E and then quotes the following result of [6] which is the analogue of theorems of Privalov and Calderón ([1], [14]).

THEOREM (2.3). *Let E be a measurable set in B and f harmonic in D. If for each u in E there exists an $\alpha > 0$ such that f is bounded in $A_\alpha(u)$, then f converges admissibly at almost every point of E.*

Korányi has pointed out that the proof of (2.3) in [6] is incorrect. Following a suggestion of C. Fefferman, Korányi outlined an independent proof of the last part of (2.1)(b) which he communicated to me, and that proof is included in the present paper. Thus (2.3) is now a corollary of (2.1).

3. **Structure of admissible domains.** For $u \in B$ there exists $g \in N$ such that $u = g \cdot 0$. Let $B_t(u) = g \cdot B_t(0)$ where $B_t(0) = \{v \in B : \text{Max}\,[|\text{Re}\, v_1|, \sum_2^n |v_k|^2] < t\}$.

In this section we use the following alternative admissible domains. If $\alpha > 0$,

$$\Gamma_\alpha(0) = \left\{z \in D : \text{Max}\left[|x_1|, \sum_2^n |z_k|^2\right] < \alpha h(z), h(z) < 1\right\},$$

$$\Gamma_\alpha(u) = \Gamma_\alpha(g \cdot 0) = g \cdot \Gamma_\alpha(0).$$

The following two results are in [6].

LEMMA (3.1). *For $\alpha > 0$, $A_\alpha(u) \subset \Gamma_{\alpha+1}(u)$ and $\Gamma_\alpha(u) \subset A_{2\alpha}(u)$.*

LEMMA (3.2). *Let E be a measurable set in B and u_0 a point of density of E with respect to the family of sets $\{B_t(u) : u \in B, t > 0\}$. Then for any $\alpha > 0$ and $\alpha_0 > 0$ there exists an $h_0 > 0$ such that*

$$\Gamma_{\alpha_0}(u_0) \cap \{z : h(z) < h_0\} \subset \bigcup_{u \in E} \Gamma_\alpha(u).$$

The next two lemmas enable us to interchange integration over admissible regions and integration over unions of admissible regions.

LEMMA (3.3). *Let $E \subset B$ be compact and $W_\alpha(E) = \bigcup_{u \in E} A_\alpha(u)$. Suppose f is non-negative and locally bounded on D and that $\int_{W_\alpha(E)} f\, d\mu < \infty$. Then $\int_{A_\gamma(u)} h^{-n} f\, d\mu < \infty$ for all $\gamma > 0$ and almost every u in E.*

Proof. By Lemma (3.1) the result is equivalent if $\Gamma_\alpha(u)$ replaces $A_\alpha(u)$ and $V_\alpha(E) = \bigcup_{u \in E} \Gamma_\alpha(u)$ replaces $W_\alpha(E)$. By Lemma (3.2) we may assume that $\gamma = \alpha$. Thus it is sufficient to prove that $I = \int_{u \in E} d\beta(u) \int_{\Gamma_\alpha(u)} h^{-n} f\, d\mu$ is finite, where

$$d\beta(u) = d(\text{Re}\, u_1)\, d(\text{Re}\, u_2)\, d(\text{Im}\, u_2) \cdots d(\text{Re}\, u_n)\, d(\text{Im}\, u_n).$$

(Throughout the paper "almost everywhere" statements are with respect to the measure β.)

$$I = \int_E d\beta(u) \int_{V_\alpha(E)} \chi_{\Gamma_\alpha(u)}(z) h(z)^{-n} f(z)\, d\mu(z)$$
$$= \int_{V_\alpha(E)} h(z)^{-n} f(z)\, d\mu(z) \int_E \chi_{\Gamma_\alpha(u)}(z)\, d\beta(u).$$

Now observe that

$$\int_E \chi_{\Gamma_\alpha(u)}(z)\, d\beta(u) = \beta(\{u : z \in \Gamma_\alpha(u)\} \cap E) \leqq \beta(\{u : z \in \Gamma_\alpha(u)\})$$
$$= \beta\left(\left\{u : \text{Max}\left[\left|\text{Re}\, z_1 - \text{Re}\, u_1 + 2\,\text{Im} \sum_2^n z_k \bar{u}_k\right|, \sum_2^n |z_k - u_k|^2\right] < \alpha h(z)\right\}\right)$$
$$= C(n) \alpha^n h(z)^n.$$

Thus $I \leqq C \int_{V_\alpha(E)} f\, d\mu$, which is finite by assumption.

LEMMA (3.4). *Let E be a compact set in B and f a locally bounded nonnegative function on D. Suppose that for each point u of E there exists an $\alpha>0$ such that $\int_{A_\alpha(u)} f\,d\mu<\infty$. Then if $\varepsilon>0$ and $\gamma>0$ there exists a compact set F contained in E such that $\beta(E-F)<\varepsilon$ and $\int_{W_\gamma(F)} h^n f\,d\mu<\infty$.*

Proof. Again we may replace $A_\alpha(u)$, $W_\alpha(E)$ by $\Gamma_\alpha(u)$, $V_\alpha(E)$. We may assume that α is fixed independently of u, and that $\int_{\Gamma_\alpha(u)} f\,d\mu \leqq M$, uniformly. Also, by Lemma (3.2) it is sufficient to prove the result for $0<\gamma<\alpha/2$. Given $\varepsilon>0$ there exists a compact set F contained in E and a $t_0>0$ such that $\beta(E-F)<\varepsilon$, and if $u \in F$ and $0<t\leqq t_0$ then $\beta(B_t(u)\cap E)\geqq \frac{1}{2}\beta(B_t(u))$. Now

$$I=\int_E d\beta(u)\int_{\Gamma_\alpha(u)} f\,d\mu\ <\ \infty,$$

$$\begin{aligned} I &= \int_E d\beta(u)\int_{V_\alpha(E)} \chi_{\Gamma_\alpha(u)}(z)f(z)\,d\mu(z)\\ &= \int_{V_\alpha(E)} f(z)\,d\mu(z)\int_E \chi_{\Gamma_\alpha(u)}(z)\,d\beta(u). \end{aligned}$$

If $z \in V_\gamma(F)$ there exists a v in F such that

$$\operatorname{Max}\left[\left|\operatorname{Re} z_1 - \operatorname{Re} v_1 + 2\operatorname{Im}\sum_2^n z_k\bar v_k\right|, \sum_2^n |z_k - v_k|^2\right] < \gamma h(z);$$

for such a z and v,

$$\begin{aligned} \int_E \chi_{\Gamma_\alpha(u)}(z)\,d\beta(u) &= \beta\left(\left\{u : \operatorname{Max}\left[\left|\operatorname{Re} z_1-\operatorname{Re} u_1+2\operatorname{Im}\sum_2^n z_k\bar u_k\right|, \sum_2^n |z_k-u_k|^2\right] < \alpha h(z)\right\}\cap E\right)\\ &\geqq \beta\left(\left\{u : \operatorname{Max}\left[\left|\operatorname{Re} v_1-\operatorname{Re} u_1+2\operatorname{Im}\sum_2^n v_k\bar u_k\right|, \sum_2^n |v_k-u_k|^2\right] < (\alpha/2-\gamma)h(z)\right\}\cap E\right)\\ &\geqq \tfrac{1}{2}\beta\left(\left\{u : \operatorname{Max}\left[\left|\operatorname{Re} v_1-\operatorname{Re} u_1+2\operatorname{Im}\sum_2^n v_k\bar u_k\right|, \sum_2^n |v_k-u_k|^2\right] < (\alpha/2-\gamma)h(z)\right\}\right)\\ &\qquad\qquad (\text{if } (\alpha/2-\gamma)h(z) \leqq t_0)\\ &= \tfrac{1}{2}C(n)(\alpha/2-\gamma)^n h(z)^n. \end{aligned}$$

Thus $I\geqq C\int_{V_\gamma(F)} h(z)^n f(z)\,d\mu(z)$ where $V_\gamma(F)$ is truncated at $(\alpha/2-\gamma)h(z)\leqq t_0$. The lemma then follows by the assumption that f is locally bounded in D.

If f is harmonic in D then f can be pulled back to $\mathscr{D}$ by the inverse Cayley transform. Thus if $\mathscr{D}_r=\{z \in \boldsymbol{C}^n : \sum_1^n |z_k|^2<r^2\}$, then the Poisson integral representation of the pulled back function on $\partial\mathscr{D}_r$ can be lifted to subdomains of D. Let Φ be the generalized Cayley transform $\Phi: \mathscr{D} \to D$ where

$$\Phi(z) = \left(i\frac{1+z_1}{1-z_1}, \frac{i\ \ z_2}{1-z_1}, \dots, \frac{i\ \ z_n}{1-z_1}\right).$$

If $w \in D$, consider $(\operatorname{Re} w_1, \tilde{w})=(\operatorname{Re} w_1, (w_2, \dots, w_n))$ and $h(w)$ as elements of N and S respectively, and let $\Phi_w=(\operatorname{Re} w_1, \tilde{w}) \circ h(w) \circ \Phi$. Then

$$\Phi_w(z)_1 = ih(w)\frac{1+z_1}{1-z_1}+\operatorname{Re} w_1-\frac{2h(w)^{1/2}}{1-z_1}\sum_2^n z_k\bar{w}_k+i\sum_2^n |w_k|^2,$$

$$\Phi_w(z)_k = \frac{ih(w)^{1/2}}{1-z_1} z_k+w_k, \qquad 2 \leqq k \leqq n.$$

Then $\Phi_w(\mathscr{D})=D$, $\Phi_w(0)=w$, and $\Phi_w(\mathscr{D}_r)=D_r(w)$ is the domain we are seeking. A direct computation gives

$$D_r(w) = \left\{z \in D : \left|i(\bar{w}_1-z_1)-2\sum_2^n z_k\bar{w}_k\right|^2 < \frac{4}{1-r} h(z)h(w)\right\}.$$

LEMMA (3.5). *Let $\alpha>\alpha'>0$ and $w \in A_{\alpha'}(u)$.*

(a) *There exists an $r>0$ such that if $z \in D_r(w)$ and $h(z)<1$, then $z \in A_\alpha(u)$.*

(b) *There exists a constant $M>0$, depending only on r, such that if $z \in D_r(w)$, then $(1/M)h(w) \leqq h(z) \leqq Mh(w)$.*

Proof. We may assume that $u=0$ since the domains $A_\alpha(u)$ and $D_r(w)$ are permuted under the action of N.

(a) Suppose $h(w)=\frac{1}{2}$. $A_\alpha(0)$ is an open set containing the compact set

$$C = \overline{A_{\alpha'}(0)} \cap \{w : h(w) = \tfrac{1}{2}\}.$$

For each w in C we can find an r such that $D_r(w)$ is contained in $A_\alpha(0)$. By compactness we can find an $r>0$ for all w in C. Now suppose $0<h(w)<1$. Since $w \in A_{\alpha'}(0)$ and $z \in D_r(w)$,

$$|w_1| < (1+a')h(w) \quad \text{and} \quad \left|i(\bar{w}_1-z_1)-2\sum_2^n z_k\bar{w}_k\right| < \frac{4}{1-r} h(z)h(w).$$

Consider $t=(2h(w))^{-1}$ as an element of S acting on z and w. Then

$$|(t\cdot w)_1| < (1+\alpha')h(t\cdot w) \quad \text{and}$$

$$\left|i((\overline{t\cdot w})_1-(\overline{t\cdot z})_1)-2\sum_2^n (t\cdot z)_k(\overline{t\cdot w})_k\right|^2 < \frac{4}{1-r} h(t\cdot z)h(t\cdot w).$$

Thus $t\cdot w \in C$ and $t\cdot z \in D_r(t\cdot w)$. By the above choice of r,

$$|(t\cdot z)_1| < (1+\alpha)h(t\cdot z).$$

Acting by t^{-1}, $|z_1|<(1+\alpha)h(z)$. Thus $z \in A_\alpha(0)$ if $h(z)<1$.

(b) Let C' be the closure of $\bigcup_{w\in C} D_r(w)$. Then C' is compact and for z in C' we can find an M such that $(1/M)h(w)\leqq h(z)\leqq Mh(w)$ for w in C. If $h(w)>0$, $z\in D_r(w)$, and t as above, then $t\cdot z\in C'$ and $t\cdot w\in C$. Thus $(1/M)h(t\cdot w)\leqq h(t\cdot z)\leqq Mh(t\cdot w)$ which gives $(1/M)h(w)\leqq h(z)\leqq Mh(w)$.

4. **Regions approximating $W_\alpha(E)$.** Write $z\in D$ as $z=[x,\tilde z,t]$ where $x=x_1$, $\tilde z=(z_2,\dots,z_n)$, $t=h(z)$. For compact $E\subset B$, $E_t=\{[x,\tilde z,t] : [x,\tilde z,0]\in E\}$ is compact. Let

$$A_\alpha(u)_{t^2} = \{[x,\tilde z,r+t^2] : [x,\tilde z,r]\in A_\alpha(u) \text{ and } r+t^2 < 1\}.$$

Then $\{A_\alpha(u)_{t^2}\cap E_t\}_{u\in E}$ forms an open cover of E_t. Choose a finite subcover for $t=t_0<1$, and then for each $t<t_0$ choose one in the following manner: If $u^1,\dots,u^{k(t)}$ are the base points chosen for the cover of E_t, and if $t'<t''\leqq t_0$ then

$$\{u^1,\dots,u^{k(t')}\}\supset\{u^1,\dots,u^{k(t'')}\}.$$

Let $W_t=\bigcup_{j=1}^{k(t)} A_\alpha(u^j)_{t^2}$.

LEMMA (4.1). (a) *$W_t\supset W_{t'}$ if $0<t<t'\leqq t_0$, and $\bigcup_{t>0} W_t=W_\alpha(E)$.*
(b) *∂W_t is a piecewise smooth surface.*
(c) *If ds is a surface measure then $\int_{\partial W_t} ds\leqq M$ independently of t.*

Proof. (a) and (b) are obvious by the construction. For (c) we divide the surface ∂W_t into two parts: $\partial W_{t,1}=\{z\in\partial W_t : h(z)=1\}$, the upper boundary, and $\partial W_{t,2}=\{z\in\partial W_t : h(z)<1\}$, the lower boundary. $\partial W_{t,1}$ is contained in a bounded piece of the surface $h(z)=1$, and thus $\int_{\partial W_{t,1}} ds\leqq M$. Let

$$\begin{aligned}\varphi_t([x,\tilde z,0]) &= 0 \quad \text{if } [x,\tilde z,r] \text{ is not in } W_t \text{ for any } r,\\ &= \min_{1\leqq j\leqq k(t)} \frac{1}{1+\alpha}\left| i(\bar z_1-u_1^j)-2\sum_1^n u_k^j\bar z_k\right|+t^2 \quad \text{otherwise.}\end{aligned}$$

Then

$$\begin{aligned}\partial W_{t,2} &= \{[x,\tilde z,\varphi_t([x,\tilde z,0])] : \varphi_t([x,\tilde z,0])\neq 0\}\\ &= \{(x_1,y_1,z_2,\dots,z_n) : y_1=\varphi_t([x,\tilde z,0])+\textstyle\sum |z_k|^2,\ \varphi_t\neq 0\}.\end{aligned}$$

For $y_1=\varphi_t([x,\tilde z,0])+\sum |z_k|^2$,

$$\frac{\partial y_1}{\partial x_1}=\frac{1}{1+\alpha}\,\frac{\operatorname{Im}[i(\bar z_1-u_1^j)-2\sum u_k^j\bar z_k]}{|i(\bar z_1-u_1^j)-2\sum u_k^j\bar z_k|},\qquad \frac{\partial y_1}{\partial z_k}=\frac{-\bar u_k^j[i(\bar z_1-u_1^j)-2\sum u_k^j\bar z_k]}{|i(\bar z_1-u_k^j)-2\sum u_k^j\bar z_k|}+\bar z_k.$$

On $\partial W_{t,2}$, $ds=(1+|\partial y_1/\partial x_1|^2+\sum_2^n |\partial y_1/\partial z_k|^2)^{1/2}\,dx_1\,dx_2\,dy_2\cdots dy_n$. Thus

$$d\beta\leqq ds\leqq M d\beta \tag{4.2}$$

where ds is the surface measure on $\partial W_{t,2}$ and $d\beta$ is the measure on the projection of $\partial W_{t,2}$ on B.

5. **Green's theorem.** Let R be a bounded domain in C^n with a smooth boundary and M be a second order differential operator

$$M = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.$$

Suppose $\psi(z)$ is a real valued C^∞ function defined in a neighborhood of R such that $R=\{z : \psi(z)>0\}$. We now define

$$Nf = \sum_{i,j} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}(a_{ij} f),$$

$$Pf = \frac{1}{|\text{grad } \psi|} \sum a_{ij} \left(\frac{\partial \psi}{\partial z_i} \frac{\partial f}{\partial \bar{z}_j} + \frac{\partial \psi}{\partial \bar{z}_j} \frac{\partial f}{\partial z_i} \right),$$

$$Q(z) = \frac{-1}{|\text{grad } \psi|} \sum \left(\frac{\partial}{\partial z_i} a_{ij} \frac{\partial \psi}{\partial \bar{z}_j} + \frac{\partial}{\partial \bar{z}_j} a_{ij} \frac{\partial \psi}{\partial z_i} \right),$$

where $|\text{grad } \psi| = [\sum_1^n |\partial\psi/\partial z_k|^2]^{1/2}$.

We then have the following (see e.g. Smirnov [9])

Green's Theorem. *If f and g are smooth functions in a neighborhood of R then*

$$\int_R \{gM(f) - fN(g)\}\, d\mu = \int_{\partial R} \{gP(f) - fP(g) + fgQ\}\, ds.$$

We will use the above with M replaced by $(1/h)L$, R replaced by W_t, f replaced by f^2, and g replaced by the function identically one. Since $W_t = \bigcup_{j=1}^{k(t)} A_\alpha(u^j)_{t^2}$ we can write $\partial W_{t,2} = \bigcup_{j=1}^{k(t)} \partial A_\alpha(u^j)_{t^2,2}$ where $\partial A_\alpha(u^j)_{t^2,2}$ is that part of $\partial A_\alpha(u^j)_{t^2}$ which is contained in $\partial W_{t,2}$. Let

$$\varphi_j(z) = (1+\alpha)(h(z)-t^2) - \left| i(\bar{z}_1 - u_1^j) - 2\sum_2^n u_k^j \bar{z}_k \right|.$$

Then $A_\alpha(u^j)_{t^2} = \{z : \varphi_j(z) > 0 \text{ and } h(z) < 1\}$ and

$$\partial A_\alpha(u^j)_{t^2} = \{z : \varphi_j(z) \geqq 0 \text{ and } h(z) = 1\}$$
$$\text{union } \{z : \varphi_j(z) = 0 \text{ and } h(z) < 1\}.$$

Let

$$\frac{\partial \psi}{\partial z_k}(z) = \frac{\partial \varphi_j(z)}{\partial z_k} \quad \text{if } z \in \partial A_\alpha(u^j)_{t^2,2},$$

$$= \frac{\partial h(z)}{\partial z_k} \quad \text{if } z \in \partial W_{t,1}.$$

This is well defined on ∂W_t except for a set of surface measure zero. We then have

$$P(f) = \frac{1}{|\text{grad }\psi|}\left(4y_1\frac{\partial\psi}{\partial\bar z_1}+2i\sum_2^n \bar z_k\frac{\partial\psi}{\partial\bar z_k}\right)\frac{\partial f}{\partial z_1}+\frac{1}{|\text{grad }\psi|}\left(4y_1\frac{\partial\psi}{\partial z_1}-2\text{i}\sum_2^n z_k\frac{\partial\psi}{\partial z_k}\right)\frac{\partial f}{\partial\bar z_1}$$

$$+\frac{1}{|\text{grad }\psi|}\sum_2^n\left(\frac{\partial\psi}{\partial\bar z_k}-2iz_k\frac{\partial\psi}{\partial\bar z_1}\right)\frac{\partial f}{\partial z_k}+\frac{1}{|\text{grad }\psi|}\sum_2^n\left(\frac{\partial\psi}{\partial z_k}+2i\bar z_k\frac{\partial\psi}{\partial z_1}\right)\frac{\partial f}{\partial\bar z_k},$$

$$Q(z) = \frac{-2ni}{|\text{grad }\psi|}\left(\frac{\partial\psi}{\partial z_1}-\frac{\partial\psi}{\partial\bar z_1}\right).$$

Finally, we observe that if $L(f)=0$, f real valued, then $(1/h)L(f^2) = 2|\nabla f|^2$. We thus obtain

GREEN'S THEOREM. $\int_{W_t} 2|\nabla f|^2\,d\mu=\int_{\partial W_t}\{P(f^2)+f^2\cdot Q\}\,ds$.

6. **Proof of Theorem (2.1)(a).** We may assume without loss of generality that f is uniformly bounded in $A_\alpha(u)$ for α fixed and all u in E, where E is compact. By Lemma (3.3) it is sufficient to prove that $\int_{W_\alpha(E)}|\nabla f|^2\,d\mu<\infty$. By Lemma (4.1)(a) it is sufficient to prove that $\int_{W_t}|\nabla f|^2\,d\mu\leqq M$ where M is independent of t. Using the formulation of Green's Theorem in §5, this is equivalent to showing that

$$\int_{\partial W_t}\{P(f^2)+f^2Q\}\,ds \leqq M.$$

Since f and Q are uniformly bounded, and $\int_{\partial W_t} ds\leqq M$ by Lemma (4.1)(c), it is sufficient to show that

$$\int_{\partial W_t} P(f)\,ds \tag{6.1}$$

is bounded independently of t.

LEMMA (6.2). *If f is bounded and harmonic in $A_\alpha(0)$, then $h(z)\,\partial f/\partial z_1$ and $h(z)^{1/2}\,\partial f/\partial z_k$, $2\leqq k\leqq n$, are bounded in $A_{\alpha'}(0)$ for $\alpha'<\alpha$.*

Proof. By Lemma (3.5)(a), there exists an $r>0$ such that $D_r(w)\subset A_\alpha(0)$ if $w\in A_{\alpha'}(0)$. As noted in §3, f has a Poisson integral representation (see Hua [4]):

$$f\circ\Phi_w(p) = \int_{\partial\mathscr{D}_r}\mathscr{P}_r(q,p)\,f\circ\Phi_w(q)\,ds(q)$$

where

$$\mathscr{P}_r(q,p) = cr((r^2-|p|^2)/|r^2-p\cdot\bar q|^2)^n, \qquad |q| = r,\ |p| < r.$$

Let $\rho=\Phi_w(p)$, $\xi=\Phi_w(q)$, and $P_r(\xi,\rho)=\mathscr{P}_r(\Phi_w^{-1}(\xi),\Phi_w^{-1}(\rho))|J\Phi_w^{-1}(\xi)|$. Then

$$f(\rho) = \int_{\partial D_r(w)} P_r(\xi,\rho)f(\xi)\,ds(\xi).$$

Assume $|p| \leqq r/2$; a straightforward computation then gives

$$|\partial P_r(\xi, \rho)/\partial \rho_1| \leqq ch(\rho)^{-1} P_r(\xi, \rho), \quad |\partial P_r(\xi, \rho)/\partial \rho_k| \leqq ch(\rho)^{-1/2} P_r(\xi, \rho), \quad k \geqq 2.$$

Thus

$$|\partial f(\rho)/\partial \rho_1| \leqq ch(\rho)^{-1} \sup_{\partial D_r(w)} |f| \int_{\partial D_r(w)} P_r(\xi, \rho)\, ds(\xi)$$

and, for $k \geqq 2$,

$$|\partial f(\rho)/\partial \rho_k| \leqq ch(\rho)^{-1/2} \sup_{\partial D_r(w)} |f| \int_{\partial D_r(w)} P_r(\xi, \rho)\, ds(\xi).$$

The lemma now follows since $\partial D_r(w) \subset A_\alpha(0)$ and f is bounded in $A_\alpha(0)$, and $\int_{\partial D_r(w)} P_r(\xi, \rho)\, ds(\xi) = 1$.

We return to the integral (6.1) which we write as $\int_{\partial W_{t,1}} P(f)\, ds + \int_{\partial W_{t,2}} P(f)\, ds$. The first integral is uniformly bounded since $\partial W_{t,1}$ is contained in a compact set in D. As in §5 we write $\partial W_{t,2} = \bigcup_{j=1}^{k(t)} \partial A_\alpha(u^j)_{t^2,2}$. If n is an element of the group N then $n \cdot \partial A_\alpha(u)_{t^2} = \partial A_\alpha(n \cdot u)_{t^2}$.

Claim. If $u_j = n_j \cdot 0$ where $n_j \in N$, then $|P(f \cdot n_j)| \leqq M$ on $\partial A_\alpha(0)_{t^2}$. If we have the claim, then

$$\begin{aligned} \int_{\partial W_{t,2}} |P(f)| ds &= \sum_{j=1}^{k(t)} \int_{\partial A_\alpha(u^j)_{t^2,2}} |P(f)|\, ds = \sum_{j=1}^{k(t)} \int_{n_j^{-1}(\partial A_\alpha(u^j)_{t^2,2})} |P(f \cdot n_j)|\, |Jn_j^{-1}|\, ds \\ &\leqq \sum_{j=1}^{k(t)} M \int_{n_j^{-1}(\partial A_\alpha(u^j)_{t^2,2})} |Jn_j^{-1}|\, ds = M \sum_{j=1}^{k(t)} \int_{\partial A_\alpha(u^j)_{t^2,2}} ds \\ &= M \int_{W_{t,2}} ds \leqq M' \end{aligned}$$

by (3) of Lemma (4.1). To complete the proof of part (a) we now verify the claim.

$$\partial A_\alpha(0)_{t^2,2} \subset \{z : \varphi_0(z) = (1+\alpha)(h(z) - t^2) - |z_1| = 0\},$$

and

$$\frac{\partial \varphi_0}{\partial z_1} = \frac{-i(1+\alpha)}{2} \pm \frac{1}{2}, \quad \frac{\partial \varphi_0}{\partial \bar{z}_1} = \frac{i(1+\alpha)}{2} \pm \frac{1}{2},$$

$$\frac{\partial \varphi_0}{\partial z_k} = -(1+\alpha)\bar{z}_k, \qquad \frac{\partial \varphi_0}{\partial \bar{z}_k} = -(1+\alpha) z_k.$$

In $A_\gamma(0)$, $y_1 \leqq Mh(z)$ and $|z_k| \leqq Mh(z)^{1/2}$. Thus there exist constants $m, M > 0$ such that $m \leqq |\text{grad}\, \varphi_0| \leqq M$, $|\partial \varphi_0/\partial z_1| \leqq M$, and $|\partial \varphi_0/\partial z_k| \leqq Mh(z)^{1/2}$. Using these estimates on the coefficients of P (see §5) we have

$$|P(f)| \leqq M\left(h(z)\left|\frac{\partial f}{\partial z_1}\right| + \sum_2^n h(z)^{1/2}\left|\frac{\partial f}{\partial z_k}\right|\right).$$

By Lemma (6.2), $|P(f)| \leqq M'$.

7. **Proof of Theorem (2.1)(b).** By Lemma (3.4) we have $\int_{W_\alpha(F)} |\nabla f|^2\, d\mu < \infty$, where $\beta(E\backslash F) < \varepsilon$, with $\varepsilon > 0$ arbitrary. To show the admissible convergence of f for almost every u in E, it is sufficient to show it for u in F. Define the regions W_t of §4 with E replaced by F. By Green's Theorem we thus have

$$\left| \int_{\partial W_t} [P(f^2) + f^2 \cdot Q]\, ds \right| \leqq M \tag{7.1}$$

with the bound independent of t.

LEMMA (7.2). *If $\int_{A_\alpha(0)} h(z)^{-n} |\nabla f|^2\, d\mu < \infty$, $0 < \alpha' < \alpha$ and $\varepsilon > 0$, then there exists a $t_0 > 0$ such that $|h(z)(\partial f/\partial z_1)| < \varepsilon$ and $|h(z)^{1/2}(\partial f/\partial z_k)| < \varepsilon$, $k \geqq 2$, for $z \in A_{\alpha'}(0)$ and $h(z) \leqq t_0$.*

Proof. Choose r, as in §3, such that $D_r(w) \subset A_\alpha(0)$ if $w \in A_{\alpha'}(0)$. Write

$$w = n \cdot h(w) \cdot (i, 0, \ldots, 0)$$

where $n \in N$ and $h(w) \in S$. Then

$$\mu(D_r(w)) = \int_{D_r(w)} d\mu = \int_{n \cdot h(w) \cdot D_r(0)} d\mu = \int_{D_r(0)} |Jn|\, |Jh(w)|\, d\mu.$$

For $n \in N$, $|Jn| = 1$, and for $t \in S$, $|Jt| = t^{n+1}$. Thus $\mu(D_r(w)) = Ch(w)^{n+1}$. Let $\delta > 0$. Since $\int_{A_\alpha(0)} h(z)^{-n} |\nabla f|^2\, d\mu < \infty$, there exists $t_0 > 0$ such that $h(w) \leqq t_0$, $w \in A_{\alpha'}(0)$, implies $\int_{D_r(w)} h(z)^{-n} |\nabla f|^2\, d\mu < \delta$. By Lemma (3.5) (b), $(1/M)h(w) \leqq h(z) \leqq Mh(w)$ and, therefore,

$$\frac{1}{\mu(D_r(w))} \int_{D_r(w)} h(z) |\nabla f|^2\, d\mu \leqq M\delta. \tag{7.3}$$

For any harmonic function g, the mean value theorem and Schwarz's inequality give

$$|g(w)|^2 \leqq \frac{1}{\mu(D_r(w))} \int_{D_r(w)} |g(z)|^2\, d\mu(z).$$

Now define the first order differential operators:

$$D_0 = z_1 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \frac{1}{2} \sum_{2}^{n} \left(z_k \frac{\partial}{\partial z_k} + \bar{z}_k \frac{\partial}{\partial \bar{z}_k} \right), \qquad D_1 = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \bar{z}_1},$$

$$D_k = -2i\bar{z}_k \frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial z_k}, \qquad \bar{D}_k = 2iz_k \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \bar{z}_k}.$$

A straightforward computation shows that these operators preserve harmonicity. Thus

$$\begin{aligned} h(w)^2 \left| \frac{\partial f}{\partial w_1} + \frac{\partial f}{\partial \bar{w}_1} \right|^2 &\leqq \frac{h(w)^2}{\mu(D_r(w))} \int_{D_r(w)} \left| \frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial \bar{z}_1} \right|^2 d\mu \\ &\leqq \frac{M}{\mu(D_r(w))} \int_{D_r(w)} h(z)^2 \left| \frac{\partial f}{\partial z_1} \right|^2 d\mu \leqq \frac{M}{\mu(D_r(w))} \int_{D_r(w)} h(z) |\nabla f|^2\, d\mu \\ &\leqq M\delta \quad \text{by (7.3).} \end{aligned}$$

Also

$$
\begin{aligned}
h(w)^2\left|\frac{\partial f}{\partial w_1}-\frac{\partial f}{\partial \bar{w}_1}\right|^2 &= \left|\left[D_0-\frac{1}{2}\sum_2^n(\bar{w}_k\bar{D}_k+w_kD_k)-\operatorname{Re} w_1D_1\right]f\right|^2\\
&\leqq M\left[|D_0f|^2+\sum_2^n h(w)|D_kf|^2+h(w)^2|D_1f|^2\right]\\
&\leqq \frac{M}{\mu(D_r(w))}\int_{D_r(w)}\left[|D_0f|^2+\sum_2^n h(z)|D_kf|^2+h(z)^2|D_1f|^2\right]d\mu\\
&\leqq \frac{M}{\mu(D_r(w))}\int_{D_r(w)}\left[h(z)^2\left|\frac{\partial f}{\partial z_1}\right|^2+\sum_2^n h(z)\left|\frac{\partial f}{\partial z_k}\right|^2\right]d\mu\\
&\leqq \frac{M}{\mu(D_r(w))}\int_{D_r(w)} h(z)|\nabla f|^2\,d\mu \leqq M\delta \quad \text{by (7.3)}.
\end{aligned}
$$

Combining the above two estimates, we have, if $w\in A_{\alpha'}(0)$ and $h(w)\leqq t_0$,

$$h(w)\left|\frac{\partial f}{\partial w_1}\right| \leqq M\delta.$$

The bound for $h(w)^{1/2}|\partial f/\partial w_k|$ is obtained by a similar argument.

In §6 the boundedness of $h(z)|\partial f/\partial z_1|$ and $h(z)^{1/2}|\partial f/\partial z_k|$ in $A_\alpha(0)$ was used to show that $\int_{\partial w_{t,2}}|Pf|\,ds\leqq M$. A repetition of that argument also gives

$$\int_{\partial w_{t,2}}|Pf|^2\,ds \leqq M. \tag{7.4}$$

From §5 recall that

$$
\begin{aligned}
Q(z) &= \frac{-2in}{|\operatorname{grad}\psi|}\left(\frac{\partial\psi}{\partial z_1}-\frac{\partial\psi}{\partial\bar{z}_1}\right) = \frac{2n}{|\operatorname{grad}\psi|}\frac{\partial\psi}{\partial y_1},\\
\frac{\partial\varphi_j}{\partial y_1} &= 1+\alpha+\frac{\operatorname{Re}\,[i(\bar{z}_1-u_1^j)-2\sum_2^n u_k^j\bar{z}_k]}{|i(\bar{z}_1-u_1^j)-2\sum_2^n u_k^j\bar{z}_k|}.
\end{aligned}
$$

Thus $Q(z)\geqq\varepsilon>0$ on $\partial w_{t,2}$. Using the fact that Q is bounded away from 0 in (7.1) gives

$$\int_{\partial w_{t,2}} f^2\,ds \leqq M\int_{\partial w_{t,2}}|fP(f)|\,ds+M.$$

An application of Schwarz's inequality and the bound (7.4) gives

$$\int_{\partial w_{t,2}} f^2\,ds \leqq M\left[\int_{\partial w_{t,2}} f^2\,ds\right]^{1/2}+M.$$

Thus

$$\int_{\partial w_{t,2}} f^2\,ds \leqq M \qquad \text{for } t>0. \tag{7.5}$$

Let

$$f_t(u) = f([\mathrm{Re}\, u_1, \tilde{u}, \varphi_t(u)]) \quad \text{if } \varphi_t(u) \neq 0,$$
$$= 0 \qquad\qquad \text{if } \varphi_t(u) = 0.$$

Then, using (4.2), (7.5) implies

$$\int_B f_t(u)^2\, d\beta(u) \leqq M. \tag{7.6}$$

Let F_t be the Poisson integral of $|f_t|$,

$$F_t(z) = \int_B P(u, z)|f_t(u)|\, d\beta(u) \quad \text{(see [6])}.$$

Claim (7.7). $|f(z)| \leqq MF_t(z) + M'$ for $z \in W_t$ where M and M' are independent of t.

By the maximum principle it is sufficient to prove this for $z \in \partial W_t$. Since $\partial W_{t,1}$ and $\partial W_{t,2} \cap \{z : h(z) \geqq t_0\}$ are contained in a compact set in D, the inequality can be made to hold there by choosing M' sufficiently large. Let $z \in \partial W_{t,2}$, with $h(z) < t_0$. Define $B_s([x, \tilde{z}, 0])$ as in §3 and let

$$E_s(z) = \{[\mathrm{Re}\, u_1, \tilde{u}, \varphi_t(u)] : [\mathrm{Re}\, u_1, \tilde{u}, 0] \in B_s([x, \tilde{z}, 0])\}.$$

Then there exists a $\delta > 0$, independent of t and z, such that $E_{\delta h(z)}(z) \subset D_r(z)$ and $E_{\delta h(z)} \subset \partial W_{t,2}$.

LEMMA (7.8). *If $w \in D_r(z)$ and $h(z) \leqq t_0$, then $|f(z) - f(w)| \leqq \varepsilon$.*

Proof. First observe that, translating by an element of N, we may assume $z, w \in A_\alpha(0)$ and the hypothesis of Lemma (7.2) is satisfied. If $h(z) = \frac{1}{2}$ and $w \in D_r(z)$ then there exists a constant M such that $|z_1 - w_1| < Mh(z)$ and $|z_k - w_k| < Mh(z)^{1/2}$, $k \geqq 2$. For $h(z) > 0$, let $t = (2h(z))^{-1}$ act on z and w as an element of the group S. Then, as in the proof of Lemma (3.5), we obtain $|z_1 - w_1| < Mh(z)$ and $|z_k - w_k| < Mh(z)^{1/2}$. Let γ be the line segment joining z and w.

$$\begin{aligned} |f(z) - f(w)| &\leqq |w_1 - z_1| \sup_\gamma \left|\frac{\partial f}{\partial \xi_1}\right| + \sum_2^n |w_k - z_k| \sup_\gamma \left|\frac{\partial f}{\partial \xi_k}\right| \\ &\leqq Mh(z) \sup_\gamma \left|\frac{\partial f}{\partial \xi_1}\right| + \sum_2^n Mh(z)^{1/2} \sup_\gamma \left|\frac{\partial f}{\partial \xi_k}\right|. \end{aligned}$$

The lemma now follows from (3.5)(b) and the estimates of (7.2).

We return to the proof of (7.7). Applying (7.8) to $w \in E_{\delta h(z)}(z)$ $(= E(z))$ we have

$$|f(z)| \leqq \frac{1}{|E(z)|} \int_{E(z)} |f(w)|\, ds + \varepsilon. \tag{7.9}$$

By (4.2), $|E(z)| = \int_{E(z)} ds \geqq \int_{B_{\delta h(z)}(z)} d\beta = M\delta^n h(z)^n$. Thus, rewriting (7.9) as an integral over $B_{\delta h(z)}(z)$ $(= B(z))$, we have

$$|f(z)| \leqq \frac{M}{h(z)^n} \int_{B(z)} |f_t(u)|\, d\beta + \varepsilon. \tag{7.10}$$

For $u \in B(z)$, $P(u, z) \geqq M/h(z)^n$. Since P is invariant under the action of N, it is sufficient to show this for $z = [0, 0]_{h(z)}$. Then it is immediate for

$$B(z) = \{u \in B : \text{Max}\,[|\text{Re}\, u_1|, \textstyle\sum |u_k|^2] < \delta h(z)\}$$

and

$$P(u, z) = \frac{ch(z)^n}{[(\text{Re}\, u_1)^2 + (h(z) + \sum |u_k|^2)^2]^n}.$$

Thus from (7.10) we have

$$|f(z)| \leqq M \int_{B(z)} P(u, z)|f_t(u)|\, d\beta + \varepsilon,$$

and the claim follows.

Now, using the uniform bound of (7.6), there exists a sequence $t_k \to 0$ such that $|f_{t_k}|$ converges weakly in $L^2(d\beta)$ to some function f_0 in $L^2(d\beta)$. Let

$$F_0(z) = \int_B P(u, z) f_0(u)\, d\beta.$$

Then by the claim, $|f(z)| \leqq MF_0(z) + M'$ for z in $W_\alpha(E)$. Since F_0 is a Poisson integral it is admissibly bounded for almost every u in E by Theorem (1.1). Thus f is admissibly bounded for almost every u in E.

It remains to be shown that f converges admissibly for almost every u in E. We use the regions $\Gamma_\alpha(u)$ and $V_\alpha(E)$ of §3, and assume that f is uniformly bounded in $V_\alpha(E)$. Let

$$V_t = \{z \in V_\alpha(E) : h(z) > t\}, \qquad \partial V_t = \partial V_{t,1} \cup \partial V_{t,2} \cup \partial V_{t,3}$$

where

$$\partial V_{t,1} = \{z \in \partial V_t : h(z) = 1\}, \qquad \partial V_{t,2} = \{z \in \partial V_t : h(z) = t\},$$
$$\partial V_{t,3} = \{z \in \partial V_t : t < h(z) < 1\}.$$

Let

$$\begin{aligned} E_t &= \{u \in B : [\text{Re}\, u_1, \tilde{u}, t] \in \partial V_{t,2}\}, \\ f_t(u) &= f([\text{Re}\, u_1, \tilde{u}, t]) \quad \text{if } u \in E_t, \\ &= 0 \qquad\qquad\qquad\quad \text{otherwise}, \\ F_t(z) &= \int_B P(u, z) f_t(u)\, d\beta(u), \\ w(z) &= \int_{B-E} P(u, z)\, d\beta(u), \\ u_t(z) &= \int_{E_t} P(u, z)\, d\beta(u). \end{aligned}$$

Claim. If $\varepsilon>0$ for t sufficiently small,

$$(7.11)\qquad |u_t(z)f(z)-F_t(z)| \leqq Mw(z)+M'h(z)^n+\varepsilon$$

for all z in V_t and some constants M and M'.

By the maximum principle it is sufficient to show this on ∂V_t. For $z \in \partial V_{t,1}$, the inequality holds by setting $M'=2\sup\{|f(z)| : z \in V_\alpha(E)\}$ and observing that $0<u_t(z)<1$ and $0<w(z)$. If $z\in\partial V_{t,3}$, then, as shown in [6], $w(z)\geqq C$ where $C>0$ depends only on α. Letting $M=M'/C$ gives (7.11) for such z.

Now consider $z\in\partial V_{t,2}$. Then

$$\begin{aligned}|u_t(z)f(z)-F_t(z)| &= \left|\int_{E_t} P(u,z)f(z)\,d\beta(u)-\int_{E_t} P(u,z)f_t(u)\,d\beta(u)\right|\\ &\leqq \int_{E_t} P(u,z)|f(z)-f_t(u)|\,d\beta(u)\\ &= \int_{E_t\cap B_{kh(z)}(z)} P(u,z)|f(z)-f_t(u)|\,d\beta(u)\\ &\quad +\int_{E_t\setminus B_{kh(z)}(z)} P(u,z)|f(z)-f_t(u)|\,d\beta(u)\\ &= I_1+I_2,\end{aligned}$$

and we estimate each of these.

$$I_2 \leqq M'\int_{B-B_{kh(z)}(z)} P(u,z)\,d\beta(u) = M'\int_{B-B_{kh(z)}([0,0,h(z)])} P(u,[0,0,h(z)])\,d\beta(u),$$

and since

$$P(u,[0,0,h(z)]) = Ch(z)^n\Big/\left((\operatorname{Re} u_1)^2+\left(h(z)+\sum_2^n |u_k|^2\right)^2\right)^n$$

and $B-B_{kh(z)}([0,0,h(z)])=\{u\in B : \operatorname{Max}\,[|\operatorname{Re} u_1|, \sum |u_k|^2]\geqq kh(z)\}$,

$$I_2 \leqq CM'/k^n.$$

Choose k sufficiently large so that $I_2\leqq\varepsilon/2$. We will now show that if t is sufficiently small, $z, w\in\partial V_{2,t}$, and $[\operatorname{Re} w_1, \tilde{w}, 0]\in B_{kh(z)}(z)$, then $|f(z)-f(w)|<\varepsilon/2$. This gives $I_1<\varepsilon/2$. Translating by an element of the group N, we have $z=[0,0,t]$, and $[\operatorname{Re} w_1, \tilde{w}, 0]\in B_{kh(z)}$ implies $\operatorname{Max}\,[|\operatorname{Re} w_1|, \sum_2^n |w_k|^2]<kt$. Since $w\in\partial V_{t,2}$, $w\in\Gamma_\alpha(u)$ for some u in E, that is,

$$\operatorname{Max}\left[\left|\operatorname{Re} w_1-\operatorname{Re} u_1+2\operatorname{Im}\sum_2^n w_k\bar{u}_k\right|, \sum_2^n |w_k-u_k|^2\right] < \alpha t.$$

Thus

$$\sum_2^n |u_k|^2 \leqq 2\left(\sum_2^n |w_k|^2+\sum_2^n |w_k-u_k|^2\right) < 2(kt+\alpha t)$$

and

$$\begin{aligned}|\mathrm{Re}\, u_1| &\leqq \left|\mathrm{Re}\, w_1 - \mathrm{Re}\, u_1 + 2\, \mathrm{Im} \sum_2^n w_k \bar{u}_k\right| + |\mathrm{Re}\, w_1| + 2\left|\sum_2^n w_k \bar{u}_k\right| \\ &< \alpha t + kt + 2\left(\sum_1^n |w_k|^2\right)^{1/2} \left(\sum_1^n |u_k|^2\right)^{1/2} \\ &< \alpha t + kt + 2k^{1/2} t^{1/2} 2^{1/2} (k+\alpha)^{1/2} t^{1/2} < 4(\alpha+k)t.\end{aligned}$$

Combining these inequalities, $\mathrm{Max}\, [|\mathrm{Re}\, u_1|\ \sum_2^n |u_k|^2] < 4(1+k/\alpha)\alpha t$, that is, $[0, 0, 4(1+k/\alpha)t] \in \Gamma_\alpha(u)$.

We have thus shown that the path γ lies in $V_\alpha(E)$, where γ consists of the two straight line segments: γ_1 connecting $[0, 0, t]$ and $[0, 0, 4(1+k/\alpha)t]$, and γ_2 connecting $[0, 0, 4(1+k/\alpha)t]$ and $[\mathrm{Re}\, w_1, \tilde{w}, t]$. Now choosing t sufficiently small and, repeating the argument of Lemma (7.8), $|f(z)-f(w)| < \varepsilon/2$. This completes the proof of (7.11).

The functions f_t are uniformly bounded and therefore there exists a sequence $t_k \to 0$ such that f_{t_k} converges weakly to some function f_0 in $L^\infty(d\beta)$. Let

$$F_0 = \int_B P(u, z) f_0(u)\, d\beta(u) \quad \text{and} \quad u_0(z) = \int_E P(u, z)\, d\beta(u).$$

Then (7.11) implies

$$|u_0(z) f(z) - F_0(z)| \leqq M w(z) + M' h(z)^n.$$

The right-hand side converges admissibly to zero almost everywhere on E, u_0 converges admissibly to one almost everywhere on E, and F_0 converges admissibly almost everywhere on E by Theorem (1.1). Thus f converges admissibly almost everywhere on E.

8. **The area theorem for the unit ball.** Theorem (2.1) is of local character, and may be pulled back to $\mathscr{D}$ by using the Cayley transform.

For $u \in \partial\mathscr{D}$ let $A'_\alpha(u) = \{z \in \mathscr{D} : |1 - \bar{z}\cdot u| < ((1+\alpha)/2)(1-|z|^2)\}$ be an admissible domain at u of aperture $\alpha > 0$. Define the gradient on $\mathscr{D}$ as

$$|\nabla' f|^2 = \sum_1^n \left|\frac{\partial f}{\partial z_k}\right|^2 - \left|\sum_1^n z_k \frac{\partial f}{\partial z_k}\right|^2.$$

THEOREM (8.1). *Let E be a measurable set in $\partial\mathscr{D}$ and suppose f is a real valued harmonic function on $\mathscr{D}$.*

(a) *If f is admissibly bounded for each point of E then*

$$\int_{A'_\alpha(u)} (1-|z|^2)^{-n} |\nabla' f|^2\, d\mu \tag{8.2}$$

is finite for all $\alpha > 0$ and almost every u in E.

(b) *If, for each point u of E, there exists an $\alpha > 0$ such that the integral* (8.2) *is finite, then f converges admissibly at almost every point of E.*

Proof. Let Φ^{-1} be the inverse Cayley transform. Then straightforward computations give

(1) If $u \in \partial D$ and $\alpha > 0$, there exist constants M, $M' > 0$ such that

$$A'_{M\alpha}(\Phi^{-1}u) \subset \Phi^{-1}A_\alpha(u) \subset A'_{M'\alpha}(\Phi^{-1}u).$$

(2) $|\nabla(f\Phi^{-1})|^2 = |1-z_1|^2|\nabla' f|^2$.

(3) $h(\Phi(z)) = (1-|z|^2)/|1-z_1|^2$.

(4) $|J\Phi| = k|1-z_1|^{-(2n+2)}$.

Let $g = f\Phi^{-1}$. Then (8.2) may be rewritten as $\int_{\Phi(A'_\alpha(u))} h(z)^{-n}|\nabla g|^2 \, d\mu$. By property (1), Theorem (2.1) can be applied to g. Pulling back to $\mathscr{D}$ gives the result.

9. **Remarks.** The present paper follows the general outline of the corresponding result in Stein [11]. Simultaneously and independently of the present work Stein [12] in fact proved an area theorem for holomorphic functions on bounded strictly pseudoconvex domains in C^n. The intersection of [12] and this paper is Theorem (8.1) for holomorphic functions.

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