

ABSTRACT EVOLUTION EQUATIONS AND THE MIXED PROBLEM FOR SYMMETRIC HYPERBOLIC SYSTEMS

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Abstract. In this paper we show that Kato's theory of linear evolution equations may be applied to the mixed problem for first order symmetric hyperbolic systems of partial differential equations.

1. Introduction. This paper is concerned with the mixed problem for the following symmetric hyperbolic system of partial differential equations:

$$(1.1) \quad \begin{aligned} \frac{\partial u}{\partial t} + \sum_{j=1}^m a_j(x, t) \frac{\partial u}{\partial x_j} + b(x, t)u &= f(x, t), & x \in \Omega, \quad 0 \leq t \leq T; \\ u(x, 0) &= \phi(x), & x \in \Omega; \\ u(x, t) &\in P(x, t), & x \in \Gamma, \quad 0 \leq t \leq T. \end{aligned}$$

The unknown $u = (u_1, \dots, u_N)$ is a real vector-valued function, the coefficients, a_j and b , are real $N \times N$ matrix-valued functions, and the a_j are symmetric. We assume a_j and b are of class C^2 and C^1 on $\bar{\Omega} \times [0, T]$, respectively. Ω is a bounded open subset of R^m with boundary Γ of class C^3 .

The results are restricted to the regular case where the *boundary matrix*

$$a_n(x, t) = \sum n_j(x) a_j(x, t), \quad x \in \Gamma, \quad 0 \leq t \leq T,$$

is nonsingular on $\Gamma \times [0, T]$. Here $n = (n_1, \dots, n_m)$ is the exterior unit normal to Ω . Limits of summation will be omitted when they are clear from the context.

The *boundary subspace* $P(x, t)$ is a linear subspace of R^N which varies in a C^3 manner with $(x, t) \in \Gamma \times [0, T]$, and it is *maximal nonnegative* for each x, t . This means

$$(a_n(x, t)u, u) \geq 0, \quad u \in P(x, t),$$

and $P(x, t)$ is not contained in any other subspace of R^N having this property. Here (u, v) denotes the usual inner product of $u, v \in R^N$.

Received by the editors May 25, 1971.

AMS 1970 subject classifications. Primary 35L50; Secondary 34G05, 35F15, 35L40, 35L45, 35R20, 47A50, 47D05.

Key words and phrases. Symmetric hyperbolic systems, evolution equations, mixed problem, C_0 -semigroups, Cauchy problem.

⁽¹⁾ This research was supported by Air Force Office of Scientific Research grant AF-AFOSR-68-1462 and National Science Foundation grant NSF GP-8257.

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Symmetric hyperbolic systems have been studied by Friedrichs ([4], [5]), Lax and Phillips [10], Cordes and Moyer [2], and others. Here we treat equation (1.1) using Kato's results [9] on linear evolution equations of the form

$$du/dt + A(t)u = f(t), \quad 0 \leq t \leq T,$$

where the linear operators $-A(t)$ are the infinitesimal generators of C_0 -semigroups in a Banach space. Kato established sufficient conditions for the existence of the evolution operator $\{U(t, s) : 0 \leq s \leq t \leq T\}$ for the family $\{A(t)\}$, and he applied these results to the equation (1.1) in the case where $\Omega = R^m$ is the whole space. In this paper we extend Kato's method to the case where Ω is a bounded domain.

Before proceeding further, we make some remarks concerning notation. We shall reserve the terminology *vector-valued* function for a function $u = (u_1, \dots, u_N)$ which has N real-valued components. X and $L^2(\Omega)$ will both be used to denote the Hilbert space of square-integrable vector-valued functions on Ω . $H^k(\Omega)$, $k=0, 1, \dots$, is the Sobolev space of those $u \in X$ whose partial derivatives of order up to k also lie in X . $H^{-k}(\Omega)$ is the dual space of $H_0^k(\Omega)$, where $H_0^k(\Omega)$ is the closure in $H^k(\Omega)$ of the set of C^∞ vector-valued functions with compact support in Ω . The spaces $L^2(R^m)$, $H^k(R^m)$, $H^{-k}(R^m)$ are the corresponding spaces on R^m . $L^2(\Gamma)$ is the space of vector-valued functions on Γ which are square integrable with respect to the natural surface measure on Γ . The Sobolev space $H^k(\Gamma)$, $0 \leq k \leq 3$, consists of those vector-valued functions on Γ which coincide on coordinate neighborhoods with functions in $H^k(R^{m-1})$, and $H^{-k}(\Gamma)$ is the dual space to $H^k(\Gamma)$. Unless otherwise stated, Sobolev spaces H^k are of integral order k , so $0 \leq k \leq 3$, for example, means $k=0, 1, 2, 3$. We use $\| \cdot \|$ and (\cdot, \cdot) to denote the norm and inner product in X , $L^2(R^m)$, or $L^2(\Gamma)$, and $\| \cdot \|_k$ to denote the norm in $H^k(\Omega)$, $H^k(R^m)$, or $H^k(\Gamma)$. The underlying space, Ω , R^m , or Γ , should be clear from the context.

If $u \in H^k(\Omega)$, $1 \leq k \leq 3$, then $u_0 \in H^{k-1}(\Gamma)$ denotes the trace of u on Γ . If $P(x)$, $x \in \Gamma$, is a linear subspace of R^N which varies continuously with x , then $H_P^1(\Omega)$ denotes the closed subspace of $H^1(\Omega)$ consisting of those u which satisfy the boundary conditions $u_0(x) \in P(x)$ for (almost all) $x \in \Gamma$. For basic properties of Sobolev spaces, see Hörmander [6], Lions and Magenes [11], Morrey [12], and Seeley [13].

If $u \in R^N$, then $|u|$ denotes the usual Euclidean norm of u . If a is an $N \times N$ matrix, then $|a| = \sup \{|au| : |u| = 1\}$, and ${}^t a$ is the transpose of a .

If X_1, X_2 are Banach spaces, then $B(X_1, X_2)$ denotes the space of bounded operators from X_1 to X_2 , and $B(X_1) \equiv B(X_1, X_1)$.

Let $A_0(t)$ be the operator defined by

$$A_0(t)u = \sum a_j(x, t)D_j u + b(x, t)u,$$

with domain, $D(A_0(t))$, equal to $H_{P_t}^1(\Omega)$. Here $P_t(x) \equiv P(x, t)$ and $D_j \equiv \partial/\partial x_j$. Let $A(t)$ denote the closure of $A_0(t)$ regarded as an unbounded operator in X . Friedrichs [5] and Lax and Phillips [10] have shown that $A(t) + \beta_t$ is m -accretive if

$$(1.2) \quad \beta_t = \sup \{|b'(x, t)| : x \in \Omega\},$$

where $b' = \frac{1}{2}(b + {}^t b - \sum \partial a_j / \partial x_j)$. (Recall that an operator A in a real Hilbert space X is *accretive* if $(Au, u) \geq 0$ for all $u \in D(A)$. A is *m-accretive* if $A + \lambda$ has range X for all $\lambda > 0$. This implies $-A$ generates a C_0 -semigroup of contractions in X .) Thus $-A(t)$ generates a C_0 -semigroup in X .

The following are the main results.

THEOREM 1. *There exists an isomorphism $S(t)$ from $H_{P_i}^1(\Omega)$ onto X such that*

$$S(t)A(t)S(t)^{-1} = A(t) + B(t),$$

where $B(t)$ is a bounded operator on X .

REMARK. According to Proposition 2.4 of [9], Theorem 1 implies that the subspace $H_{P_i}^1(\Omega)$ is *admissible* with respect to $A(t)$. This means (see [9, Definition 2.1]) the semigroup generated by $-A(t)$ leaves $H_{P_i}^1(\Omega)$ invariant and forms a C_0 -semigroup in this space.

THEOREM 2. *If $P(x, t) = P(x)$ does not vary with t , then $S(t)$ in Theorem 1 may be chosen so that it is continuously differentiable on $[0, T]$ to $B(H_{P_i}^1(\Omega), X)$ and $B(t)$ is continuous on $[0, T]$ to $B(X)$.*

REMARK. Suppose that $P(x, t) = P(x)$ does not vary with t so that Theorems 1 and 2 are true. Then Theorems 4.1 and 6.1 of [9] may be applied to the family $\{A(t)\}$, taking $H_{P_i}^1(\Omega)$ for the subspace Y in those two theorems. Note that the stability condition (i) of Theorem 4.1 is true with $M = 1$ and $\beta = \sup_i \{\beta_i\}$, since the operators $A(t) + \beta_i$ are *m-accretive*. The condition (iii) of Theorem 4.1 is also easily seen to be true.

We then have the following result for equation (1.1).

THEOREM 3. *Suppose $\phi \in H_{P_0}^1(\Omega)$ and the map $t \rightarrow f(\cdot, t)$ is continuous on $[0, T]$ to $H^1(\Omega)$ so that $f(\cdot, t)$ belongs to $H_{P_i}^1(\Omega)$ for $0 \leq t \leq T$. Then (1.1) has a unique solution $u(x, t)$ such that the map $t \rightarrow u(\cdot, t)$ is continuously differentiable on $[0, T]$ to X and $u(\cdot, t)$ belongs to $H_{P_i}^1(\Omega)$ for $0 \leq t \leq T$.*

REMARK. If $P(x, t) = P(x)$ is independent of t , then the conclusions of Theorem 3 follow directly from Theorem 7.1 of [9]. We shall show that the general case where $P(x, t)$ varies with t may be reduced to the case $P(x, t) = P(x, 0)$ by an orthogonal transformation of the dependent variables.

The remainder of the paper is devoted to proving the above results. In §2 we construct the operator $S(t)$. §3 contains inequalities involving commutators. These are used in §4 to show that $S(t)$ is an isomorphism from $H_{P_i}^1(\Omega)$ onto X and to establish a regularity result for $S(t)$. Theorems 1, 2, and 3 are proved in §5.

The author wishes to thank Professor T. Kato, who suggested this problem, for his assistance and guidance in this research.

2. Construction of the operator S . We first consider Theorem 1. Here the variable t is only a parameter, and we shall omit it in the discussion and simply write $a_j(x)$, S, \dots , for $a_j(x, t)$, $S(t)$, etc.

By the Stone-Weierstrass Theorem, the set of all real-valued functions of class C^3 on Γ is dense in the Banach space of real-valued, continuous functions on Γ . It follows that, given any $\varepsilon > 0$, there exists a C^3 vector field ν such that $|-n(y) - \nu(y)| \leq \varepsilon$ for all $y \in \Gamma$, where n is the exterior unit normal to Γ . We choose such a ν so that this inequality is satisfied for some $\varepsilon < 1$. Then $\nu(y)$ points into the interior of Ω for each $y \in \Gamma$, i.e. $(\nu(y), n(y)) < 0$.

Using this ν , we introduce new coordinates near Γ as follows. Let $\omega = \Gamma \times [0, \sigma]$, where σ is chosen small enough so that certain conditions stated below are satisfied. Consider the mapping $\omega \rightarrow R^m$ defined by

$$(2.1) \quad (y, s) \rightarrow y + s\nu(y), \quad (y, s) \in \omega.$$

Using the fact that $\nu(y)$ is nowhere tangential to Γ , it follows that the derivative of (2.1) is nonsingular for $s=0$. Using the inverse function theorem, one may then show that (2.1) is a diffeomorphism if σ is chosen sufficiently small. Denoting the range of (2.1) by Ω' (note that $\Omega' \subset \bar{\Omega}$), the inverse $\Omega' \rightarrow \omega$ has the form

$$(2.2) \quad x \rightarrow (y(x), s(x)), \quad x \in \Omega'.$$

Thus, $y(x) \in \Gamma$ and $s(x) \in [0, \sigma]$ may be thought of as new coordinates for $x \in \Omega'$.

The matrix

$$(2.3) \quad c(x) = \sum a_j(x)(\partial s / \partial x_j), \quad x \in \Omega',$$

has the property that it is a strictly negative scalar multiple of the boundary matrix $a_n(x)$ for $x \in \Gamma$. This is because the vector $(\partial s / \partial x_1, \dots, \partial s / \partial x_m)$ is an interior normal to Ω , since $\Gamma = \{x \in \Omega' : s(x) = 0\}$. Since a_n is nonsingular, σ may be chosen so that $c(x)$ is nonsingular for $x \in \Omega'$.

The spaces $L^2(\Omega')$ and $H^k(\Omega')$ are defined in the same way as $L^2(\Omega)$ and $H^k(\Omega)$. $L^2(\omega)$ denotes the space of vector-valued functions on $\omega = \Gamma \times [0, \sigma]$ which are square integrable with respect to the product measure on ω . We consider ω as a compact, C^3 manifold-with-boundary, and we shall use the Sobolev spaces $H^k(\omega)$, $-3 \leq k \leq 3$. For the definition and basic properties of these spaces, see Hörmander [6].

Let the operators $U_0, U: L^2(\Omega') \rightarrow L^2(\omega)$ be defined by

$$(2.4) \quad \begin{aligned} U_0 u(y, s) &= u(y + s\nu(y)), & (y, s) \in \omega, \\ U u &= U_0 h u, \end{aligned}$$

for $u \in L^2(\Omega')$. Here $h(x) = |j(x)|^{-1/2}$, where $j(x)$ is the Jacobian of the mapping (2.2). Since the map (2.1) is of class C^3 , U_0 is an isomorphism from $H^k(\Omega')$ onto $H^k(\omega)$ for $0 \leq k \leq 3$. However h is only of class C^2 , so U is an isomorphism between $H^k(\Omega')$ (resp. $H_0^k(\Omega')$) and $H^k(\omega)$ (resp. $H_0^k(\omega)$) only for $0 \leq k \leq 2$. Using the change of variables formula for integrals, one sees that U is unitary from $L^2(\Omega')$ to $L^2(\omega)$. By duality, U extends to an isomorphism between $H^k(\Omega')$ and $H^k(\omega)$ for $k = -1, -2$.

Let $\phi^4 + \psi^4 = 1$ be a C^3 partition of unity for R^m with the following properties:

- (i) $\phi = 1$ and $\psi = 0$ in a neighborhood of $\Omega \sim \Omega'$.
- (ii) The support of ϕ is compact in Ω , so $\phi = 0$ and $\psi = 1$ near Γ .
- (iii) For $x \in \Omega'$, $\phi(x)$ and $\psi(x)$ depend only on $s(x)$. The reason for this last assumption will be discussed in a moment.

In the definition of S we shall use certain matrix-valued functions defined on Γ which locally transform the boundary subspace $P(y)$ into a subspace which does not vary with y . To construct these functions, we shall use the following lemma.

LEMMA 1. *Given $y_0 \in \Gamma$, there exists a neighborhood \mathcal{U} of y_0 (with respect to Γ) and an orthogonal matrix-valued function $r \in C^2(\Gamma)$ such that, for $y \in \mathcal{U}$, $r(y)$ maps $P(y)$ onto $P = \{u \in R^N : u_1 = \dots = u_p = 0\}$, where p is the common codimension of $P(y)$ for y belonging to the connected component of Γ which contains y_0 .*

Proof. Clearly there exists \mathcal{U} and r of class C^2 on \mathcal{U} with the property that $r(y)P(y) = P$ for $y \in \mathcal{U}$. The problem is to extend r to all of Γ . This can be done by modifying r near $\partial\mathcal{U}$ so that it is equal to $r(y_0)$ there. For example, by shrinking the neighborhood \mathcal{U} , if necessary, and introducing local coordinates, we may assume that we are working in R^{m-1} , $y_0 = 0$, and \mathcal{U} is the ball about y_0 of radius 1. We choose a C^∞ real-valued function ρ on $[0, 1]$ with the property that $\rho(t) = 1$ for t near 0 and $\rho(t) = 0$ for t near 1. Then $r'(y) \equiv r(\rho(|y|)y)$ has the desired properties. \square

Using this lemma, we can find an open covering $\mathcal{U}_1, \dots, \mathcal{U}_K$ of Γ together with orthogonal matrix-valued functions $r_1, \dots, r_K \in C^2(\Gamma)$ such that, for $k = 1, \dots, K$, and $y \in \mathcal{U}_k$, $r_k(y)$ maps $P(y)$ onto

$$(2.5) \quad P_k = \{u \in R^N : u_1 = \dots = u_{p_k} = 0\}.$$

(p_k is the same for those \mathcal{U}_k lying in the same component of Γ .)

We choose a partition of unity for Γ

$$(2.6) \quad \sum_{k=1}^K \zeta_k^2 = 1$$

such that ζ_k has support in \mathcal{U}_k for each k .

Let Δ_Γ denote the Laplace-Beltrami operator on Γ . This is a negative self-adjoint operator in $L^2(\Gamma)$ if we choose $D(\Delta_\Gamma) = H^2(\Gamma)$. Let $\Lambda_\Gamma = (1 - \Delta_\Gamma)^{1/2}$. Then Λ_Γ is an isomorphism from $H^k(\Gamma)$ onto $H^{k-1}(\Gamma)$ for $1 \leq k \leq 3$. By duality Λ_Γ extends to an isomorphism from $H^k(\Gamma)$ onto $H^{k-1}(\Gamma)$ for $-2 \leq k \leq 0$.

We shall frequently use the natural correspondence whereby a function $u(y, s)$ on $\omega = \Gamma \times [0, \sigma]$ is regarded as a function $u(\cdot, s)$ on $[0, \sigma]$ whose values are functions on Γ . Under this correspondence $L^2(\omega)$ is naturally isomorphic to $L^2([0, \sigma]; L^2(\Gamma))$, the space of square-integrable functions on $[0, \sigma]$ with values in $L^2(\Gamma)$. We also have

$$(2.7) \quad H^k(\omega) \cong H^0([0, \sigma]; H^k(\Gamma)) \cap \dots \cap H^k([0, \sigma]; H^0(\Gamma)),$$

for $k=1, 2, 3$. Using this correspondence Λ_Γ may be regarded as a bounded operator from $H^k(\omega)$ to $H^{k-1}(\omega)$ for $1 \leq k \leq 3$. Λ_Γ maps $H_0^k(\omega)$ into $H_0^{k-1}(\omega)$, so, by duality, Λ_Γ extends to a bounded operator from $H^k(\omega)$ to $H^{k-1}(\omega)$ for $-2 \leq k \leq 0$.

Let $M=U^{-1}\Lambda_\Gamma U$. Then M is a bounded operator from $H^k(\Omega')$ to $H^{k-1}(\Omega')$ for $-1 \leq k \leq 2$. Since U is unitary, M is symmetric and bounded below by 1 if it is considered as an operator in $L^2(\Omega')$ by restricting its domain to $H^1(\Omega')$. If we regard ϕ and ψ as functions on Ω' , then it follows from the assumption (iii) above that $M\phi u = \phi Mu$ for $u \in H^{-1}(\Omega')$ and similarly for ψ .

Let $\Lambda=(1-\Delta)^{1/2}$ where Δ is the Laplacian in the whole space R^m . Λ is an isomorphism from $H^k(R^m)$ onto $H^{k-1}(R^m)$ for all k .

The expressions $u \rightarrow \phi \Lambda \phi u$ and $u \rightarrow \psi \zeta_k r_k^{-1} M r_k \zeta_k \psi u$ define bounded operators from $H^k(\Omega)$ to $H^{k-1}(\Omega)$ for $-2 \leq k \leq 3$ in the first case and for $-1 \leq k \leq 2$ in the second. We are regarding multiplication by ϕ as a bounded operator from $H^k(\Omega)$ to $H^k(R^m)$ and also from $H^k(R^m)$ to $H^k(\Omega)$ since it has compact support in Ω . In the second case ψ vanishes in a neighborhood of $\Omega \sim \Omega'$ so that it is a bounded operator from $H^k(\Omega)$ to $H^k(\Omega')$ and also from $H^k(\Omega')$ to $H^k(\Omega)$.

The operator S is now defined by

$$(2.8) \quad Su = (A_0 + \beta)u + \phi \Lambda \phi u + \sum \psi \zeta_k r_k^{-1} M r_k \zeta_k \psi u,$$

for $u \in D(S) \equiv H^{\frac{1}{2}}(\Omega)$. Here $\beta = \beta_t$ has the value given by (1.2). One sees that S is a bounded operator from $H^{\frac{1}{2}}(\Omega)$ to X . We shall show in §§4 and 5 that S fulfills the requirements of Theorem 1.

3. Inequalities involving commutators. This section contains results which will be used later to prove Theorems 1 and 2. We begin with the following proposition which is due to T. Kato (unpublished).

PROPOSITION 1. *Let A be a strictly positive selfadjoint operator in a Hilbert space H , and let $B \in B(H)$ be such that B maps $D(A)$ into itself with $ABA^{-1} \in B(H)$. If*

$$(3.1) \quad \|[A, B]u\| \leq C \|A^{1/2}u\|, \quad \|A^{-1/2}[A, B]u\| \leq C \|u\|,$$

for $u \in D(A)$, with a constant C , then

$$(3.2) \quad \|[A^{1/2}, B]u\| \leq (C/2)\|u\|, \quad u \in D(A^{1/2}),$$

$$(3.3) \quad \|[A^{1/4}, B]u\| \leq (C/2)\pi^{-1/2}\|A^{-1/4}u\|, \quad u \in D(A^{1/4}).$$

Here $[A, B] = AB - BA$ denotes the commutator.

Proof. Using interpolation (see Kato [7]), one sees that B maps $D(A^\alpha)$ into itself with $A^\alpha B A^{-\alpha} \in B(H)$ for $0 \leq \alpha \leq 1$. Thus $[A^{1/2}, B]u$ and $[A^{1/4}, B]u$ are well defined for $u \in D(A^{1/2})$ and $u \in D(A^{1/4})$ respectively.

We claim that (3.1) implies

$$(3.4) \quad \|A^{-1/4}[A, B]u\| \leq C \|A^{1/4}u\|, \quad u \in D(A).$$

In order to see this, let $Tu = A^{-1/2}[A, B]u$, $u \in D(A)$. By (3.1), T and $A^{1/2}TA^{-1/2}$ extend to bounded operators on H with norm bounded by C . Using interpolation, we obtain $\|A^{1/4}TA^{-1/4}\| \leq C$, which implies (3.4).

We now consider $[A^{1/2}, B]$. Since $A^{1/2}BA^{-1/2}$ belongs to $B(H)$ and $D(A)$ is a core of $A^{1/2}$, it suffices to prove (3.2) only for $u \in D(A)$. For such u , $A^{1/2}u$ is given by

$$A^{1/2}u = \pi^{-1} \int_0^\infty \lambda^{-1/2} R(\lambda) Au \, d\lambda,$$

where $R(\lambda) = (\lambda + A)^{-1}$ is the resolvent of $-A$. For fractional powers, see Kato [8] and Yosida [14]. Since $[R(\lambda)A, B] = \lambda[B, R(\lambda)] = \lambda R(\lambda)[A, B]R(\lambda)$, it follows that

$$[A^{1/2}, B]u = \pi^{-1} \int \lambda^{1/2} R(\lambda)[A, B]R(\lambda)u \, d\lambda.$$

Here and in the following, integrals are from 0 to ∞ . If $v \in D(A)$, then

$$([A^{1/2}, B]u, v) = \pi^{-1} \int \lambda^{1/2} (A^{-1/4}[A, B]R(\lambda)u, A^{1/4}R(\lambda)v) \, d\lambda.$$

Using (3.4), it follows that

$$\begin{aligned} |([A^{1/2}, B]u, v)| &\leq \pi^{-1} C \int \lambda^{1/2} \|A^{1/4}R(\lambda)u\| \|A^{1/4}R(\lambda)v\| \, d\lambda \\ &\leq \pi^{-1} C \left\{ \int \lambda^{1/2} (R(\lambda)^2 A^{1/2}u, u) \, d\lambda \right\}^{1/2} \\ &\quad \times \left\{ \int \lambda^{1/2} (R(\lambda)^2 A^{1/2}v, v) \, d\lambda \right\}^{1/2}. \end{aligned}$$

Using the spectral theorem, one has $\int \lambda^{1/2} R(\lambda)^2 \, d\lambda = (\pi/2)A^{-1/2}$. It follows that $|([A^{1/2}, B]u, v)| \leq (C/2)\|u\| \|v\|$, from which (3.2) follows.

The inequality (3.3) is proved in a similar manner. Starting from the formula

$$A^{1/4}u = 2^{-1/2}\pi^{-1} \int_0^\infty \lambda^{-3/4} R(\lambda) Au \, d\lambda,$$

one proceeds as before to obtain

$$([A^{1/4}, B]u, v) = 2^{-1/2}\pi^{-1} \int \lambda^{1/4} (R(\lambda)u, [B, A]R(\lambda)v) \, d\lambda.$$

Using the first half of (3.1), one obtains

$$\begin{aligned} |([A^{1/4}, B]u, v)| &\leq 2^{-1/2}\pi^{-1} C \left\{ \int \lambda^{1/2} (R(\lambda)^2 u, u) \, d\lambda \right\}^{1/2} \left\{ \int (R(\lambda)^2 Av, v) \, d\lambda \right\}^{1/2} \\ &= (C/2)\pi^{-1/2} \|A^{-1/4}u\| \|v\|, \end{aligned}$$

from which (3.3) follows. \square

COROLLARY. *Let A be as in Proposition 1. Let T be an unbounded operator in H with $D(T) = D(A^{1/2})$ and $TA^{-1/2} \in B(H)$ and such that T maps $D(A^{3/2})$ into $D(A)$ with $ATA^{-3/2} \in B(H)$. If*

$$(3.5) \quad \|[A, T]v\| \leq C\|Av\|, \quad \|A^{-1/2}[A, T]v\| \leq C\|A^{1/2}v\|,$$

$v \in D(A^{3/2})$, with a constant C , then

$$\begin{aligned} \|[A^{1/2}, T]v\| &\leq (C/2)\|A^{1/2}v\|, & v \in D(A), \\ \|[A^{1/4}, T]v\| &\leq (C/2)\pi^{-1/2}\|A^{1/4}v\|, & v \in D(A^{3/4}). \end{aligned}$$

Proof. This follows by applying Proposition 1 to $B = TA^{-1/2}$. \square

The next theorem is a consequence of results from the theory of singular integrals proved by Calderón [1], and it will be used frequently later.

THEOREM (CALDERÓN). *Let Δ be the Laplacian on R^m and $\Lambda = (1 - \Delta)^{1/2}$. If $a \in C^1(R^m)$, then*

$$\|[\Lambda, a]u\| \leq C\|a\|_{C^1}\|u\|, \quad u \in H^1(R^m),$$

with a constant C independent of a and u . If $a \in C^2(R^m)$, then

$$\|[\Lambda, a]u\|_1 \leq C'\|a\|_{C^2}\|u\|_1, \quad u \in H^2(R^m).$$

Here $C^k(R^m)$, $k = 1, 2$, is the class of real $N \times N$ matrix-valued functions on R^m which together with their first k derivatives are continuous and bounded on R^m . $\| \cdot \|_{C^k}$ is the usual supremum norm in this space. Also, $\| \cdot \|$ and $\| \cdot \|_1$ are the norms in $L^2(R^m)$ and $H^1(R^m)$, respectively.

The following result is essentially due to Seeley [13], but we give another proof here.

LEMMA 2. *Let Δ_Γ be the Laplace-Beltrami operator on Γ and $\Lambda_\Gamma = (1 - \Delta_\Gamma)^{1/2}$. If $a \in C^2(\Gamma)$, then*

$$(3.6) \quad \|[\Lambda_\Gamma, a]u\|_k \leq C\|a\|_{C^2}\|u\|_k, \quad u \in H^{k+1}(\Gamma), \quad k = 0, 1,$$

with a constant C independent of a and u . Here $C^2(\Gamma)$ is the Banach space of C^2 matrix-valued functions on Γ , and $\| \cdot \|_k$ denotes the norm in $H^k(\Gamma)$.

Proof. For $k = 0$ this can be established using Proposition 1 with $H = L^2(\Gamma)$, $A = 1 - \Delta_\Gamma$, and B being the operator of multiplication by a . The hypothesis (3.1) reduces to showing

$$(3.7) \quad \|[\Delta_\Gamma, a]u\|_{k-1} \leq \text{const.} \|a\|_{C^2}\|u\|_k, \quad u \in H^2(\Gamma), \quad k = 0, 1.$$

We have

$$[\Delta_\Gamma, a]u = 2(\text{grad}(a), \text{grad}(u)) + u\Delta_\Gamma a,$$

and the inequality (3.7) follows from this. Therefore (3.6) is true for $k = 0$.

The case $k=1$ may be proved from the case $k=0$ using the relation $\Lambda_\Gamma[\Lambda_\Gamma, a]u = [a, \Delta_\Gamma]u - [\Lambda_\Gamma, a]\Lambda_\Gamma u$. \square

LEMMA 3. Let $\Delta_\Gamma, \Lambda_\Gamma$ be as in Lemma 2. If $a \in C^2(\Gamma)$, then

$$\|[\Lambda_\Gamma^{1/2}, a]u\| \leq C \|a\|_{C^2} \|\Lambda_\Gamma^{-1/2}u\|, \quad u \in H^1(\Gamma),$$

with a constant C independent of a and u . Here $\| \cdot \|$ denotes the norm of $L^2(\Gamma)$.

Proof. This is proved using Proposition 1 in the same way that (3.6) was established in the case $k=0$. \square

The following lemma is again due to Seeley [13], but before stating it we make some notational comments. If V is a vector field on Γ and u is a function on Γ , then $Vu = (V, \text{grad}(u))$ is the directional derivative of u in the direction V . The set of vector fields of class C^k on Γ ($k=0, 1, 2$) is a Banach space, and a norm for this space may be defined as follows. Let $\{U_i\}$ be a finite covering of Γ such that each U_i is the domain of a coordinate chart which maps U_i onto the unit ball $B \subset R^{m-1}$. With respect to the local coordinates for U_i the vector field V can be represented as $(V_{i,1}, \dots, V_{i,m-1})$, where the $V_{i,j}$ are C^k real-valued functions on B . We define

$$\|V\|_{C^k} = \sum_{i,j} \|V_{i,j}\|_{C^k(B)}.$$

LEMMA 4. Let $\Delta_\Gamma, \Lambda_\Gamma$ be as in Lemma 2. If V is a C^2 vector field on Γ , then

$$(3.8) \quad \|[\Lambda_\Gamma, V]u\|_{k-1} \leq C \|V\|_{C^2} \|u\|_k, \quad u \in H^{k+1}(\Gamma), \quad k = 1, 2,$$

where the constant C is independent of V and u .

Proof. For $k=1$ the inequality (3.8) can be established using the Corollary to Proposition 1 where one takes $H=L^2(\Gamma)$, $A=1-\Delta_\Gamma$, and $Tu=Vu$. The hypothesis (3.5) reduces to showing

$$(3.9) \quad \|[\Delta_\Gamma, V]u\|_{k-2} \leq \text{const.} \|V\|_{C^2} \|u\|_k, \quad u \in H^3(\Gamma), \quad k = 1, 2.$$

This inequality can be proved by showing that it holds in the domain of any coordinate chart. When restricted to such a domain we may assume we are working in R^{m-1} . The operators V and Δ_Γ become first and second order differential operators respectively, and the inequality (3.9) is shown to be true. Thus (3.8) holds for $k=1$.

The inequality (3.8) for $k=2$ follows from the case $k=1$ together with (3.9) and the relation $\Lambda_\Gamma[\Lambda_\Gamma, V]u = [V, \Delta_\Gamma]u - [\Lambda_\Gamma, V]\Lambda_\Gamma u$. \square

Now we obtain formulas for the transformation of the differential operators $D_j = \partial/\partial x_j$ under the mappings U, U_0 defined by (2.4). Using these, we consider commutators involving the operator M .

Let $(y, s) \rightarrow y + sv(y)$, $x \rightarrow (y(x), s(x))$, $c(x)$, and $h(x)$ be as in §2. Define $h_j = h \partial(h^{-1})/\partial x_j$, $y_j = \partial y/\partial x_j$, $s_j = \partial s/\partial x_j$. Put $\alpha_j(y, s) = a_j(y + sv(y))$ for $(y, s) \in \omega$,

and let $\gamma, \delta_j, \sigma_j, V_j$ be defined in the same way from c, h_j, s_j, y_j , respectively. (Note that V_j is a vector field on ω .) Let $\delta = \sum \alpha_j \delta_j$ and

$$(3.10) \quad \mathcal{L}v = \gamma \partial v / \partial s + \sum \alpha_j V_j v + \delta v, \quad v \in H^1(\omega).$$

Then we have

LEMMA 5.

$$\begin{aligned} U_0 D_j U_0^{-1} v &= V_j v + \sigma_j (\partial v / \partial s), & U D_j U^{-1} v &= V_j v + \sigma_j (\partial v / \partial s) + \delta_j v, \\ U \sum a_j D_j U^{-1} v &= \mathcal{L}v, & \|V_j v\| &\leq C \|\Lambda_\Gamma v\|, \end{aligned}$$

$v \in H^1(\omega)$. Here $\| \cdot \|$ denotes the norm in $L^2(\omega)$.

Proof. In order to show the first relation, note that $U_0^{-1} v(x) = v(y(x), s(x))$. Using the chain rule, we have $D_j U_0^{-1} v = V_j v + (\partial v / \partial s)(\partial s / \partial x_j)$, where $V_j v$ and $\partial v / \partial s$ are evaluated at $(y, s) = (y(x), s(x))$. The formula for $U_0 D_j U_0^{-1}$ then follows. Since $U = U_0 h$, one may obtain the formula for $U D_j U^{-1}$ from the formula for $U_0 D_j U_0^{-1}$. Using this and (2.3), one arrives at the third formula.

Note that the vector field V_j is tangential to Γ , i.e. $V_j(y, s)$ is a tangent vector to Γ for each $(y, s) \in \omega$. Therefore $(V_j v)(s) = V_j(s)v(s)$ for $s \in [0, \sigma]$. Here $V_j(s)$ denotes the vector field on Γ defined by $V_j(s)(y) = V_j(y, s)$, $y \in \Gamma$. It follows that $\|V_j v(s)\|_\Gamma^2 \leq C \|\Lambda_\Gamma v(s)\|_\Gamma^2$, where C is a constant independent of s and $\| \cdot \|_\Gamma$ is the norm in $L^2(\Gamma)$. Integrating from 0 to σ gives the last inequality in the lemma. \square

LEMMA 6. If $a \in C^2(\Omega')$, then

$$\|[M, a]u\|_k \leq C \|a\|_{C^2} \|u\|_k, \quad u \in H^{k+1}(\Omega'), \quad k = 0, 1,$$

with a constant C independent of a and u . Here $\| \cdot \|_k$ is the norm in $H^k(\Omega')$.

Proof. Under the mapping U this inequality corresponds to the inequality $\|[\Lambda_\Gamma, \alpha]v\|_k \leq C \|\alpha\|_{C^2} \|v\|_k$, where $\alpha(y, s) = a(y + sv(y))$, $v = Uu$, and the norms are now with respect to ω . This inequality follows from Lemma 2. \square

LEMMA 7.

$$\|[M, D_j]u\| \leq C \|u\|_1, \quad u \in H^2(\Omega').$$

Here $\| \cdot \|$ is the norm in $L^2(\Omega')$.

Proof. Using Lemma 5 and transforming to ω by the mapping U , one sees that it suffices to prove

$$\|[\Lambda_\Gamma, V_j + \sigma_j (\partial / \partial s) + \delta_j]v\| \leq \text{const.} \|v\|_1,$$

where $v = Uu$. This inequality is easily obtained from Lemmas 2 and 4. \square

LEMMA 8. If ρ is a C^3 function with support in the interior of Ω' , then

$$\|[\rho M \rho, \Lambda]u\| \leq C \|u\|_1, \quad \|[\rho M \rho, \Lambda^{1/2}]u\| \leq C \|\Lambda^{1/2}u\|,$$

$u \in H^2(R^m)$. Here $\| \cdot \|$ and $\| \cdot \|_1$ denote the norms in $L^2(R^m)$ and $H^1(R^m)$. We are regarding $u \rightarrow \rho M \rho u$ as a bounded operator from $H^k(R^m)$ to $H^{k-1}(R^m)$ for $k=1, \frac{3}{2}, 2$.

Proof. Let $M' = U_0^{-1} \Lambda_\Gamma U_0 = h M h^{-1}$. M' has the advantage that it maps $H^3(\Omega')$ into $H^2(\Omega')$, while M does not. Note that $(M - M')u = h[h^{-1}, M]u$, $u \in H^1(\Omega')$. It follows from Lemma 6 that $M - M'$ extends to a bounded operator on $H^k(\Omega')$, $k=0, 1$. Thus $\rho M \rho - \rho M' \rho$ extends to a bounded operator on $H^k(R^m)$, $k=0, 1$. Using interpolation, one sees that this is also true for $k=\frac{1}{2}$. Consequently it suffices to prove the inequalities in the lemma with M replaced by M' .

We can use the Corollary to Proposition 1 with $H=L^2(R^m)$, $A=1-\Delta$, and $Tu = \rho M' \rho u$. The hypothesis reduces to showing

$$(3.11) \quad \|[\Delta, \rho M' \rho]u\|_{k-2} \leq C \|u\|_k, \quad u \in H^3(R^m), \quad k = 1, 2.$$

One has

$$(3.12) \quad \begin{aligned} [\Delta, \rho M' \rho]u &= \sum (D_j [D_j, \rho M' \rho]u + [D_j, \rho M' \rho] D_j u), \\ [D_j, \rho M' \rho]u &= (\partial \rho / \partial x_j) M' \rho u + \rho [D_j, M'] \rho u + \rho M' (\partial \rho / \partial x_j) u. \end{aligned}$$

Using Lemma 5, we have

$$[D_j, M']v = U_0^{-1} [V_j, \Lambda_\Gamma] U_0 v + U_0^{-1} [\sigma_j, \Lambda_\Gamma] (\partial / \partial s) U_0 v,$$

$v \in H^2(\Omega')$. It follows from Lemmas 2 and 4 that

$$\| [D_j, M']v \|_{k-1} \leq C \|v\|_k, \quad v \in H^{k+1}(\Omega'), \quad k = 1, 2.$$

Hence

$$\| [D_j, \rho M' \rho]u \|_{k-1} \leq C \|u\|_k, \quad u \in H^{k+1}(R^m), \quad k = 1, 2.$$

Using duality and the fact that $(M'u, v) = (u, h^{-2} M' h^2 v)$, $u, v \in H^1(\Omega')$, one can show that this inequality holds for $k=0$. The inequality (3.11) can now be proved by combining this with (3.12). \square

4. Properties of the operator S .

PROPOSITION 2. S is an isomorphism from $H^1_p(\Omega)$ onto X .

In the proof of this proposition we shall use the operator R defined by

$$Ru = (A_0 + \beta)u + \phi \Lambda \phi u + \psi M \psi u, \quad u \in D(R) \equiv D(S).$$

LEMMA 9. $S-1$ and $R-1$ are accretive when considered as operators in X , and $S-R$ extends to a bounded operator on X . It follows that Proposition 2 is true if and only if R is an isomorphism from $H^1_p(\Omega)$ onto X .

Proof. If we expand (Su, u) using the formula (2.8), then $((A_0 + \beta)u, u)$ is non-negative because $A_0 + \beta$ is accretive, and $(\phi \Lambda \phi u, u)$ is bounded below by $\|\phi u\|^2$

because Λ is selfadjoint and bounded below by 1 when considered as an operator in $L^2(R^m)$. We have

$$\sum (Mr_k\zeta_k\psi u, r_k\zeta_k\psi u) \geq \sum (r_k\zeta_k\psi u, r_k\zeta_k\psi u) = \|\psi u\|^2.$$

The inequality on the left is a consequence of the fact that M is symmetric and bounded below by 1 when considered as an operator in $L^2(\Omega')$, and the equality on the right follows from (2.6) and the fact that the r_k are orthogonal. Therefore

$$(Su, u) \geq \|\phi u\|^2 + \|\psi u\|^2 \geq \|u\|^2, \quad u \in D(S).$$

Here we have used the fact that $\phi^2 + \psi^2 \geq \phi^4 + \psi^4 = 1$. The proof that $R - 1$ is accretive is the same except the r_k and ζ_k do not appear.

One has

$$(4.1) \quad (S - R)u = \sum \psi \zeta_k r_k^{-1} [M, r_k \zeta_k] \psi u, \quad u \in D(S).$$

By Lemma 6, $S - R$ extends to a bounded operator on X .

Since $S - 1$ and $R - 1$ are accretive, it follows that S has range equal to X precisely when $S - 1$ is m -accretive and similarly for R . In general, if two accretive operators differ by a bounded operator, then one is m -accretive if the other is. Thus, Proposition 2 is equivalent to the range of R being X . \square

We prove that R has range equal to X by showing (1) R is closed when regarded as an operator in X , and (2) its adjoint R^* is one-to-one. The first assertion is a consequence of the following lemma.

LEMMA 10. *The operator R satisfies*

$$(4.2) \quad \|u\|_1 \leq C \|Ru\|, \quad u \in D(R),$$

with a constant C .

Proof. Since $R - 1$ is accretive, one has $\|u\| \leq \|Ru\|$, $u \in D(R)$. Therefore, it suffices to show

$$(4.3) \quad \|u\|_1^2 \leq C(\|Ru\|^2 + \|u\|^2), \quad u \in D(R).$$

Here and in the following, C denotes a constant.

Since $\phi^4 + \psi^4 = 1$, we have

$$(4.4) \quad \|u\|_1^2 \leq C(\|\phi^2 u\|_1^2 + \|\psi^2 u\|_1^2), \quad u \in H^1(\Omega).$$

Using Calderón's theorem, it is not hard to show

$$(4.5) \quad \|\phi^2 u\|_1^2 \leq C(\|\phi \Lambda \phi u\|^2 + \|u\|^2), \quad u \in H^1(\Omega).$$

For the term $\psi^2 u$, we shall establish the following estimate:

$$(4.6) \quad \|\psi^2 u\|_1^2 \leq C(\|\mathcal{A}u\|^2 + \|\psi M \psi u\|^2 + \|u\|^2), \quad u \in H^1(\Omega),$$

where \mathcal{A} denotes the formal differential operator $\sum a_j D_j$. In order to show (4.6), we first note that

$$\|w\|_1 \leq C(\|\partial w/\partial s\| + \|\Lambda_\Gamma w\| + \|w\|), \quad w \in H^1(\omega),$$

where $\| \cdot \|$ and $\| \cdot \|_1$ are the norms in $L^2(\omega)$ and $H^1(\omega)$. This inequality follows from (2.7) and the fact that Λ_Γ is an isomorphism from $H^1(\Gamma)$ onto $L^2(\Gamma)$. Now let \mathcal{L} be the operator defined by (3.10). Since γ is nonsingular, one has

$$\|\partial w/\partial s\| \leq C\left(\|\mathcal{L}w\| + \sum \|V_j w\| + \|w\|\right), \quad w \in H^1(\omega).$$

If one combines this with the preceding inequality and uses the fact $\|V_j w\| \leq C\|\Lambda_\Gamma w\|$, proved in Lemma 5, one obtains

$$\|w\|_1 \leq C(\|\mathcal{L}w\| + \|\Lambda_\Gamma w\| + \|w\|), \quad w \in H^1(\omega).$$

Using the formula $U\mathcal{A} = \mathcal{L}U$, proved in Lemma 5, it follows that

$$\|v\|_1 \leq C(\|\mathcal{A}v\| + \|Mv\| + \|v\|), \quad v \in H^1(\Omega'),$$

where the norms are with respect to Ω' . Putting $v = \psi^2 u$ and using the fact that M commutes with multiplication by ψ , one obtains (4.6).

Combination of (4.4), (4.5), and (4.6) leads to

$$(4.7) \quad \|u\|_1^2 \leq C(\|\mathcal{A}u\|^2 + \|\phi \Lambda \phi u\|^2 + \|\psi M \psi u\|^2 + \|u\|^2),$$

$u \in H^1(\Omega)$. Keeping this in mind, we expand $\|Ru\|^2$. Note that in proving (4.3), we may ignore terms in R which act as bounded operators on X . Thus we may assume without loss of generality that $b=0, \beta=0$. Then

$$(4.8) \quad \|Ru\|^2 = \|\mathcal{A}u\|^2 + \|\phi \Lambda \phi u\|^2 + \|\psi M \psi u\|^2 + 2(\mathcal{A}u, Ku) + 2(\phi \Lambda \phi u, \psi M \psi u),$$

where $Ku = \phi \Lambda \phi u + \psi M \psi u$. In a moment we shall prove

$$(4.9a) \quad -(\phi \Lambda \phi u, \psi M \psi u) \leq C\|u\|_1\|u\|,$$

$$(4.9b) \quad -(\mathcal{A}u, Ku) \leq C\|u\|_1\|u\|,$$

$u \in D(R)$. Combining (4.7), (4.8), and (4.9ab), one arrives at

$$\|u\|_1^2 \leq C(\|Ru\|^2 + \|u\|_1\|u\|).$$

Since $2\|u\|_1\|u\| \leq C^{-1}\|u\|_1^2 + C\|u\|^2$, one obtains (4.3).

We now show (4.9a). Let ρ be a C^3 , real-valued function on R^m such that (1) $\rho(x) = \psi(x)$ for x belonging to the support of ϕ , and (2) the support of ρ is contained in the interior of Ω' . Then $\phi\psi = \phi\rho$. It follows that $\phi\psi M \psi u = \rho M \rho \phi u$ since ϕ and M commute. Therefore, it suffices to show $-(\Lambda w, \rho M \rho w) \leq C\|w\|_1\|w\|$, where $w = \phi u \in H^1(R^m)$ and the inner product and norms are with respect to R^m . We may assume $w \in H^2(R^m)$ since this set is dense in $H^1(R^m)$. We have

$$-(\Lambda w, \rho M \rho w) = -(\rho \Lambda^{1/2} w, M \rho \Lambda^{1/2} w) - ([\rho M \rho, \Lambda^{1/2}] \Lambda^{1/2} w, w).$$

The first term on the right is nonpositive since $M \geq 1$. Using Lemma 8, one sees that the second term is bounded in absolute value by $C \|w\|_1 \|w\|$.

Finally we show (4.9b). We may assume $u \in H^2(\Omega) \cap H^1_p(\Omega)$ since this set is dense in $H^1_p(\Omega)$. Since K is symmetric, one has $(\mathcal{A}u, Ku) = (\mathcal{A}Ku, u) + ([K, \mathcal{A}]u, u)$. If one integrates the first term by parts, one obtains $(\mathcal{A}Ku, u) = -(Ku, (\mathcal{A} + e)u) + ((Ku)_0, a_n v)_\Gamma$, where $e = \sum \partial a_j / \partial x_j$. Here $v = u_0$ and $(Ku)_0$ are the traces of u and Ku on Γ , and $(\cdot, \cdot)_\Gamma$ denotes the inner product in $L^2(\Gamma)$. Since $\phi = 0$ on Γ , we have $(Ku)_0 = (\psi M \psi u)_0$. Using the fact that $Mu = h^{-1} U_0^{-1} \Lambda_\Gamma U_0 h u$, one can show $(Ku)_0 = h_0^{-1} \Lambda_\Gamma h_0 v$, where h_0 is the restriction of h to Γ . It follows that

$$2(\mathcal{A}u, Ku) = ([K, \mathcal{A}]u, u) - (Ku, eu) + (h_0^{-1} \Lambda_\Gamma h_0 v, a_n v)_\Gamma.$$

Using Calderón's theorem and Lemmas 6 and 7, one can show

$$(4.10) \quad \|[K, \mathcal{A}]w\| \leq C \|w\|_1 \sum \|a_j\|_{C^2}, \quad w \in H^2(\Omega).$$

Therefore, in order to finish the proof of (4.9b), it suffices to show

$$(4.11) \quad -(h_0^{-1} \Lambda_\Gamma h_0 v, a_n v)_\Gamma \leq C \|u\|_1 \|u\|.$$

Let $\pi(y)$, $y \in \Gamma$, be the orthogonal projection of R^N onto $P(y)$. Then $\pi(y)v(y) = v(y)$ since $u \in D(R)$. Thus

$$(h_0^{-1} \Lambda_\Gamma h_0 v, a_n v)_\Gamma = (\Lambda_\Gamma v, \alpha v)_\Gamma + (h_0^{-1} [\Lambda_\Gamma, h_0 \pi] v, a_n v)_\Gamma,$$

where $\alpha = \pi a_n \pi$. Note that $\alpha(y) \geq 0$ for $y \in \Gamma$. By Lemma 2, the second term on the right is bounded in absolute value by $C \|v\|_\Gamma^2$, where $\|\cdot\|_\Gamma$ denotes the norm in $L^2(\Gamma)$. For the first term on the right we have

$$(\Lambda_\Gamma v, \alpha v)_\Gamma = (\Lambda_\Gamma^{1/2} v, \alpha \Lambda_\Gamma^{1/2} v)_\Gamma + ([\alpha, \Lambda_\Gamma^{1/2}] \Lambda_\Gamma^{1/2} v, v)_\Gamma.$$

The first term on the right is nonnegative since $\alpha \geq 0$, and the second term is bounded by $C \|v\|_\Gamma^2$, by Lemma 3. This proves $-(h_0^{-1} \Lambda_\Gamma h_0 v, a_n v)_\Gamma \leq C \|v\|_\Gamma^2$. In order to complete the proof of (4.11), it suffices to show $\|v\|_\Gamma^2 \leq C \|u\|_1 \|u\|$. Using a partition of unity and change of variables, this inequality can be reduced to the case where $\Omega = \{x \in R^m : x_m > 0\}$ and $\Gamma = \{x \in R^m : x_m = 0\}$. Using integration by parts, we have $(v, v)_\Gamma = -2(D_m u, u)$, and the desired inequality follows from this. \square

In order to finish the proof that the range of R is X , it remains to show that R^* is one-to-one, where R^* is the adjoint of R regarded as an operator in X .

The formal adjoint of A_0 is given by

$$B_0 v = -\sum D_j a_j v + {}^t b v, \quad v \in D(B_0) \equiv H^1_Q(\Omega),$$

where $Q(x) = (a_n(x)P(x))^1$, $x \in \Gamma$, is the boundary subspace formally adjoint to P . The formal adjoint of R is then defined by

$$T v = (B_0 + \beta)v + \phi \Lambda \phi v + \psi M \psi v, \quad v \in D(T) \equiv H^1_Q(\Omega).$$

R and T are formally adjoint to each other, i.e. $(Ru, v) = (u, Tv)$, $u \in D(R)$, $v \in D(T)$. It follows that T is closable when regarded as an operator in X and $\tilde{T} \subseteq R^*$, where \tilde{T} is the closure of T .

The operator $B_0 + \beta$ is accretive; this is proved by Friedrichs [5]. Using the argument in Lemma 9, one sees that $T - 1$ is accretive. Hence $\|u\| \leq \|\tilde{T}u\|$, $u \in D(\tilde{T})$, and \tilde{T} is one-to-one. In order to show R^* is one-to-one (and to complete the proof of Proposition 2), it suffices to prove the following lemma.

LEMMA 11. $\tilde{T} = R^*$.

Proof. In the terminology of Friedrichs [3], \tilde{T} is the strong extension of T and R^* is the weak extension of T , so the proposition asserts the equivalence of the weak and strong extensions of T . Friedrichs ([3], [4], [5]) and Lax and Phillips [10] have shown the identity of weak and strong extensions of first order partial differential operators, and this proof is an extension of their methods to the case at hand.

Let $\xi^2 + \eta^2 = 1$ be a C^3 partition of unity for R^m with the following properties:

(i) $\xi = 1$ and $\eta = 0$ in a neighborhood of the support of ϕ , so that $\xi\phi = \phi$, $\eta\phi = 0$, and $\eta\psi = \eta$.

(ii) The support of ξ is relatively compact in Ω ; in particular, $\xi = 0$ and $\eta = 1$ near Γ .

(iii) For $x \in \Omega'$, $\xi(x)$ and $\eta(x)$ depend only on $s(x)$. This implies that $M\xi u = \xi Mu$, $u \in H^{-1}(\Omega')$, and similarly for η .

Since the inclusion $\tilde{T} \subseteq R^*$ has already been shown, it remains to prove the opposite inclusion. Let $v \in D(R^*)$ with $R^*v = f$. We shall show $\xi^2 v, \eta^2 v \in D(\tilde{T})$.

We first show $\xi^2 v \in D(\tilde{T})$. Let $J_\epsilon = (1 + \epsilon\Lambda)^{-1}$, $\epsilon > 0$. For each k , J_ϵ maps $H^k(R^m)$ into itself with norm uniformly bounded in ϵ . If $u \in H^k(R^m)$, then $J_\epsilon u \rightarrow u$ in $H^k(R^m)$ as $\epsilon \rightarrow 0$. Furthermore J_ϵ maps $H^k(R^m)$ into $H^{k+1}(R^m)$.

Let $H_\epsilon u = \xi J_\epsilon \xi u$, where we regard multiplication by ξ as an operator from $H^k(\Omega)$ to $H^k(R^m)$ and also from $H^k(R^m)$ to $H^k(\Omega)$, $|k| \leq 3$. Then, for $|k| \leq 3$, H_ϵ maps $H^k(\Omega)$ into itself with norm uniformly bounded in ϵ . If $u \in H^k(\Omega)$, then $H_\epsilon u \rightarrow \xi^2 u$ in $H^k(\Omega)$. (From now on we omit " $\epsilon \rightarrow 0$ " when it is clear from the context.) For $-3 \leq k \leq 2$, H_ϵ maps $H^k(\Omega)$ into $H^{k+1}(\Omega)$.

Let $v_\epsilon = H_\epsilon v$. Then $v_\epsilon \in D(T)$ since $\xi = 0$ near Γ . Also $v_\epsilon \rightarrow \xi^2 v$ in X . In order to show $\xi^2 v \in D(\tilde{T})$, it remains to show Tv_ϵ converges in X .

Let $T_1: X \rightarrow H^{-1}(\Omega)$ be defined by

$$T_1 u = -\sum D_j a_j u + (b + \beta)u + \phi \Lambda \phi u + \psi M \psi u, \quad u \in X.$$

Note that T_1 is an extension of T . It is not hard to see that $T_1 v = f$. Then we have $Tv_\epsilon = H_\epsilon f + [T_1, H_\epsilon]v$. Since $H_\epsilon f \rightarrow \xi^2 f$ in X , it remains to show $[T_1, H_\epsilon]v$ converges in X . In fact we shall show

$$(4.12) \quad [T_1, H_\epsilon]u \text{ converges in } X \text{ for all } u \in X.$$

The first step is to show

$$(4.13) \quad \|[T_1, H_\varepsilon]u\| \leq C\|u\|, \quad u \in X, \quad \varepsilon > 0,$$

with a constant C independent of ε .

Let χ be a C^∞ real-valued function with compact support in Ω with the property that $\chi(x)=1$ for x belonging to the support of ξ . Then $\chi\xi=\xi$ and

$$(4.14) \quad [T_1, H_\varepsilon]u = [T_1, \xi]\chi J_\varepsilon \xi u + \xi[\chi T_1 \chi, J_\varepsilon] \xi u + \xi J_\varepsilon \chi [T_1, \xi]u.$$

In the first and third terms on the right we have

$$(4.15) \quad \|[T_1, \xi]w\| \leq C\|w\|, \quad w \in H^1(\Omega).$$

This follows from Calderón's theorem and Lemma 6. In the second term we have

$$[\chi T_1 \chi, J_\varepsilon]w = \varepsilon J_\varepsilon [\Lambda, \chi T_1 \chi] J_\varepsilon w = J_\varepsilon [\Lambda, \chi T_1 \chi] \Lambda^{-1} (1 - J_\varepsilon) w,$$

$w \in L^2(R^m)$. It follows from Calderón's theorem and Lemma 8 that $\|[\Lambda, \chi T_1 \chi]v\| \leq C\|v\|_1, v \in H^2(R^m)$. Therefore

$$\|[\chi T_1 \chi, J_\varepsilon]w\| \leq C\|w\|, \quad w \in H^1(R^m), \quad \varepsilon > 0.$$

Since $H^1(R^m)$ is dense in $L^2(R^m)$, this holds for $u \in L^2(R^m)$. Combining this inequality and (4.15) with (4.14) proves (4.13).

Having established the inequality (4.13), we can prove (4.12) using the following well-known principle:

(4.16) Let $A_\varepsilon: X_1 \rightarrow X_2, \varepsilon > 0$, be a family of continuous linear operators between Banach spaces X_1 and X_2 . Suppose the A_ε are uniformly bounded in norm with respect to ε and $A_\varepsilon u$ converges in X_2 as $\varepsilon \rightarrow 0$ for all u belonging to a dense subset of X_1 . Then $A_\varepsilon u$ converges in X_2 for all u in X_1 .

In the particular case of (4.12) one sees that $[T_1, H_\varepsilon]u$ converges in X for all $u \in H^1(\Omega)$. Since $H^1(\Omega)$ is dense in X , the assertion (4.12) follows. This completes the proof that $\xi^2 v \in D(\tilde{T})$.

In order to show $\eta^2 v \in D(\tilde{T})$, we transform from Ω to ω . However, we first make the following observation. Suppose C is a bounded operator on X . Then Lemma 11 is true if and only if $(R+C)^*$ coincides with the closure of $T+C^*$. Thus we may assume without loss of generality that $b=0, \beta=0$ in R and T .

Let

$$(4.17) \quad Eu = \mathcal{L}u + \Lambda_\Gamma u,$$

where the domain of E consists of those $u \in H^1(\omega)$ which satisfy the boundary conditions $u(y, 0) \in P(y), u(y, \sigma) = 0$ for a.a. $y \in \Gamma$. If $u \in D(E)$, then $\eta U^{-1}u \in D(R)$ and

$$(4.18) \quad R\eta U^{-1}u = \eta U^{-1}Eu + \sum a_j(\partial\eta/\partial x_j)U^{-1}u.$$

Here we have used Lemma 5 and the fact that $\eta\phi=0$. It follows that if E^* is the adjoint of E regarded as an operator in $L^2(\omega)$, then $U\eta v \in D(E^*)$ with

$$E^*U\eta v = f' \equiv U\left(\eta f - \sum (\partial\eta/\partial x_j)a_j v\right).$$

The formal adjoint of E is given by $Fw = \mathcal{M}w + \Lambda_\Gamma w$, where

$$\mathcal{M}w = -(\partial/\partial s)\gamma w - \sum V_j \alpha_j w + \delta_1 w,$$

with $\delta_1 = \delta - \sum \text{div}(V_j)\alpha_j$. Here we are using the fact that if V is a vector field on Γ , then V and $-V - \text{div}(V)$ are formally adjoint as operators in $L^2(\Gamma)$. The domain of F consists of those $w \in H^1(\omega)$ which satisfy the boundary conditions $w(y, 0) \in Q(y)$ for a.a. $y \in \Gamma$. We note that $(Eu, w) = (u, Fw)$, $u \in D(E)$, $w \in D(F)$. Thus $\tilde{F} \subseteq E^*$. In a moment we shall show $E^* = \tilde{F}$. Assuming this, it follows that there exists a sequence $\{w_n\} \subset D(\tilde{F})$ with $w_n \rightarrow U\eta v$ and $Fw_n \rightarrow f'$ in $L^2(\omega)$. Then $\eta U^{-1}w_n \in D(T)$, $\eta U^{-1}w_n \rightarrow \eta^2 v$ in X . Using the fact that $U(-\sum D_j a_j)U^{-1} = \mathcal{M}$, one can show

$$T\eta U^{-1}w_n = \eta U^{-1}Fw_n - \sum (\partial\eta/\partial x_j)a_j U^{-1}w_n.$$

Thus $T\eta U^{-1}w_n$ converges in X . We conclude $\eta^2 v \in D(\tilde{T})$.

It remains to show $E^* = \tilde{F}$. We first reduce the problem to the case where the boundary subspace $P(y)$ is independent of y . Let r_k, ζ_k be as in §2. Let $v \in D(E^*)$. In order to show $v \in D(\tilde{F})$, it suffices to show $\zeta_k^2 v \in D(\tilde{F})$, $k=1, \dots, K$. Fix k and let $\zeta = \zeta_k, r = r_k$ and

$$Lu = \gamma r^{-1} \partial u/\partial s + \sum \alpha_j r^{-1} V_j u + \delta r^{-1} u + \Lambda_\Gamma r^{-1} u,$$

with the domain of L consisting of those $u \in H^1(\omega)$ which satisfy the boundary conditions

$$(4.19) \quad u(y, 0) \in P_k, \quad u(y, \sigma) = 0, \quad \text{for a.a. } y \in \Gamma.$$

If $u \in D(L)$, then $\zeta r^{-1} u \in D(E)$ and

$$E\zeta r^{-1} u = \zeta Lu + \sum \alpha_j V_j (\zeta r^{-1}) u + [\Lambda_\Gamma, \zeta] r^{-1} u.$$

By Lemma 2, $[\Lambda_\Gamma, \zeta]$ extends to a bounded operator on $L^2(\omega)$. It follows that $\zeta v \in D(L^*)$ with

$$L^*\zeta v = \zeta r E^* v - \sum V_j (\zeta r) \alpha_j v - r [\Lambda_\Gamma, \zeta] v.$$

The formal adjoint of L is

$$Mw = -(\partial/\partial s)r\gamma w - \sum V_j r \alpha_j w + r\delta_1 w + r\Lambda_\Gamma w,$$

where the domain of M consists of those $w \in H^1(\omega)$ which satisfy the adjoint boundary conditions $w(y, 0) \in Q_k(y)$, where

$$Q_k(y) = (\gamma(y, 0)r(y))^{-1} P_k^\perp.$$

We have $(Lu, w) = (u, Mw)$, $u \in D(L)$, $w \in D(M)$, so $\tilde{M} \subseteq L^*$. In a moment we shall show $L^* = \tilde{M}$. Assuming this, it follows that there exists a sequence $\{w_n\} \subset D(M)$ with $w_n \rightarrow \zeta v$, $Mw_n \rightarrow L^*\zeta v$ in $L^2(\omega)$. For y belonging to the support of ζ , we have $r^{-1}(y)P_k = P(y)$. For such y , one has $Q_k(y) = Q(y)$. Thus $\{\zeta w_n\} \subset D(F)$. Also $\zeta w_n \rightarrow \zeta^2 v$ and

$$F\zeta w_n = \zeta r^{-1}Mw_n - \sum V_j(\zeta r^{-1})r\alpha_j w_n + [\Lambda_\Gamma, \zeta]w_n,$$

which converges in $L^2(\omega)$ since $[\Lambda_\Gamma, \zeta]$ extends to a bounded operator on $L^2(\omega)$. It follows that $\zeta^2 v \in D(\tilde{F})$.

We now show $L^* = \tilde{M}$. By multiplying L by $r\gamma^{-1}$, we may assume L has the form $Lu = \partial u / \partial s + Gu$, where $Gu = \sum a_j V_j u + bu + c_1 \Lambda_\Gamma c_2 u$, where $a_j = r\gamma^{-1}\alpha_j r^{-1}$, $b = r\gamma^{-1}\delta r^{-1}$, $c_1 = r\gamma^{-1}$, and $c_2 = r^{-1}$. Then $Mv = -\partial v / \partial s + Hv$, where

$$Hv = -\sum V_j {}^t a_j v + \left({}^t b - \sum \operatorname{div} (V_j) {}^t a_j \right) v + {}^t c_2 \Lambda_\Gamma {}^t c_1 v,$$

and the adjoint boundary conditions become $v(y, 0) \in P_k^\perp$.

To show $L^* \subseteq \tilde{M}$, we use the mollifier $K_\epsilon = (1 + \epsilon \Lambda_\Gamma)^{-1}$, $\epsilon > 0$. For $-1 \leq k \leq 1$, K_ϵ maps $H^k(\Gamma)$ into itself with norm uniformly bounded in ϵ . For $u \in H^k(\Gamma)$, $K_\epsilon u \rightarrow u$ in $H^k(\Gamma)$. Furthermore K_ϵ maps $H^k(\Gamma)$ into $H^{k+1}(\Gamma)$ for $k = -1, 0$.

Using the natural correspondence (2.7), we shall regard K_ϵ as mapping $H^k(\omega)$ into itself for $-1 \leq k \leq 1$. Then, for $u \in H^k(\omega)$, $K_\epsilon u \rightarrow u$ in $H^k(\omega)$.

Given $v \in D(L^*)$, let $v_\epsilon = K_\epsilon v$. Then $v_\epsilon \rightarrow v$ in $L^2(\omega)$. Since K_ϵ maps $H^0(\Gamma)$ into $H^1(\Gamma)$, it follows that the first order derivatives of v_ϵ along directions tangential to Γ lie in $L^2(\omega)$. In order to show v_ϵ belongs to $H^1(\omega)$, it suffices to show $\partial v_\epsilon / \partial s$ belongs to $L^2(\omega)$.

If $L^*v = f$, then $-\partial v / \partial s + Hv = f$ when we regard $\partial v / \partial s$ and Hv as elements of $H^{-1}(\omega)$. Then $\partial v_\epsilon / \partial s = K_\epsilon(\partial / \partial s)v = -K_\epsilon f + K_\epsilon Hv$. One has $K_\epsilon Hv \in L^2(\omega)$, since K_ϵ maps $H^{-1}(\Gamma)$ and $L^2(\Gamma)$ and the operator H only involves differentiation in the y component of a function $v(y, s)$ on ω . Thus $v_\epsilon \in H^1(\omega)$.

We claim v_ϵ satisfies the boundary conditions $v_\epsilon(y, 0) \in P_k^\perp$. Suppose $u \in D(L)$. Let $u(0)$, $(K_\epsilon u)(0)$, denote the traces of u and $K_\epsilon u$ on $\Gamma \times \{0\}$. Then $(K_\epsilon u)(0) = K_\epsilon(u(0))$. It follows that $K_\epsilon u$ satisfies the boundary conditions (4.19) and hence belongs to $D(L)$. We have

$$(4.20) \quad (LK_\epsilon u, v) = (K_\epsilon u, f) = (u, K_\epsilon f),$$

since $L^*v = f$. On the other hand, we have $(LK_\epsilon u, v) = (\partial u / \partial s, v_\epsilon) + (GK_\epsilon u, v)$, since K_ϵ and $\partial / \partial s$ commute. Integrating by parts, one obtains

$$(\partial u / \partial s, v_\epsilon) = -(u, \partial v_\epsilon / \partial s) - (u(0), v_\epsilon(0))_\Gamma.$$

We use here the fact that $u(y, \sigma) = 0$ for a.a. $y \in \Gamma$. Thus

$$(LK_\epsilon u, v) = -(u, \partial v_\epsilon / \partial s) - (u(0), v_\epsilon(0))_\Gamma + (v, K_\epsilon Hv) = (u, K_\epsilon f) - (u(0), v_\epsilon(0))_\Gamma.$$

Combining this with (4.20) gives $(u(0), v_\varepsilon(0))_\Gamma = 0, u \in D(L)$. Since the values of $u(0)$ can be chosen arbitrarily subject to the restriction (4.19), it follows that v_ε satisfies the boundary conditions $v_\varepsilon(y, 0) \in P_k^\perp$. Therefore v_ε belongs to $D(M)$.

It remains to show $Mv_\varepsilon \rightarrow f$ in $L^2(\omega)$. We have $Mv_\varepsilon = K_\varepsilon f + [H, K_\varepsilon]v$. Since $f \in L^2(\omega), K_\varepsilon f \rightarrow f$ in $L^2(\omega)$. We claim $[H, K_\varepsilon]v \rightarrow 0$ in $L^2(\omega)$. In fact, we shall show

$$(4.21) \quad [H, K_\varepsilon]w \rightarrow 0 \quad \text{in } L^2(\omega) \quad \text{for all } w \in L^2(\omega).$$

The proof of this is very similar to (4.12). The crucial step is to establish the inequality

$$(4.22) \quad \|[H, K_\varepsilon]w\| \leq C\|w\|, \quad w \in L^2(\omega), \quad \varepsilon > 0,$$

and then we can apply the principle (4.16). We have

$$[H, K_\varepsilon]w = K_\varepsilon[\Lambda_\Gamma, H]\Lambda_\Gamma^{-1}(1 - K_\varepsilon)w.$$

Using Lemmas 2 and 4, one sees that $\|[\Lambda_\Gamma, H]\Lambda_\Gamma^{-1}v\| \leq C\|v\|, v \in H^1(\omega)$. Using this inequality, one easily obtains (4.22).

Since $[H, K_\varepsilon]w \rightarrow 0$ in $L^2(\omega)$ for $w \in H^1(\omega)$, and $H^1(\omega)$ is dense in $L^2(\omega)$, one obtains (4.21). Thus $Mv_\varepsilon \rightarrow f$ and $v \in D(\tilde{M})$. \square

This concludes the proof of Proposition 2. We now prove a regularity result for S .

PROPOSITION 3. *If $v \in H^1(\Omega)$ then $S^{-1}v \in H^2(\Omega)$.*

Proof. Let $u = S^{-1}v$. Then $u \in H_k^1(\Omega), Su \in H^1(\Omega)$, and we must show $u \in H^2(\Omega)$. $(S - R)u$ is given by (4.1) and, by Lemma 6, $[M, r_k \zeta_k]$ is a bounded operator on $H^1(\Omega')$. Therefore $(S - R)u$ belongs to $H^1(\Omega)$. Since $Su \in H^1(\Omega)$, it follows that $Ru \in H^1(\Omega)$.

Let $\xi^2 + \eta^2 = 1$ be the partition of unity introduced in the proof of Lemma 11. We must show $\xi^2 u, \eta^2 u \in H^2(\Omega)$.

Let $\{J_\varepsilon : \varepsilon > 0\}$ be the operators introduced in the proof of Lemma 11 and $G_\varepsilon u = \varepsilon^{-1} \xi(1 - J_\varepsilon)\xi u$. Note that $\varepsilon^{-1}(1 - J_\varepsilon) = \Lambda J_\varepsilon$. We claim that in order to show $\xi^2 u \in H^2(\Omega)$, it suffices to show $G_\varepsilon u$ converges in $H^1(\Omega)$. Suppose the latter is true. Then

$$G_\varepsilon u - \varepsilon^{-1}(1 - J_\varepsilon)\xi^2 u = \varepsilon^{-1}[J_\varepsilon, \xi]\xi u = J_\varepsilon[\xi, \Lambda]J_\varepsilon \xi u.$$

By Calderón's theorem, $[\xi, \Lambda]$ is a bounded operator on $H^1(R^m)$, so the right side converges in $H^1(R^m)$. Therefore $J_\varepsilon \Lambda \xi^2 u$ converges in $H^1(R^m)$. So $\Lambda \xi^2 u \in H^1(R^m)$ which implies $\xi^2 u \in H^2(\Omega)$. This proves the claim.

It remains to show $G_\varepsilon u$ converges in $H^1(\Omega)$. Note that $G_\varepsilon u \in D(R)$ since $\xi = 0$ near Γ . By (4.2), we are reduced to proving $RG_\varepsilon u$ converges in X . We have

$$RG_\varepsilon u = G_\varepsilon Ru + [R, G_\varepsilon]u.$$

Since $Ru \in H^1(\Omega), G_\varepsilon Ru \rightarrow \xi \Lambda \xi Ru$ in X .

It remains to show that $[R, G_\epsilon]u$ converges in X . In fact we shall show that $[R, G_\epsilon]w$ converges in X for all $w \in H^1(\Omega)$. (Note that in proving this we do not use the fact that w satisfies the boundary conditions $w_0(y) \in P(y)$, $y \in \Gamma$, so we shall assume that $P(y) = R^N$ and $D(R) = H^1(\Omega)$.) The proof is similar to the proof of (4.12). We establish the inequality

$$(4.23) \quad \|[R, G_\epsilon]w\| \leq C \|w\|_1, \quad w \in H^1(\Omega), \quad \epsilon > 0,$$

and then apply the principle (4.16). Let χ be the function introduced in connection with the proof of (4.13). Then

$$(4.24) \quad [R, G_\epsilon]w = [R, \xi]\chi \Lambda J_\epsilon \xi w + \xi J_\epsilon [\chi R \chi, \Lambda] J_\epsilon \xi w + \xi \Lambda J_\epsilon \chi [R, \xi]w.$$

Application of Calderón's theorem and Lemma 6 shows that

$$\|[R, \xi]w\|_k \leq C \|w\|_k, \quad w \in H^{k+1}(\Omega), \quad k = 0, 1.$$

Using Calderón's theorem and Lemma 8, one obtains

$$\|[\chi R \chi, \Lambda]v\| \leq C \|v\|_1, \quad v \in H^2(R^m).$$

Combination of these last two inequalities with (4.24) proves (4.23).

Note that if $w \in H^2(\Omega)$, then $G_\epsilon w \rightarrow \xi \Lambda \xi w$ in $H^1(\Omega)$. Thus $[R, G_\epsilon]w \rightarrow [R, \xi \Lambda \xi]w$ in X for $w \in H^2(\Omega)$. Since $H^2(\Omega)$ is dense in $H^1(\Omega)$, it follows that $[R, G_\epsilon]w$ converges in X for all $w \in H^1(\Omega)$. Thus we have proved $\xi^2 u \in H^2(\Omega)$.

We now show $\eta^2 u \in H^2(\Omega)$. It suffices to show $U\eta^2 u \in H^2(\omega)$ which in turn reduces to showing $\Lambda_\Gamma U\eta^2 u$ and $(\partial/\partial s)U\eta^2 u$ belong to $H^1(\omega)$. To show this, we shall use the operators $\{K_\epsilon : \epsilon > 0\}$ introduced in the proof of Lemma 11. In order to show $\Lambda_\Gamma U\eta^2 u \in H^1(\omega)$, it suffices to show $K_\epsilon \Lambda_\Gamma U\eta^2 u = \epsilon^{-1}(1 - K_\epsilon)U\eta^2 u$ converges in $H^1(\omega)$. Since η commutes with $U^{-1}K_\epsilon U$, it suffices to show $\epsilon^{-1}\eta U^{-1}(1 - K_\epsilon)w$ converges in $H^1(\Omega)$, where $w = U\eta u$.

Let $L_\epsilon = \sum \zeta_k r_k^{-1} K_\epsilon r_k \zeta_k$. Using (2.6), we have

$$(4.25) \quad \epsilon^{-1}(K_\epsilon - L_\epsilon)w = \epsilon^{-1} \sum \zeta_k r_k^{-1} [r_k \zeta_k, K_\epsilon]w.$$

Note that $\epsilon^{-1}[r_k \zeta_k, K_\epsilon]w = K_\epsilon [\Lambda_\Gamma, r_k \zeta_k] K_\epsilon w$. This converges in $H^1(\omega)$ as $\epsilon \rightarrow 0$ by Lemma 2. Thus it suffices to show

$$(4.26) \quad \epsilon^{-1}\eta U^{-1}(1 - L_\epsilon)w \text{ converges in } H^1(\Omega).$$

We claim that $\epsilon^{-1}\eta U^{-1}(1 - L_\epsilon)w$ belongs to $D(R)$. Since $u \in D(R)$, it follows that $w(y, 0) \in P(y)$ for a.a. $y \in \Gamma$. Therefore $r_k(y)\zeta_k(y)w(y, 0) \in P_k$ for a.a. $y \in \Gamma$, which implies $(K_\epsilon r_k \zeta_k w)(y, 0) \in P_k$ for a.a. $y \in \Gamma$. From this one concludes $(L_\epsilon w)(y, 0) \in P(y)$ for a.a. $y \in \Gamma$. The claim then follows.

Using (4.2), one sees that the assertion (4.26) reduces to showing $\epsilon^{-1}R\eta U^{-1}(1 - L_\epsilon)w$ converges in X . We may assume without loss of generality that $b=0, \beta=0$ in R . Let E be the operator given by (4.17). Using (4.18), one sees that it suffices to show $\epsilon^{-1}E(1 - L_\epsilon)w$ converges in $L^2(\omega)$. For this we do not use

the fact that w satisfies the boundary conditions $w(y, 0) \in P(y)$, $y \in \Gamma$, so we shall assume that $P(y) = R^N$ and $D(E) = H^1(\omega)$. Again using (4.25), one sees that it suffices to show $\varepsilon^{-1}E(1 - K_\varepsilon)w$ converges in $L^2(\omega)$. One has

$$\varepsilon^{-1}E(1 - K_\varepsilon)w = \varepsilon^{-1}(1 - K_\varepsilon)Ew + \varepsilon^{-1}[K_\varepsilon, E]w.$$

It is not hard to show that $Ru \in H^1(\Omega)$ implies $Ew \in H^1(\omega)$. Therefore the first term on the right converges to $\Lambda_\Gamma Ew$ in $L^2(\omega)$.

It remains to show $\varepsilon^{-1}[K_\varepsilon, E]w$ converges in $L^2(\omega)$. We shall show that $\varepsilon^{-1}[K_\varepsilon, E]v$ converges in X for all $v \in H^1(\omega)$. We first establish the inequality

$$(4.27) \quad \|\varepsilon^{-1}[K_\varepsilon, E]v\| \leq C\|v\|_1, \quad v \in H^1(\omega), \quad \varepsilon > 0,$$

and then apply (4.16). We have

$$(4.28) \quad \varepsilon^{-1}[K_\varepsilon, E]v = K_\varepsilon[E, \Lambda_\Gamma]K_\varepsilon v.$$

Using Lemmas 2 and 5, one sees that $\|[E, \Lambda_\Gamma]u\| \leq C\|u\|_1$, $u \in H^2(\omega)$. Combining this with (4.28), one obtains (4.27). Since $\varepsilon^{-1}[K_\varepsilon, E]v = \varepsilon^{-1}[(I - K_\varepsilon), E]v$ converges to $[\Lambda_\Gamma, E]v$ in $L^2(\omega)$ for $v \in H^2(\omega)$ and $H^2(\omega)$ is dense in $H^1(\omega)$, it follows that $\varepsilon^{-1}[K_\varepsilon, E]v$ converges in X for all $v \in H^1(\omega)$. This concludes the proof that $\Lambda_\Gamma U\eta^2 u \in H^1(\omega)$.

It remains to show $(\partial/\partial s)U\eta^2 u \in H^1(\omega)$. It was noted above that $Ew = EU\eta u \in H^1(\omega)$. It follows that $EU\eta^2 u \in H^1(\omega)$. Using the fact that $\Lambda_\Gamma U\eta^2 u \in H^1(\omega)$ and the fact that the $V_j \Lambda_\Gamma^{-1}$ are bounded operators in $H^1(\omega)$, it follows that $\gamma(\partial/\partial s)U\eta^2 u \in H^1(\omega)$. Since γ is nonsingular, this implies $(\partial/\partial s)U\eta^2 u \in H^1(\omega)$. \square

5. Proof of Theorems 1–3. In order to show that S defined by (2.8) satisfies the requirements of Theorem 1, it remains to prove the following proposition.

PROPOSITION 4. $SAS^{-1} = A + B$, where $B \in B(X)$.

Proof. We first show that $u \in H^1_P(\Omega)$ implies $AS^{-1}u \in H^1_P(\Omega)$. Let $v = S^{-1}u$. Then $Sv = u \in H^1_P(\Omega)$ and, by Proposition 3, $v \in H^1_P(\Omega) \cap H^2(\Omega)$. We must show $Av \in H^1_P(\Omega)$.

Note that $\phi \Lambda \phi v \in H^1_P(\Omega)$ since $\phi = 0$ near Γ . Next we claim

$$(5.1) \quad \psi \zeta_k r_k^{-1} M r_k \zeta_k \psi v \in H^1_P(\Omega), \quad k = 1, \dots, K.$$

To prove this, note that $v \in H^1_P(\Omega) \cap H^2(\Omega)$ implies $r_k \zeta_k \psi v \in H^2(\Omega')$ and $(r_k \zeta_k \psi v)(y) \in P_k$ for a.a. $y \in \Gamma$ since $r_k(y)$ maps $P(y)$ onto P_k for y belonging to the support of ζ_k . Therefore $w = U r_k \zeta_k \psi v \in H^2(\omega)$ and $w(y, 0) \in P_k$ for a.a. $y \in \Gamma$. It follows that $(\Lambda_\Gamma w)(y, 0) \in P_k$ for a.a. $y \in \Gamma$ and $(M r_k \zeta_k \psi v)(y) \in P_k$ for a.a. $y \in \Gamma$. The assertion (5.1) follows from this. We conclude $Av \in H^1_P(\Omega)$.

It follows that $H^1_P(\Omega) \subset D(SAS^{-1})$, and for $u \in H^1_P(\Omega)$ we have

$$SAS^{-1}u - Au = [S, A]S^{-1}u = [Z, A]S^{-1}u,$$

with

$$Zv = \phi \Lambda \phi v + \sum \psi \zeta_k r_k^{-1} M r_k \zeta_k \psi v, \quad v \in H^1(\Omega).$$

Note that Z is an extension of $S - A - \beta$. We claim

$$(5.2) \quad \|[Z, A]v\| \leq C \|v\|_1 \left(\sum \|a_j\|_{C^2} + \|b\|_{C^1} \right), \quad v \in H^1_p(\Omega) \cap H^2(\Omega).$$

If $Kv = \phi \Lambda \phi v + \psi M \psi v$ is as in Lemma 10, then $(Z - K)v = (S - R)v$ is given by (4.1) with u replaced by v . We have already noted that $S - R$ is a bounded operator on both X and $H^1(\Omega)$. Therefore (5.2) follows from (4.10) since $Av = \mathcal{A}v + bv$.

It follows from (5.2) that $B = [Z, A]S^{-1}$ extends to a bounded operator on X . Thus we have $SAS^{-1}u = Au + Bu$ for $u \in H^1_p(\Omega)$. Since $H^1_p(\Omega)$ is a core of A , it follows that this holds for $u \in D(A)$. So SAS^{-1} is an extension of $A + B$. It follows that $S(A + \lambda)S^{-1} \supset A + B + \lambda$ for all λ . If λ is large, the right side has range X and the left side is one-to-one. Therefore we have equality: $SAS^{-1} = A + B$. \square

Theorem 1 is a direct consequence of Propositions 2 and 4.

Proof of Theorem 2. Let

$$S(t)u = (A_0(t) + \beta)u + \phi \Lambda \phi u + \sum \psi \zeta_k r_k^{-1} M r_k \zeta_k \psi u,$$

for $u \in D(S(t)) \equiv H^1_p(\Omega)$. Here $\beta = \sup_t \{\beta_t\}$, where β_t has the value (1.2). The matrix

$$c(x, t) = \sum a_j(x, t) \partial s / \partial x_j, \quad x \in \Omega', \quad 0 \leq t \leq T,$$

defined by (2.3), is now a function of t , as well as x . Therefore the value of σ in (2.1) should be chosen so that $c(x, t)$ is nonsingular for $x \in \Omega', 0 \leq t \leq T$.

Note that the only part of $S(t)$ which varies with t is $A_0(t)$. Thus $S(t)$ is continuously differentiable on $[0, T]$ to $B(H^1_p(\Omega), X)$, since this is true for $A_0(t)$.

According to Propositions 2 and 4, $S(t)$ is an isomorphism from $H^1_p(\Omega)$ onto X , and $S(t)A(t)S(t)^{-1} = A(t) + B(t)$, where $B(t) \in B(X)$. It follows from the inequality (5.2) that the map $t \rightarrow B(t) = [Z, A(t)]S(t)^{-1}$ is continuous on $[0, T]$ to $B(X)$. \square

Proof of Theorem 3. We noted in the remark after Theorem 3 that the conclusions of Theorem 3 are true if $P(x, t) = P(x)$ is independent of t . We turn now to the general case where $P(x, t)$ varies with t .

Suppose $r(x, t)$ is an orthogonal matrix-valued function of class C^2 on $\bar{\Omega} \times [0, T]$ with the property that

$$(5.3) \quad r(y, t) \text{ maps } P(y, 0) \text{ onto } P(y, t) \text{ for } y \in \Gamma, \quad 0 \leq t \leq T.$$

If one makes the change of variables

$$v(x, t) = r(x, t)^{-1}u(x, t), \quad x \in \bar{\Omega}, \quad 0 \leq t \leq T,$$

then equation (1.1) for u corresponds to the following equation for v .

$$\begin{aligned} \partial v / \partial t + \sum r^{-1} a_j r \partial v / \partial x_j + b'v &= r^{-1}f, \\ v(x, 0) &= r(x, 0)^{-1}\phi(x), \\ v(x, t) \in P(x, 0), \quad x \in \Gamma, \quad 0 \leq t \leq T, \end{aligned}$$

where $b' = r^{-1}(\partial r / \partial t + \sum a_j \partial r / \partial x_j + br)$. In this equation the boundary subspace $P(x, 0)$ does not vary with t . We also note that the assumption $f(\cdot, t) \in H_{\bar{P}_i}^1(\Omega)$ implies $r(\cdot, t)^{-1}f(\cdot, t) \in H_{\bar{P}_0}^1(\Omega)$. Therefore, as noted above, the conclusions of Theorem 3 hold for this equation. It follows that $u = rv$ is the desired solution to equation (1.1).

To complete the argument, one must show that there is a function $r(x, t)$ with the above properties. The condition (5.3) can be restated as

$$(5.4) \quad r(y, t)e(y, 0)r(y, t)^{-1} = e(y, t), \quad y \in \Gamma, \quad 0 \leq t \leq T,$$

where $e(y, t)$ is the orthogonal projection of R^N onto $P(y, t)$. Such an r is called a transformation function for $e(y, t)$, and it may be constructed using Kato's method (see [8, p. 99]). We take for r the solution of the differential equation

$$(\partial/\partial t)r(x, t) = q(x, t)r(x, t), \quad 0 \leq t \leq T,$$

$r(x, 0) = 1$, where $q = [\partial e / \partial t, e]$. Since q is antisymmetric, this r is an orthogonal matrix-valued function of class C^2 on $\Gamma \times [0, T]$ which satisfies (5.4) with $r(y, 0) = 1$. We must extend r to $\bar{\Omega} \times [0, T]$. Let $r'(y, s, t) = r(y, t\rho(s))$, where ρ is a function mapping $[0, \sigma]$ into $[0, 1]$ with $\rho(0) = 1$ and $\rho(s) = 0$ for $s \geq \sigma/2$. r' is defined on $\Gamma \times [0, \sigma] \times [0, T]$ with $r'(y, 0, t) = r(y, t)$ and $r'(y, s, t) = 1$ for $s \geq \sigma/2$. Letting $y(x), s(x)$ be the coordinates in Ω' introduced in (2.2), one may change variables to obtain $r(x, t) = r'(y(x), s(x), t)$. This r is defined on $\Omega' \times [0, T]$ with $r(x, t) = 1$ if $s(x) \geq \sigma/2$. r may then be extended to $\bar{\Omega} \times [0, T]$ by setting $r(x, t) = 1$ for $x \in \bar{\Omega} \sim \Omega'$. \square

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