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OPEN MAPPINGS OF THE UNIVERSAL CURVE ONTO CONTINUOUS CURVES

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Abstract. A criterion for the existence of an open mapping from one compact metric space onto another is established in this paper. This criterion is then used to establish the existence of a monotone open mapping of the universal curve onto any continuous curve and the existence of a light open mapping of the universal curve onto any nondegenerate continuous curve. These examples show that if f is a monotone open or a light open mapping of one compact space X onto another Y, then it will not necessarily be the case that $\dim Y \leqq \dim X + k$, where k is some positive integer.

1. **Introduction.** The two main theorems of this paper are the following:

THEOREM 1. *There exists a monotone open map of the universal curve onto any continuous curve such that each point-inverse set is homeomorphic to the universal curve.*

THEOREM 2. *There exists a light open map of the universal curve onto any nondegenerate continuous curve such that each point-inverse set is a Cantor set.*

R. D. Anderson announced Theorem 1 in 1956 [5]. However, since he never published a proof, the details are supplied here. In 1958 he conjectured [10] that there exists a light open map of the universal curve onto any n-cell. This question is answered by Theorem 2.

The existence of open dimension raising mappings has been of interest for some time. The first light open dimension raising mapping was given by Kolmogoroff [16] in 1937. In this example the domain is a 1-dimensional continuous curve and the range is 2-dimensional. In 1954 Keldyš [15] constructed a similar example where the range is a 2-cell. In 1952 Anderson [6] constructed a monotone open map from a 1-dimensional continuum onto the Hilbert cube. The techniques of this paper are basic to the proofs of Theorems 1 and 2.

Theorem 1 is of particular interest because of the following theorem of Dyer [13]: *If M and N are compact metric spaces, f is an open map of M onto N, $f^{-1}(y)$ is a nondegenerate continuous curve for each $y \in Y$, and there exists $\varepsilon > 0$ such that no simple closed curve in M of diameter less than ε is mapped to a point, then* $\dim M = \dim N + 1$. A theorem of Alexandroff [1] states that *if X and Y are continuous curves and f is an open map of X onto Y such that each point-inverse set is countable, then*

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dim $Y \leqq$ dim X. Theorem 2 shows that if a light open map has uncountable point-inverse sets, then the dimension of the range can be any positive integer or even infinite.

2. **Open mappings.** Let (X, d) be any metric space. Let G be any collection of subsets of X.

Notation. Let G^* denote the subset of X consisting of all points of X which are in some member of G. If A is a subset of X and ε a real number, then let $N_\varepsilon(A) = \{x \in X : \text{there exists } a \in A \text{ such that } d(x, a) < \varepsilon\}$. Let $d[A]$ denote the diameter of A. If B is another subset of X, then let $d[A, B]$ denote the Hausdorff distance between A and B. Let $\mu(G) = \max_{g \in G} \{d[g]\}$.

A compact metric space is called a *compactum*. A space is called a *continuous curve* if it is a connected and locally connected compactum. Int (A) will denote the interior of A relative to X.

PROPOSITION 1. *Let X, Y, Z_1, and Z_2 be compacta such that $X \subseteq Z_1$ and $Y \subseteq Z_2$. Suppose there exist two sequences of finite collections of compacta, $F = \{F_n\}_{n=1}^\infty$ and $G = \{G_n\}_{n=1}^\infty$, with the following properties*:

1. *$Z_1 \supseteq F_n^* \supseteq F_{n+1}^* \supseteq X$ for all n and $\bigcap_{n=1}^\infty F_n^* = X$.*
2. *$Z_2 \supseteq G_n^* \supseteq G_{n+1}^* \supseteq Y$ for all n and $\bigcap_{n=1}^\infty G_n^* = Y$.*
3. *Given $\varepsilon > 0$ there exists N such that $n > N$ implies $\mu(G_n) < \varepsilon$.*
4. *T_n is a function of F_n into G_n such that*
   (a) *if $f_n \in F_n$, $f_{n-1} \in F_{n-1}$, and $f_n \subseteq f_{n-1}$, then $T_n(f_n) \subseteq T_{n-1}(f_{n-1})$,*
   (b) *if $x \in X$, then there exists a nested sequence $\{f_n\}_{n=1}^\infty$ such that $x \in f_n \in F_n$.*
5. *If $f_n, f_n' \in F_n$ and $f_n \cap f_n' \neq \varnothing$, then $T_n(f_n) \cap T_n(f_n') \neq \varnothing$.*

Then there exists a continuous function of X into Y defined by $g(\bigcap_{n=1}^\infty f_n) = \bigcap_{n=1}^\infty T_n(f_n)$, where $f_n \in F_n$ and the sequence $\{f_n\}_{n=1}^\infty$ is nested.

Proof. We leave the proof to the reader.

DEFINITION. A continuous function from X onto Y is called *open* if and only if the image of every open subset of X is open in Y.

The next theorem is a generalization of Theorem 1 in [6].

PROPOSITION 2. *Let X, Y, and Z be compacta such that $X \subseteq Z$. Suppose there exist two sequences of finite collections of compacta $F = \{F_n\}_{n=1}^\infty$ and $G = \{G_n\}_{n=1}^\infty$ with the following properties*:

1. *$G_n^* = Y$ for all n.*
2. *$Z \supseteq F_{n-1}^* \supseteq F_n^*$ for all n and $\bigcap_{n=1}^\infty F_n^* = X$.*
3. *Given $\varepsilon > 0$ there exists integer n_1 such that $n > n_1$ implies $\mu(G_n) < \varepsilon$.*
4. *There exists a one-to-one and onto correspondence between F_n and G_n given by T_n such that*
   (a) *if $f_n \in F_n$, $f_{n-1} \in F_{n-1}$, and $f_n \subseteq f_{n-1}$, then $T_n(f_n) \subseteq T_{n-1}(f_{n-1})$,*
   (b) *if $x \in X$, then there exists a nested sequence $\{f_n\}_{n=1}^\infty$ such that $x \in f_n \in F_n$,*
   (c) *if $y \in Y$, then there exists a nested sequence $\{g_n\}_{n=1}^\infty$ such that $y \in g_n \in G_n$ and $T_n^{-1}(g_n) \subseteq T_{n-1}^{-1}(g_{n-1})$.*
5. *If $f_n, f_n' \in F_n$, then $f_n \cap f_n' \neq \varnothing$ if and only if $T_n(f_n) \cap T_n(f_n') \neq \varnothing$.*

6. *There exists* $\eta>0$ *such that if* $f_n, f_n' \in F_n$ *and* $f_n \cap f_n' \neq \varnothing$, *then* $f_n \subseteq N_{\eta/2^n}(f_n')$.

7. *There exists* $\mu>0$ *such that if* $f_n \in F_n$, $f_{n-1} \in F_{n-1}$, *and* $f_n \subseteq f_{n-1}$, *then* $f_{n-1} \subseteq N_{\mu/2^{n-1}}(f_n)$.

Then there exists an open mapping of X *onto* Y *defined by*

$$g\left(\bigcap_{n=1}^{\infty} f_n\right) = \bigcap_{n=1}^{\infty} T_n(f_n),$$

where $f_n \in F_n$ *and the sequence* $\{f_n\}_{n=1}^{\infty}$ *is nested. Moreover,* $g^{-1}(g(\bigcap_{n=1}^{\infty} f_n)) = \bigcap_{n=1}^{\infty} f_n$.

Proof. By Proposition 1, we know that g is a continuous map of X into Y. It is easy to show that the map g is onto.

If $\{f_n\}_{n=1}^{\infty}$ and $\{f_n'\}_{n=1}^{\infty}$ are nested sequences which have the property that there exists $f_m \in F_m$ such that $f_n \cap f_m \neq \varnothing$ and $f_n' \cap f_m \neq \varnothing$, then Properties 5, 6, and 7 can be combined to show that $\bigcap_{n=1}^{\infty} f_n' \subseteq N_{(1/2)^m(2\eta+3\mu)}(\bigcap_{n=1}^{\infty} f_n)$.

The map g is open. Let $x \in X$ and let V be any open subset of X containing x. We must show that $g(x)$ is interior to $g(V)$. Choose m large enough that $N_{(1/2)^m(2\eta+3\mu)}(x) \subseteq V$. Let $g_m^1, \ldots, g_m^r$ be all the members of G_m which contain $g(x)$. Let $\{f_n\}_{n=1}^{\infty}$ be a nested sequence such that $x \in f_n \in F_n$ for all n. Let $y \in g_m^i$, where $1 \leqq i \leqq r$. Let $\{f_n'\}_{n=1}^{\infty}$ be a nested sequence such that $f_n \in F_n$ for all n and $g(\bigcap_{n=1}^{\infty} f_n') = y$. Since $y \in g_m^i \cap T_m(f_m')$, $T_m^{-1}(g_m^i) \cap f_m' \neq \varnothing$. Since $g(x) \in g_m^i \cap T_m(f_m)$ $T_m^{-1}(g_m^i) \cap f_m \neq \varnothing$. Therefore, $x \in \bigcap_{n=1}^{\infty} f_n \subseteq N_{(1/2)^m(2\eta+3\mu)}(\bigcap_{n=1}^{\infty} f_n')$, and there must exist $x' \in \bigcap_{n=1}^{\infty} f_n'$ such that $d(x, x') < (\frac{1}{2})^m(2\eta+3\mu)$. Thus, $x' \in V$. Since $g(x')=y$, by the definition of g, we have shown that $g(V)$ contains $g_m^1 \cup \cdots \cup g_m^r$. Since $g(x) \in \operatorname{Int}(g_m^1 \cup \cdots \cup g_m^r)$, g is an open mapping.

We now want to show that if $\{f_n\}_{n=1}^{\infty}$ is a nested sequence such that $f_n \in F_n$, then $g^{-1}(g(\bigcap_{n=1}^{\infty} f_n)) = \bigcap_{n=1}^{\infty} f_n$.

By the definition of g, we know that $g^{-1}(g(\bigcap_{n=1}^{\infty} f_n))$ contains $\bigcap_{n=1}^{\infty} f_n$. If $g^{-1}(g(\bigcap_{n=1}^{\infty} f_n)) \neq \bigcap_{n=1}^{\infty} f_n$, then there exists $x' \in g^{-1}(g(\bigcap_{n=1}^{\infty} f_n)) - \bigcap_{n=1}^{\infty} f_n$. Let $\{f_n'\}_{n=1}^{\infty}$ be a nested sequence such that $x' \in f_n' \in F_n$ for all n. Choose m large enough that $(\frac{1}{2})^m(2\eta+3\mu) < \frac{1}{2}d[x', \bigcap_{n=1}^{\infty} f_n]$. Since $g(x') = g(\bigcap_{n=1}^{\infty} f_n)$, $T_m(f_m) \cap T_m(f_m') \neq \varnothing$. Therefore, $f_m \cap f_m' \neq \varnothing$ so that $\bigcap_{n=1}^{\infty} f_n' \subseteq N_{(1/2)^m(2\eta+3\mu)}(\bigcap_{n=1}^{\infty} f_n)$. Since $x' \in \bigcap_{n=1}^{\infty} f_n'$, there exists $x \in \bigcap_{n=1}^{\infty} f_n$ such that $d(x, x') < (\frac{1}{2})^m(2\eta+3\mu)$. Therefore, $d(x', \bigcap_{n=1}^{\infty} f_n) \leqq d(x, x') < \frac{1}{2}d[x', \bigcap_{n=1}^{\infty} f_n]$, a contradiction. Thus, $g^{-1}(g(\bigcap_{n=1}^{\infty} f_n)) = \bigcap_{n=1}^{\infty} f_n$, and we have established our proposition.

Notation. If G is a collection of subsets of X and A is a subset of X, then let $\operatorname{St}(A, G) = \{g \in G : g \cap A \neq \varnothing\}$. This collection is called the *star* of A. Inductively, let $\operatorname{St}^n(A, G) = \operatorname{St}(\operatorname{St}^{n-1}(A, G)^*, G)$. For convenience let $\operatorname{St}^0(A, G)$ denote the collection of all members of G contained in A. Let $\operatorname{St}^k(A, G)$ be the empty collection for all negative integers k.

When R. D. Anderson proved Theorem 1 in [6], he added additional inductive conditions to the original hypotheses in order to prove the theorem. For our purposes it is convenient to isolate these conditions into a separate proposition.

PROPOSITION 3. *Let X, Y, and Z be compacta such that* $X \subseteq Z$. *Suppose there exist two sequences* $J=\{J_n\}_{n=1}^{\infty}$ *and* $K=\{K_n\}_{n=1}^{\infty}$ *of finite collections of compacta with the following properties*:

1. $J_1=\{Y\}$ *and* $J_n^*=Y$ *for all n.*
2. $K_1=\{Z\}$, $K_{n-1}^* \supseteq K_n^*$, *and* $\bigcap_{n=1}^{\infty} K_n^*=X$.
3. *Given* $\varepsilon>0$ *there exists m such that* $n>m$ *implies* $\mu(J_n)<\varepsilon$.
4. *The members of* J_n *and* K_n *have disjoint nonempty interiors.*
5. *There exists an integer* $L_n>1$ *with the property that if* $j_n \in J_n$ *and* $j_{n-1}^1, \ldots, j_{n-1}^r$ *are all the members of* J_{n-1} *which meet* $\mathrm{St}^{L_n+1}(j_n, J_n)^*$, *then* $j_{n-1}^1 \cap \cdots \cap j_{n-1}^r \neq \varnothing$.
6. *There exists a one-to-one and onto correspondence between* J_n *and* K_n *given by* R_n *such that* $R_n(j_n) \cap R_n(j_n') \neq \varnothing$ *if and only if* $j_n \cap j_n' \neq \varnothing$. *There exists* $\eta>0$ *such that if* $j_n \cap j_n' \neq \varnothing$, *then* $R_n(j_n) \subseteq N_{\eta/2^{n-1}}(R_n(j_n'))$.
7. *If* $j_n \in J_n$ *and* $j_{n-1} \in J_{n-1}$, *then* $R_n(j_n) \cap R_{n-1}(j_{n-1}) \neq \varnothing$ *if and only if* j_{n-1} *meets* $\mathrm{St}^{L_n}(j_n, J_n)^*$. *Also if* $k_n \in K_n$ *and* $k_{n-1} \in K_{n-1}$ *and* $k_n \cap k_{n-1} \neq \varnothing$, *then* $k_n \cap \mathrm{Int}(k_{n-1}) \neq \varnothing$.
8. *There exists* $\lambda>0$, *such that if* $j_n \cap j_{n-1} \neq \varnothing$, *then* $R_{n-1}(j_{n-1}) \subseteq N_{\lambda/2^n}(R_n(j_n))$.

Then there exists an open map g of X onto Y, which has the property that $g^{-1}(g(\bigcap_{n=1}^{\infty} f_n))=\bigcap_{n=1}^{\infty} f_n$, *where* $\{f_n\}_{n=1}^{\infty}$ *is a nested sequence such that* $f_n=R_n(j_n^1) \cup \cdots \cup R_n(j_n^r)$, *where* $j_n^1 \cup \cdots \cup j_n^r \in G_n$. ($G_n$ *is defined below.*)

Proof. Let $G_n'=\{j_n^1 \cup \cdots \cup j_n^r : j_n^1 \cap \cdots \cap j_n^r \neq \varnothing, j_n^i \in J_n\}$. A member g of G_n' is in G_n if and only if $g=j_n^1 \cup \cdots \cup j_n^r$ and $j_n^1 \cap \cdots \cap j_n^r \cap j=\varnothing$ for all $j \notin \{j_n^1, \ldots, j_n^r\}$. Note that $G_n^*=Y$.

Since the members of J_n have nonempty disjoint interiors, each member of G_n can be written uniquely as a union of members of J_n. If $g_n=j_n^1 \cup \cdots \cup j_n^r$, then define $T_n^{-1}(g_n)=R_n(j_n^1) \cup \cdots \cup R_n(j_n^r)$. Since g_n is written uniquely as a union of members of J_n, T_n^{-1} is a well-defined function from G_n onto the collection $F_n=\{T_n^{-1}(g_n) : g_n \in G_n\}$. Since distinct members of K_n have disjoint nonempty interiors, T_n^{-1} is one-to-one. Therefore, T_n is well defined.

The proof of Proposition 3 will be a verification of the seven properties listed in the hypotheses of Proposition 2. Since most of this checking is routine, only a few properties are verified here.

Property 4b. Let $x \in X$. We must find a nested sequence $\{f_n\}_{n=1}^{\infty}$ such that $x \in f_n \in F_n$ for all n.

Let $f_1=Z$. For $n>2$ choose $k_n \in K_n$ such that $x \in k_n$. Let $k_{n-1}^1, \ldots, k_{n-1}^r$ be all the members of K_{n-1} which have the property that $R_{n-1}^{-1}(k_{n-1}^i)$ meets $\mathrm{St}^{L_n+1}(R_n^{-1}(k_n), J_n)^*$. If k_{n-1} is a member of K_{n-1} which contains x, then $x \in k_n \cap k_{n-1}$, so that by hypothesis 7, $R_{n-1}^{-1}(k_{n-1})$ meets $\mathrm{St}^{L_n}(R_n^{-1}(k_n), J_n)^*$. Therefore, $k_{n-1} \in \{k_{n-1}^1, \ldots, k_{n-1}^r\}$ and $x \in k_{n-1}^1 \cup \cdots \cup k_{n-1}^r$.

Now choose f_{n-1} to be any member of F_{n-1} which contains $k_{n-1}^1 \cup \cdots \cup k_{n-1}^r$. (There does exist such a member of F_{n-1}, because by hypothesis 5, we know that $R_{n-1}^{-1}(k_{n-1}^1) \cap \cdots \cap R_{n-1}^{-1}(k_{n-1}^r) \neq \varnothing$.) We want to show that if f_{n-1} has been chosen as above for all n, then the sequence $\{f_{n-1}\}_{n=2}^{\infty}$ is nested. Since $x \in k_n$, we know that k_n will be contained in f_n. Thus, $f_n \subseteq \mathrm{St}(k_n, K_n)^*$. Therefore, it is sufficient to show that $\mathrm{St}(k_n, K_n)^* \subseteq k_{n-1}^1 \cup \cdots \cup k_{n-1}^r$. If $k'_n \in \mathrm{St}(k_n, K_n)$ and $k'_n \cap k_{n-1} \neq \varnothing$, where $k_{n-1} \in K_{n-1}$, then $R_{n-1}^{-1}(k_{n-1})$ meets $\mathrm{St}^{L_n}(R_n^{-1}(k'_n), J_n)^*$. Since $k_n \cap k'_n \neq \varnothing$, $R_n^{-1}(k_n) \cap R_n^{-1}(k'_n) \neq \varnothing$, and thus $R_{n-1}^{-1}(k_{n-1})$ meets $\mathrm{St}^{L_n+1}(R_n^{-1}(k_n), J_n)^*$. Therefore, k_{n-1} is a member of $\{k_{n-1}^1, \ldots, k_{n-1}^r\}$. Therefore, $k'_n \subseteq k_{n-1}^1 \cup \cdots \cup k_{n-1}^r$.

Property 4c. Let $y \in Y$. Choose $j_n \in J_n$ such that $y \in j_n$. Let $j_{n-1}^1, \ldots, j_{n-1}^r$ be all the members of J_{n-1} which meet $\mathrm{St}^{L_n+1}(j_n, J_n)^*$. Since $j_{n-1}^1 \cap \cdots \cap j_{n-1}^r \neq \varnothing$, there exists $g_{n-1} \in G_{n-1}$ which contains $j_{n-1}^1 \cup \cdots \cup j_{n-1}^r$. We want to show that the sequence $\{g_{n-1}\}_{n=2}^{\infty}$ has the required properties.

Note that $y \in g_{n-1}$. For if $y \in j_{n-1}$, then $j_n \cap j_{n-1} \neq \varnothing$, and j_{n-1} meets $\mathrm{St}(j_n, J_n)^*$. Therefore, j_{n-1} meets $\mathrm{St}^{L_n+1}(j_n, J_n)^*$ and $j_{n-1} \in \{j_{n-1}^1, \ldots, j_{n-1}^r\}$. Thus, $y \in j_{n-1}^1 \cup \cdots \cup j_{n-1}^r \subseteq g_{n-1}$.

We must show that $g_n \subseteq g_{n-1}$. Since $y \in j_n$, $j_n \subseteq g_n$. Therefore, it is sufficient to show that $\mathrm{St}(j_n, J_n)^* \subseteq j_{n-1}^1 \cup \cdots \cup j_{n-1}^r$. If $j'_n \in \mathrm{St}(j_n, J_n)$ and $j'_n \cap j_{n-1} \neq \varnothing$, where $j_{n-1} \in J_{n-1}$, then $j_{n-1} \cap \mathrm{St}(j'_n, J_n)^* \neq \varnothing$. Hence, $j_{n-1} \cap \mathrm{St}^2(j_n, J_n)^* \neq \varnothing$ and $j_{n-1} \in \{j_{n-1}^1, \ldots, j_{n-1}^r\}$. Thus, $\mathrm{St}(j_n, J_n)^* \subseteq j_{n-1}^1 \cup \cdots \cup j_{n-1}^r$.

We want to prove that $T_n^{-1}(g_n) \subseteq T_{n-1}^{-1}(g_{n-1})$. It is sufficient to show that if $j'_n \in \mathrm{St}(j_n, J_n)$, then $R_n(j'_n) \subseteq R_{n-1}(j_{n-1}^1) \cup \cdots \cup R_{n-1}(j_{n-1}^r)$. If $j'_n \in \mathrm{St}(j_n, J_n)$ and $R_n(j'_n) \cap R_{n-1}(j_{n-1}) \neq \varnothing$, then $j_{n-1} \cap \mathrm{St}^{L_n}(j'_n, J_n)^* \neq \varnothing$. Thus, $j_{n-1} \cap \mathrm{St}^{L_n+1}(j_n, J_n)^* \neq \varnothing$ and $j_{n-1} \in \{j_{n-1}^1, \ldots, j_{n-1}^r\}$. Therefore, $R_n(j'_n) \subseteq R_{n-1}(j_{n-1}^1) \cup \cdots \cup R_{n-1}(j_{n-1}^r)$.

Property 6. Let $f_n, f'_n \in F_n$ such that $f_n \cap f'_n \neq \varnothing$. Thus, there exist $k_n, k'_n \in K_n$ such that $k_n \subseteq f_n$, $k'_n \subseteq f'_n$, and $k_n \cap k'_n \neq \varnothing$. By hypothesis 6, we have

$$f_n \subseteq N_{\eta/2^{n-1}}(k_n) \subseteq N_{\eta/2^{n-1}}(N_{\eta/2^{n-1}}(k'_n)) \subseteq N_{\eta/2^n}(k'_n) \subseteq N_{\eta/2^n}(f'_n).$$

Property 7. Let $f_n \in F_n$ and $f_{n-1} \in F_{n-1}$ be chosen such that $f_n \subseteq f_{n-1}$. If $j_n \in J_n$ and $j_n \subseteq T_n(f_n)$, then since $T_n(f_n) \subseteq T_{n-1}(f_{n-1})$, there exists $j_{n-1} \subseteq T_{n-1}(f_{n-1})$ such that $j_n \cap j_{n-1} \neq \varnothing$. Since $R_{n-1}(j_{n-1}) \subseteq N_{\lambda/2^n}(R_n(j_n))$, $f_{n-1} \subseteq N_{\eta/2^{n-1}}(R_{n-1}(j_{n-1})) \subseteq N_{\eta/2^{n-1}}(N_{\lambda/2^n}(R_n(j_n))) \subseteq N_{(1/2)^n(2\eta+\lambda)}(R_n(j_n)) \subseteq N_{(1/2)^n(2\eta+\lambda)}(f_n)$. If we let

$$\mu = \tfrac{1}{2}(2\eta+\lambda),$$

we have Property 7.

Proposition 3 now follows from Proposition 2.

3. **The two main theorems.**

Definition of the universal curve. Let N be the set of points in E^3 for which $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, and $0 \leqq z \leqq 1$. For $w = x, y, z$ and $i = 1, 2, 3, \ldots$, let $D_i(w)$ be the collection of all open intervals on the w-axis of length $1/3^i$ whose endpoints have w-coordinates which are positive rational numbers less than 1, the expression for each such rational number having 3^i as a denominator when in lowest terms. Let M be the set of all points (x, y, z) of N for which for no i do two of the points $(x, 0, 0)$, $(0, y, 0)$, and $(0, 0, z)$ belong to the set $D_i^*(x) \cup D_i^*(y) \cup D_i^*(z)$. The set M is called the *universal curve*.

Notation. If $A \subseteq X$, then Bd (A) will denote the boundary of A relative to X. Cl (A) will denote the closure of A relative to X. If G is a collection of subsets of X, then let $G(A)$ denote the collection of all members of G which are contained in A. Let $G'(A)$ denote the collection of elements of G not contained in A. Let $Z_G(A)$ be those elements of $G(A)$ which meet elements of $G'(A)$. Let $|G|$ denote the cardinality of G.

DEFINITION. If G is a collection of point sets, then G is said to be *simple* provided

1. For each $g \in G$, $g - \text{Bd}(g)$ is connected and $\text{Cl}(g - \text{Bd}(g)) = g$.
2. Distinct members of G have disjoint interiors relative to G^*.

DEFINITION. A finite collection of sets $G = \{g_1, \ldots, g_s\}$ is called a *simple chain* if $g_i \cap g_k \neq \varnothing$ if and only if $|i-k| \leq 1$.

Note. In this paper it will always be the case that simple chains are simple collections.

The terms interlace, 1-dimensional collection, A-defining sequence, and B-defining sequence are all defined in [2] so that we will not define them here. In the same paper R. D. Anderson showed that every 1-dimensional continuum for which there exists a B-defining sequence is homeomorphic to the universal curve.

PROPOSITION 4. *If Y is any continuous curve, then there exist two sequences of finite collections of continua $J = \{J_n\}_{n=1}^{\infty}$ and $K = \{K_n\}_{n=1}^{\infty}$ with the following properties:*

1. *$J_1 = \{Y\}$ and $J_n^* = Y$ for all n.*
2. *$K_1 = \{I^3\}$ and $K_n^* \subseteq K_{n-1}^*$ for all n.*
3. *$\mu(J_n) < 1/n$.*
4. *J_n and K_n are simple collections.*
5. *There exists an integer $L_n > 1$ with the property that if $j_n \in J_n$ and $j_{n-1}^1, \ldots, j_{n-1}^r$ are all the members of J_{n-1} which meet $\text{St}^{L_n+1}(j_n, J_n)^*$, then $j_{n-1}^1 \cap \cdots \cap j_{n-1}^r \neq \varnothing$.*
6. *There exists a one-to-one and onto correspondence between J_n and K_n given by R_n such that $R_n(j_n) \cap R_n(j_n') \neq \varnothing$ if and only if $j_n \cap j_n' \neq \varnothing$. If $j_n \cap j_n' \neq \varnothing$, then $R_n(j_n) \subseteq N_{8/2^{n-1}}(R_n(j_n'))$.*
7. *If $j_n \in J_n$ and $j_{n-1} \in J_{n-1}$, then $R_n(j_n) \cap R_{n-1}(j_{n-1}) \neq \varnothing$ if and only if j_{n-1} meets $\text{St}^{L_n}(j_n, J_n)^*$. Also, if $k_n \in K_n$, $k_{n-1} \in K_{n-1}$, and $k_n \cap k_{n-1} \neq \varnothing$, then*
$$k_n \cap \text{Int}(k_{n-1}) \neq \varnothing.$$
8. *If $j_n \in J_n$, $j_{n-1} \in J_{n-1}$, and $j_n \cap j_{n-1} \neq \varnothing$, then $R_n(j_n)$ meets every member of $H_{n-1}(R_{n-1}(j_{n-1}))$ and $R_{n-1}(j_{n-1}) \subseteq N_{8/2^n}(R_n(j_n))$.*
9. *There exists a finite simple collection of polyhedral 3-cells H_n such that H_n refines K_n, $K_n^* = H_n^*$, and $H = \{H_n\}_{n=1}^{\infty}$ is a B-defining sequence. Also, $\mu(H_n) < 4/2^n$. Distinct members of H_n meet in the empty set or in a 2-cell.*
10. *For each $j_n \in J_n$ there exists a collection $A_{j_n} \subseteq H_n$ such that $A_{j_n}^* \subseteq R_n(j_n)$ and such that if $j_n' \in J_n$ and $j_n \cap j_n' \neq \varnothing$, then $R_n(j_n')$ meets each member of A_{j_n} and each component of $R_n(j_n')$ meets some member of A_{j_n}. Each component of $R_n(j_n)$ will contain exactly one member of A_{j_n}. No member of H_n will meet two members of the collection $\{a \in A_{j_n} : j_n \in J_n\}$.*

Proof. We can choose the metric on Y so that $d[Y]<1$. Define R_1 between J_1 and K_1 by $R_1(Y)=I^3$. Since $d[Y]<1$, $\mu(J_1)<1$.

Let $H_1=\{I^3\}$. Since $d[I^3]=3^{1/2}<2$, $\mu(H_1)<4/2^1$. All of the other conditions in the first stage of the induction are trivially satisfied. Assume the theorem for the integer n.

We now want to define a function γ_n from the collection $E=\{h \cap h' : h, h' \in H_n$ and $h \cap h' \neq \varnothing\}$ into the subsets of I^3 with the following properties:

1. $\gamma_n(h \cap h')$ is a polyhedral 3-cell contained in Int $(h \cup h')$.
2. The members of the collection $\{\gamma_n(h \cap h') : h \neq h'\}$ are pairwise disjoint.
3. If h and h' are distinct members of H_n and $h \cap h' \neq \varnothing$, then $\gamma_n(h \cap h') \cap \gamma_n(h) =$ 2-cell and $\gamma_n(h \cap h') \cap h \cap h' =$ 2-cell.

Moreover, we want to require that there exists a simple chain of polyhedral 3-cells, $\Gamma_n(h \cap h')$ such that

1. The first member of $\Gamma_n(h \cap h')$ is $\gamma_n(h)$ and the last is $\gamma_n(h')$ (or vice versa).
2. $\Gamma_n(h \cap h')^* = \gamma_n(h) \cup \gamma_n(h \cap h') \cup \gamma_n(h')$.
3. $\Gamma_n(h \cap h')^*$ refines $\{h, h'\}$ and has an equal number of members in each of h and h'.
4. Consecutive members of $\Gamma_n(h \cap h')$ meet in 2-cells.
5. $\mu(\Gamma_n(h \cap h')) < 4(\frac{1}{2})^{n+1}$ and if $h \neq h'$, then $|\Gamma_n(h \cap h')| > 50$.

Let $\gamma_n(h)$ be a polyhedral 3-cell in the interior of h such that $d[\gamma_n(h)] < 4(\frac{1}{2})^{n+1}$. (The set $\gamma_n(h)$ can be taken to be a cube.) Let $\Gamma_n(h)=\{\gamma_n(h)\}$. Let $h \cap h'$ be a member of E, where $h \neq h'$. Let A be a polygonal arc in the interior of $h \cup h'$ such that A meets the 2-cell $h \cap h'$ in one point, and A meets each of $\gamma_n(h)$ and $\gamma_n(h')$ in an endpoint of A. We can assume the members of the collection of all such arcs are pairwise disjoint with the same properties as the old. "Fatten" each arc slightly so that the fattened arcs remain disjoint and meet $\gamma_n(h)$ in a 2-cell. The set $\gamma_n(h \cap h')$ will denote the fattened arc between $\gamma_n(h)$ and $\gamma_n(h')$. It is now clear that we can find a simple chain of 3-cells $\Gamma_n(h \cap h')$ with the desired properties. Let Γ_n denote the collection of all 3-cells which are members of some $\Gamma_n(h \cap h')$.

Let $y \in Y$. Since $y \in$ Int (St $(y, J_n)^*$), there exists $\varepsilon_y > 0$ such that $N_{\varepsilon_y}(y) \subseteq$ St $(y, J_n)^*$. The collection $\{N_{\varepsilon_y}(y) : y \in Y\}$ covers Y. Since Y is compact, there exists a number $\varepsilon' > 0$ such that every subset of Y of diameter less than ε' will be contained in some member of this cover.

Let $L_{n+1} = \max_{k \in K_n} \{|H_n(k)|\} + 1$.

Pick $\varepsilon < \min\{1/(n+1), \varepsilon'/(2L_{n+1}+3)\}$ and let J'_{n+1} be any ε-partitioning of Y. Let $J_{n+1} = \{\mathrm{Cl}\,(j'_{n+1}) : j'_{n+1} \in J'_{n+1}\}$.

For each $j_{n+1} \in J_{n+1}$ we want to construct a polyhedron in H_n^*. This polyhedron will be denoted by $R_{n+1}(j_{n+1})$ and will in fact be contained in $\bigcup_{h \cap h' \in E} \gamma(h \cap h')$.

First we must decide which members of H_n the set $R_{n+1}(j_{n+1})$ is to meet. Let $j_n \in J_n$ and let A_{j_n} be the collection given in Property 10 of the induction. If $j_{n+1} \in J_{n+1}$ and $j_n \cap \mathrm{St}^k(j_{n+1}, J_{n+1})^* \neq \varnothing$, but $j_n \cap \mathrm{St}^{k-1}(j_{n+1}, J_{n+1})^* = \varnothing$, then $R_{n+1}(j_{n+1})$ is to meet each member of $\mathrm{St}^{L_{n+1}-k}(A^*_{j_n}, H_n(R_n(j_n)))$ and $R_{n+1}(j_{n+1})$

is to miss every member of $H_n(R_n(j_n))$ not in this collection. From now on we will denote this collection by $\mathrm{St}^{L_{n+1}-k}(A^*_{j_n})$.

If $h \in H_n$, then let $J(h)=\{j_{n+1} \in J_{n+1} : R_{n+1}(j_{n+1})$ is to meet $h\}$. Let $h \cap h' \in E$. We want to define two functions $\theta_1^{h,h'}=\theta_1^{h',h}$ and $\theta_2^{h,h'}=\theta_2^{h',h}$ from $J(h) \cap J(h')$ onto collections of 3-cells in $\mathrm{Int}\,(\Gamma_n(h \cap h')^*)$ with the following properties:

1. If $C_i(h, h')$ denotes the range of $\theta_i^{h,h'}$, then $C_1(h, h') \cup C_2(h, h')$ is a finite simple collection of polyhedral 3-cells such that $|J(h) \cap J(h')|=|C_i(h, h')|$ for $i=1, 2$.

2. Each member of $C_i(h, h')$ meets each member of $\Gamma_n(h \cap h')$ in a 3-cell.

3. If j_{n+1}, $j'_{n+1} \in J(h) \cap J(h')$ and $\gamma \in \Gamma_n(h \cap h')$, then $j_{n+1} \cap j'_{n+1} \neq \varnothing$ iff for $i, k=1, 2$, $\theta_i^{h,h'}(j_{n+1}) \cap \theta_k^{h,h'}(j'_{n+1}) \cap \gamma$ is a 2-cell. If $j_{n+1} \cap j'_{n+1}=\varnothing$, then

$$\theta_i^{h,h'}(j_{n+1}) \cap \theta_k^{h,h'}(j_{n+1}) = \varnothing \quad \text{for } i, k = 1, 2.$$

4. Denote $\theta_i^{h,h}$ by θ_i^h and $C_i(h, h)$ by $C_i(h)$. If $h \neq h'$, then $\theta_i^h(j_{n+1})$ meets exactly one member of $C_k(h, h')$, where $k=1$ or 2. In particular, $\theta_i^h(j_{n+1}) \cap \theta_k^{h,h'}(j_{n+1})$ is a 2-cell.

To construct these collections first let $C'_1(h, h') \cup C'_2(h, h')$ denote a disjoint collection of polyhedral 3-cells in $\mathrm{Int}\,(\Gamma_n(h, h')^*)$ such that $|C_1(h, h')|=|C'_2(h, h')| =|J(h) \cap J(h')|$ and such that each member of $C'_i(h, h')$ meets each member of $\Gamma_n(h \cap h')$ in a 3-cell. Let $C'=\bigcup_{h\cap h' \neq \varnothing} C'_1(h, h') \cup C'_2(h, h')$. We will also assume that the members of C' are pairwise disjoint. Let $\theta_i^{h,h'}$ be any one-to-one and onto correspondence between $J(h) \cap J(h')$ and $C'_i(h, h')$. If $j_{n+1} \in J(h) \cap J(h')$ and $\gamma \in \Gamma_n(h, h')$, then construct a polygonal arc α in $\mathrm{Int}\,(\gamma)$ such that α meets each of $\theta_1^{h,h'}(j_{n+1}) \cap \gamma$ and $\theta_2^{h,h'}(j_{n+1}) \cap \gamma$ in an endpoint. The arc will meet no members of C' other than $\theta_1^{h,h'}(j_{n+1})$ and $\theta_2^{h,h'}(j_{n+1})$. If j_{n+1} and j'_{n+1} are distinct members of $J(h) \cap J(h')$ with a point in common and $\gamma \in \Gamma_n(h, h')$, then construct an arc $\alpha_{i,k}$ in $\mathrm{Int}\,(\gamma)$ such that $\alpha_{i,k}$ meets each of $\theta_i^{h,h'}(j_{n+1}) \cap \gamma$ and $\theta_k^{h,h'}(j'_{n+1}) \cap \gamma$ in an endpoint of $\alpha_{i,k}$. The arc $\alpha_{i,k}$ will meet no members of C' other than $\theta_i^{h,h'}(j_{n+1})$ and $\theta_k^{h,h'}(j'_{n+1})$. Adjust the collection of arcs so that no two meet. "Fatten" each in such a way that the collection of arcs remains disjoint, and meets $(C')^*$ in exactly two 2-cells. Associate each fattened arc to one of the two members of $C'_i(h, h')$ that it meets. A member of $C_i(h, h')$ will be a 3-cell which is the union of a member of $C'_i(h, h')$ and the fattened arcs associated with it. Let

$$C = \bigcup_{h\cap h' \neq \varnothing} C_1(h, h') \cup C_2(h, h').$$

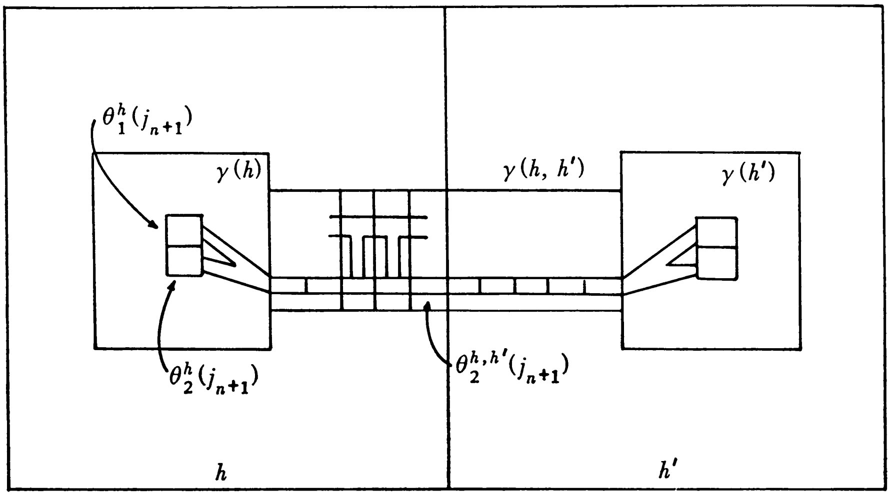
Let $R_{n+1}(j_{n+1})=\bigcup_{h\cap h' \in E} \theta_1^{h,h'}(j_{n+1}) \cup \theta_2^{h,h'}(j_{n+1})$. Let

$$H_{n+1} = \{c \cap \gamma : c \in C, \gamma \in \Gamma_n, \text{ and } c \cap \gamma \neq \varnothing\}.$$

Note that $\mu(H_{n+1})<4/2^{n+1}$. Let $K_{n+1}=\{R_{n+1}(j_{n+1}) : j_{n+1} \in J_{n+1}\}$.

Due to space limitations we will check only a few of the properties at the next step of the induction.

DIAGRAM



Property 8. Let $j_{n+1} \in J_{n+1}$ and $j_n \in J_n$ be such that $j_n \cap \operatorname{St}(j_{n+1}, J_{n+1})^* \neq \varnothing$. By the starring rules we know that $R_{n+1}(j_{n+1})$ meets every member of $\operatorname{St}^{L_{n+1}-1}(A^*_{j_n})$. Since $R_n(j_n)$ is connected and since $L_{n+1} \geqq |H_n(R_n(j_n))|+1$,

$$H_n(R_n(j_n)) = \operatorname{St}^{L_{n+1}}(A^*_{j_n}).$$

Therefore, $R_{n+1}(j_{n+1})$ meets every member of $H_n(R_n(j_n))$. This stronger version of Property 8 will be needed in the proof of Property 10.

Since $\mu(H_n) < 4/2^n$ and since $R_{n+1}(j_{n+1})$ meets every member of $H_n(R_n(j_n))$, we see that $R_n(j_n) \subseteq N_{4/2^n}(R_{n+1}(j_{n+1}))$.

Property 5. Let $j_{n+1} \in J_{n+1}$ and let $j_n^1, \ldots, j_n^r$ be all the members of J_n which meet $\operatorname{St}^{L_{n+1}+1}(j_{n+1}, J_{n+1})^*$. Since $\mu(J_{n+1}) < \varepsilon$,

$$d[\operatorname{St}^{L_{n+1}+1}(j_{n+1}, J_{n+1})^*] < (2(L_{n+1}+1)+1)\varepsilon < \varepsilon'.$$

Thus, there exists $y \in Y$ such that $\operatorname{St}^{L_{n+1}+1}(j_{n+1}, J_{n+1})^* \subseteq N_{\varepsilon_y}(y)$. By the choice of ε_y we know that $y \in j_n^1 \cap \cdots \cap j_n^r$. Therefore, $j_n^1 \cap \cdots \cap j_n^r \neq \varnothing$.

Property 6. By the definition of K_{n+1}, R_{n+1} maps J_{n+1} onto K_{n+1}. By construction distinct members of J_{n+1} are mapped to distinct members of K_{n+1}. Thus, R_{n+1} is one-to-one.

If $j_{n+1}, j'_{n+1} \in J_{n+1}$ and $j_{n+1} \cap j'_{n+1} = \varnothing$, then $R_{n+1}(j_{n+1}) \cap R_{n+1}(j'_{n+1}) = \varnothing$. If $j_{n+1} \cap j'_{n+1} \neq \varnothing$, then pick $j_n \in J_n$ such that $j_{n+1} \cap j'_{n+1} \cap j_n \neq \varnothing$. In Property 8 we showed that each of $R_{n+1}(j_{n+1})$ and $R_{n+1}(j'_{n+1})$ will meet every member of $H_n(R_n(j_n))$. Let h be any member of $H_n(R_n(j_n))$. Since $j_{n+1} \cap j'_{n+1} \neq \varnothing$, $\theta_1^h(j_{n+1}) \cap \theta_1^h(j'_{n+1}) \neq \varnothing$. Therefore, $R_{n+1}(j_{n+1}) \cap R_{n+1}(j'_{n+1}) \neq \varnothing$.

If $j_{n+1} \cap j'_{n+1} \neq \varnothing$, then we will show that $R_{n+1}(j_{n+1}) \subseteq N_{16/2^{n+1}}(R_{n+1}(j'_{n+1}))$. Let $x \in R_{n+1}(j_{n+1})$. There exist $h \in H_n$ which contains x, and $j_n \in J_n$ such that $h \subseteq R_n(j_n)$. If $h \in A_{j_n}$, then let j'_n be a member of J_n such that $j_{n+1} \cap j'_{n+1} \cap j'_n \neq \varnothing$.

By Property 8, we know that $R_{n+1}(j'_{n+1})$ meets every member of $H_n(R_n(j'_n))$. Since $R_{n+1}(j_{n+1}) \cap R_n(j_n) \neq \varnothing$, we know by 7 that $j_n \cap \mathrm{St}^{L_{n+1}}(j_{n+1}, J_{n+1})^* \neq \varnothing$. Thus, $j_n \cap j'_n \neq \varnothing$, and $R_n(j'_n) \cap h \neq \varnothing$. Let h' be a member of $H_n(R_n(j'_n))$ which meets h. Since $R_{n+1}(j'_{n+1})$ meets h', $d[x, R_{n+1}(j'_{n+1})] < d[h \cup h'] \leqq d[h] + d[h'] < 16/2^{n+1}$.

If $h \notin A_{j_n}$, then there exists an integer k such that $0 < k < L_{n+1}$ and $h \in \mathrm{St}^{L_{n+1}-k}(A^*_{j_n})$ and $j_n \cap \mathrm{St}^k(j_{n+1}, J_{n+1})^* \neq \varnothing$. Since $j_{n+1} \cup j'_{n+1} \neq \varnothing$,

$$j_n \cap \mathrm{St}^{k+1}(j'_{n+1}, J_{n+1})^* \neq \varnothing.$$

Therefore, $R_{n+1}(j'_{n+1})$ meets every member of $\mathrm{St}^{L_{n+1}-k-1}(A^*_{j_n})$. This collection will be nonempty because $L_{n+1}-k-1 \geqq 0$. The remainder of the proof is the same as when $h \in A_{j_n}$. Therefore, $R_{n+1}(j_{n+1}) \subseteq N_{16/2^{n+1}}(R_{n+1}(j'_{n+1}))$.

Property 4. The members of K_{n+1} have disjoint nonempty interiors by construction. We must show that the interiors are connected.

LEMMA. *If L is a finite collection of closed subsets and $A \subseteq L^*$, then A is connected provided*:

1. *L^* is connected.*
2. *$A \cap h$ is connected for all $h \in L$.*
3. *If $h, h' \in L$ and $h \cap h' \neq \varnothing$, then $A \cap h \cap h' \neq \varnothing$.*

Proof. The proof is routine and thus is omitted.

Let $j_{n+1} \in J_{n+1}$. Let j_n be a member of J_n which meets j_{n+1}. We know that $R_n(j_n)$ is connected by our induction assumption. We showed in Property 8 that $R_{n+1}(j_{n+1})$ meets every member of H_n in $R_n(j_n)$. To apply the lemma, let $L = H_n(R_n(j_n))$ and $A = R_{n+1}(j_{n+1}) \cap R_n(j_n)$. Therefore, $R_{n+1}(j_{n+1}) \cap R_n(j_n)$ is connected.

If $j'_n \in J_n$ and $R_{n+1}(j_{n+1}) \cap R_n(j'_n) \neq \varnothing$, then there exists an integer k such that $R_{n+1}(j_{n+1})$ meets exactly those members of $H_n(R_n(j'_n))$ in $\mathrm{St}^k(A^*_{j'_n})$. If $h \in A_{j'_n}$, then $\mathrm{St}^k(h)^*$ is connected because the star of a connected set is connected if the links are connected. To apply the lemma let $L = \mathrm{St}^k(h)$ and $A = R_{n+1}(j_{n+1}) \cap L^*$. Therefore, $R_{n+1}(j_{n+1}) \cap L^*$ is connected. Since $R_n(j_n) \cap h \neq \varnothing$, $R_{n+1}(j_{n+1}) \cap (R_n(j_n) \cup R_n(j'_n))$ is connected. Therefore, $R_{n+1}(j_{n+1})$ is connected. Since distinct members of H_{n+1} meet in 2-cells, $\mathrm{Int}(R_{n+1}(j_{n+1}))$ is also connected. Therefore, K_{n+1} is a simple collection.

Property 9. Since most of the properties of a B-defining sequence are obvious, we will only check the interlacing axiom and the fact that H^*_{n+1} is connected.

Let $h \in H_n$. Pick $j_n \in J_n$ so that $h \subseteq R_n(j_n)$. There exists an integer k such that $h \in \mathrm{St}^{L_{n+1}-k}(A^*_{j_n}) - \mathrm{St}^{L_{n+1}-k-1}(A^*_{j_n})$. Therefore, if $j_{n+1} \in J_{n+1}$, then $R_{n+1}(j_{n+1}) \cap h \neq \varnothing$ if and only if $j_n \cap \mathrm{St}^k(j_{n+1}, J_{n+1})^* \neq \varnothing$. Note that $j_n \cap \mathrm{St}^k(j_{n+1}, J_{n+1})^* \neq \varnothing$ if and only if $j_{n+1} \in \mathrm{St}^{k+1}(j_n, J_{n+1})$. Therefore, $J(h) = \mathrm{St}^{k+1}(j_n, J_{n+1})$. Since j_n is connected and the members of J_{n+1} are connected, $J(h)^*$ is connected. Since θ^h_i maps $J(h)$ onto $C_i(h)$ with the property that $j_{n+1} \cap j'_{n+1} \neq \varnothing$ if and only if

$\theta_i^h(j_{n+1}) \cap \theta_i^h(j'_{n+1}) \neq \varnothing$, $C_i(h)^*$ is also connected for $i=1, 2$. Moreover, since $\theta_1^h(j_{n+1}) \cap \theta_2^h(j_{n+1}) \neq \varnothing$ for each $j_{n+1} \in J(h)$, $C_1(h)^* \cup C_2(h)^*$ is connected.

Let Z denote $Z_{H_{n+1}}(h)$. If $z \in Z$, then there exists a unique $h' \in H_n$ different from h such that $z \cap h' \neq \varnothing$. In fact, z is contained in exactly one member of $C_1(h, h') \cup C_2(h, h')$. Denote this member by $C(z)$.

Let Z_1 and Z_2 denote two disjoint subcollections of Z such that $Z_1 \cup Z_2 = Z$. Let $W_1 = C_1(h)^* \cup (\bigcup_{z \in Z_1} C(z) \cap h)$ and $W_2 = C_2(h)^* \cup (\bigcup_{z \in Z_2} C(z) \cap h)$. Since each $C(z) \cap h$ is a 3-cell and meets each of $C_i(h)^*$, $i=1, 2$, in a 2-cell, both W_1 and W_2 are connected. Since $C_1(h) \cup C_2(h)$ is a simple collection and since $\operatorname{Int}(C(z)) \cap \operatorname{Int}(C(z')) = \varnothing$ if $z \neq z'$, the collection $\{W_1, W_2\}$ is simple. In the terminology of [2] $\{W_1, W_2\}$ is a simple complete amalgam of $H_{n+1}(h)$ such that $Z_i^* \subseteq W_i$. Therefore, $H_{n+1}(h)$ is interlaced in h.

Since $H_{n+1}(h)^*$ is connected for all $h \in H_n$, and since H_n^* is connected, the lemma used in Property 4 tells us that H_{n+1}^* is connected.

Property 10. Let $j_{n+1} \in J_{n+1}$. Pick $j_n \in J_n$ so that $j_{n+1} \cap j_n \neq \varnothing$. If $j'_{n+1} \in J_{n+1}$ and $j_{n+1} \cap j'_{n+1} \neq \varnothing$, then $j_n \cap \operatorname{St}(j'_{n+1}, J_{n+1})^* \neq \varnothing$. Thus, by Property 8, $R_{n+1}(j'_{n+1})$ meets every member of $H_n(R_n(j_n))$. In particular, $R_{n+1}(j'_{n+1})$ meets every member of A_{j_n}. If $h \in A_{j_n}$ and j'_{n+1} meets j_{n+1}, then by the construction $\theta_i^h(j_{n+1}) \cap \theta_i^h(j'_{n+1}) \neq \varnothing$. Thus, $\theta_i^h(j_{n+1})$ meets $R_{n+1}(j'_{n+1})$. The desired collection is

$$A_{j_{n+1}} = \{\theta_i^h(j_{n+1}) : h \in A_{j_n} \text{ and } i = 1, 2\}.$$

Note that $A_{j_{n+1}}^* \subseteq A_{j_n}^*$. We have now completed the proof of Proposition 4.

Let K be the class of all 1-dimensional continuous curves up to topological equivalence. Let M be the subclass of K consisting of those elements of K having no local cut points. Anderson [8] proved that a necessary and sufficient condition for a member of M to be the universal curve is that it contain no open subset imbeddable in the plane.

DEFINITION. A map will be called *monotone* if each point-inverse set is compact and connected.

THEOREM 1. *There exists a monotone open map of the universal curve onto any continuous curve such that each point-inverse set is also a universal curve.*

REMARK. Since the monotone image of a 2-sphere is a cactoid [1, p. 172], the monotone image of a compact subset of the plane will always have dimension less than or equal to two.

Proof. Let Y be any continuous curve. We can find sequences of continua $\{J_n\}_{n=1}^{\infty}$, $\{K_n\}_{n=1}^{\infty}$, and B-defining sequence $\{H_n\}_{n=1}^{\infty}$ with the ten properties stated in Proposition 4. Let $M = \bigcap_{n=1}^{\infty} H_n^* = \bigcap_{n=1}^{\infty} K_n^*$. Proposition 3 and Proposition 4 now combine to give an open map g of M onto Y. Since $\{H_n\}_{n=1}^{\infty}$ is a B-defining sequence, M is homeomorphic to the universal curve. Since the members of K_n are connected, g will be monotone.

To show that each point-inverse set is a universal curve, it is sufficient to show that for each $y \in Y$, $g^{-1}(y)$ is locally connected, $g^{-1}(y)$ has no local cut points, and $g^{-1}(y)$ contains no open subset imbeddable in the plane(2).

LEMMA 1. *If* $j_{n+3} \in J_{n+3}$, $j_{n+2} \in J_{n+2}$, *and* $j_{n+2} \cap j_{n+3} \neq \varnothing$, *then* $R_{n+3}(j_{n+3}) = R_{n+3}(j_{n+3}) \cap R_{n+2}(j_{n+2}) \cup T_1 \cup \cdots \cup T_s$ *where*

1. *each* T_i *is a connected union of members of* H_{n+3},
2. $T_i \cap R_{n+3}(j_{n+3}) \cap R_{n+2}(j_{n+2}) \neq \varnothing$ *for* $i=1, \ldots, s$,
3. *the set* $\mathrm{St}^5(T_i, H_{n+2})^*$ *is contained in five members of* H_{n+1}.

LEMMA 2. *Let* $j_{n+i} \in J_{n+i}$ *for* $i=2, \ldots, m$. *If* $j_{n+2} \cap \cdots \cap j_{n+m} \neq \varnothing$, *then* $R_{n+m}(j_{n+m}) = R_{n+m}(j_{n+m}) \cap \cdots \cap R_{n+2}(j_{n+2}) \cup S_1 \cup \cdots \cup S_r$, *where*

1. *each* S_i *is connected,*
2. *each* S_i *meets* $R_{n+m}(j_{n+m}) \cap \cdots \cap R_{n+2}(j_{n+2})$,
3. *each* S_i *lies in five members of* H_{n+1}.

Proof. The proof of Lemma 2 follows from Lemma 1 and the construction.

We now will show that $g^{-1}(y)$ is locally connected for all $y \in Y$. The proof given here follows the technique of [3].

Let $x \in g^{-1}(y)$ and let V be an open subset of $g^{-1}(y)$ containing x. It is sufficient to show that x is interior to the component of V containing x.

Pick n large enough so that $\mathrm{St}^3(x, H_n)^* \cap g^{-1}(y) \subseteq V$. Let $\rho = \mathrm{St}(x, H_n)^*$. If $h, h' \in H_n$, then we will show that there is at most one component of $g^{-1}(y) \cap \gamma(h \cap h')$ which meets every member of $\Gamma_n(h \cap h')$. This will prove that there are only a finite number of components of $g^{-1}(y) \cap (\mathrm{St}(\rho, H_n)^* \cup \bigcup_{\rho \cap h \neq \varnothing} \gamma(h \cap h'))$ meeting ρ. Therefore, V has only a finite number of components meeting ρ.

Let $\{f_n\}_{n=1}^{\infty}$ be a nested sequence of sets given by the conclusion of Proposition 3 such that $\bigcap_{n=1}^{\infty} f_n = g^{-1}(y)$.

Let $h \cap h' \in E$, where $h \neq h'$. If $R_{n+1}(j_{n+1})$ meets $\gamma(h \cap h')$, then $R_{n+1}(j_{n+1}) \cap \gamma(h \cap h') = \theta_1^{h,h'}(j_{n+1}) \cup \theta_2^{h,h'}(j_{n+1}) \cap \gamma(h \cap h')$. From now on we will denote $\theta_i^{h,h'}(j_{n+1})$ by $\theta_i(j_{n+1})$. Note that $f_{n+1} \cap \gamma(h \cap h')$ is connected.

Let $\gamma(h) = \gamma_1, \gamma_2, \ldots, \gamma_k = \gamma(h')$ be the members of $\Gamma_n(h \cap h')$ listed so that $\gamma_i \cap \gamma_j \neq \varnothing$ if and only if $|i-j| \leqq 1$.

Let F denote any continuum in H_{n+1}^* which is not contained in $\gamma(h \cap h')$. If S is a component of $F \cap \gamma(h \cap h')$ meeting γ_i, then note that S will either meet the interior of every member of $\{\gamma_1, \ldots, \gamma_i\}$ or the interior of every member of $\{\gamma_i, \ldots, \gamma_k\}$.

Suppose $f_{n+m} \cap \gamma(h \cap h')$ contains two components K_1 and K_2 which meet every member of $\Gamma_n(h \cap h')$. Let h_{n+m} be a member of H_{n+m} contained in $K_1 \cap \gamma_{25}$. Pick $j_{n+m} \in J_{n+m}$ such that $h_{n+m} \subseteq R_{n+m}(j_{n+m})$. We can find $j_{n+i} \in J_{n+i}$ for $i=2, \ldots, m-1$, such that $j_{n+m} \cap j_{n+m-1} \cap \cdots \cap j_{n+2} \neq \varnothing$. Therefore, by Lemma 2,

(2) The author is grateful to Professor Anderson for suggesting this approach.

we can find a connected subset of T of $R_{n+m}(j_{n+m}) \cap \gamma(h \cap h')$ which contains h_{n+m} and meets $(R_{n+m}(j_{n+m}) \cap \cdots \cap R_{n+2}(j_{n+2})) \cap (\gamma_{21} \cup \cdots \cup \gamma_{29})$.

Let h^1_{n+m} be a member of H_{n+m} contained in $R_{n+m}(j_{n+m}) \cap \cdots \cap R_{n+2}(j_{n+2})$ which meets T. Note that $h^1_{n+m} \subseteq K_1 \cap (\gamma_{20} \cup \cdots \cup \gamma_{30})$. Since $A^*_{j_{n+2}} \cap \gamma(h \cap h') = \varnothing$, we know that $R_{n+m}(j_{n+m}) \cap \cdots \cap R_{n+2}(j_{n+2})$ is not contained in $\gamma(h \cap h')$. Since $R_{n+m}(j_{n+m}) \cap \cdots \cap R_{n+2}(j_{n+2})$ is connected, we know that the component of $(R_{n+m}(j_{n+m}) \cap \cdots \cap R_{n+2}(j_{n+2})) \cap \gamma(h \cap h')$ containing h^1_{n+m}, denoted by K'_1, either meets every member of $\{\gamma_1, \ldots, \gamma_{20}\}$ or $\{\gamma_{30}, \ldots, \gamma_k\}$. Since $|\Gamma_n(h \cap h')| > 50$, both collections contain at least twenty members so that we can assume K'_1 meets every member of $\{\gamma_1, \ldots, \gamma_{20}\}$. Since K_2 meets every member of $\Gamma_n(h \cap h')$, K_2 meets γ_{11}.

Let h^2_{n+m} be a member of H_{n+m} contained in $K_2 \cap \gamma_{11}$. Choose $j'_{n+m} \in J_{n+m}$ and $j'_{n+m-1} \in J_{n+m-1}$ so that $h^2_{n+m} \subseteq R_{n+m}(j'_{n+m}) \cap R_{n+m-1}(j'_{n+m-1})$. Since $R_{n+m}(j'_{n+m})$ meets every member of H_{n+m-1} in $R_{n+m-1}(j_{n+m-1})$ that $R_{n+m}(j_{n+m})$ does (or vice versa), we know that K_2 contains a member of $H_{n+m}(R_{n+m}(j_{n+m}))$ in $\gamma_{10} \cup \gamma_{11} \cup \gamma_{12}$. Thus, applying Lemma 2 as before, we can find $h^3_{n+m} \in H_{n+m}$ such that $h^3_{n+m} \subseteq K_2 \cap R_{n+m}(j_{n+m}) \cap \cdots \cap R_{n+2}(j_{n+2}) \cap (\gamma_5 \cup \cdots \cup \gamma_{17})$. Therefore, for some $\gamma_i \in \{\gamma_5, \ldots, \gamma_{17}\}$ there exists two members h_{n+1} and h'_{n+1} of H_{n+1} such that $h_{n+1} \cup h'_{n+1} \subseteq \gamma_i$, $K_1 \cap R_{n+m}(j_{n+m}) \cap \cdots \cap R_{n+2}(j_{n+2}) \cap h_{n+1} \neq \varnothing$, and

$$K_2 \cap R_{n+m}(j_{n+m}) \cap \cdots \cap R_{n+2}(j_{n+2}) \cap h'_{n+1} \neq \varnothing .$$

Since $h_{n+1} \cup h'_{n+1} \subseteq \gamma_i \cap f_{n+1}$, we know by the rules of construction that $h_{n+1} \cap h'_{n+1} \neq \varnothing$. Thus, $R_{n+m}(j_{n+m}) \cap \cdots \cap R_{n+2}(j_{n+2}) \cap (h_{n+1} \cup h'_{n+1})$ is connected and $K_1 \cap K_2 \neq \varnothing$. This contradicts the assumption that K_1 and K_2 are distinct components. Therefore, $f_{n+m} \cap \gamma(h \cap h')$ contains at most one component which meets every member of $\Gamma_n(h \cap h')$, and $g^{-1}(y)$ is locally connected.

The set $g^{-1}(y)$ has no local cut points. Let $x \in g^{-1}(y)$ and let U be a connected open subset of $g^{-1}(y)$ containing x. Suppose $U - \{x\}$ is not connected. Let U_1 and U_2 be two different components of $U - \{x\}$. Let $u_1 \in U_1$ and $u_2 \in U_2$. Choose n large enough that $\mathrm{St}^2(x, H_n)^* \cap g^{-1}(y) \subseteq U - \{u_1, u_2\}$.

Choose h and h' in H_n and $j_{n+1} \in J_{n+1}$ such that $x \in \theta_i^{h,h'}(j_{n+1})$. There is no loss of generality in assuming that $x \in \mathrm{Int}\,(\theta_1^{h,h'}(j_{n+1}))$.

Let $S_1 = \theta_1^{h,h'}(j_{n+1})$.

LEMMA 3. *If $m > n$, then there exists a function T_m of H_m into H_m with the following properties*:

1. *If $h_m, h'_m \in H_m$ and $h_m \cap h'_m \neq \varnothing$, then $T_m(h_m) \cap T_m(h'_m) \neq \varnothing$.*
2. *If $h_m \in H_m$, $h_{m+1} \in H_{m+1}$ and $h_{m+1} \subseteq h_m$, then $T_{m+1}(h_{m+1}) \subseteq T_m(h_m)$.*
3. *If $h_m \in H_m$, then h_m and $T_m(h_m)$ lie in the same member of K_m.*
4. *If $h_m \in \mathrm{St}^k(A^*_{j_m})$, then $T_m(h_m) \in \mathrm{St}^k(A^*_{j_m})$ for all $j_m \in J_m$.*
5. *If $h_m \subseteq \mathrm{St}(S_1, H_n)^*$, then $T_m(h_m) \subseteq \mathrm{St}(S_1, H_n)^*$. If $h_m \nsubseteq \mathrm{St}(S_1, H_n)^*$, then $T_m(h_m) = h_m$.*
6. *$x \notin T_m(h_m)$ for all $h_m \in H_m$.*

Proof. The proof is left to the reader.

Combining Lemma 3 and Proposition 1 we have a continuous function f from M into M defined by $f(\bigcap_{m=n+1}^{\infty} h_m)=\bigcap_{m=n+1}^{\infty} T_m(h_m)$, where the sequence $\{h_m\}_{m=n+1}^{\infty}$ is any nested sequence with $h_m \in H_m$. Moreover, Properties 3, 5 and 6 in the lemma tell us that

1. $f(g^{-1}(y))\subseteq g^{-1}(y)$,
2. $f|M-\mathrm{St}\,(S_1, H_n)^*=$identity and $f(\mathrm{St}\,(S_1, H_n)^* \cap M)\subseteq \mathrm{St}\,(S_1, H_n)^*$,
3. $x \notin f(M)$.

Let $f_1=f|g^{-1}(y)$. Note that $f_1(U)\subseteq U$. Moreover, since u_1 and u_2 are not contained in $\mathrm{St}\,(S_1, H_n)^*$, $f_1(u_1)=u_1$ and $f_1(u_2)=u_2$. Therefore, $f_1(U)$ is a connected subset of U which contains both u_1 and u_2. But $x\notin f_1(U)$ which contradicts the assumption that U_1 and U_2 are distinct components of $U-\{x\}$. Therefore, $g^{-1}(y)$ has no local cut points.

The set $g^{-1}(y)$ is not locally imbeddable in the plane. Let U be any nonempty open subset of $g^{-1}(y)$ and let $x\in U$. Pick n large enough so that $\mathrm{St}^2\,(x, H_n)^* \cap g^{-1}(y)\subseteq U$. For each integer $m>n$ choose $j_m\in J_m$ so that $y\in j_m$. Since $j_{m+1}\cap j_m \neq\varnothing$, $R_{m+1}(j_{m+1})$ meets every member of H_m in $R_m(j_m)$. Therefore, the same argument as that used to show that $\{H_n\}_{n=1}^{\infty}$ is a B-defining sequence can be used to show that $\{H_m(R_m(j_m)\cap\cdots\cap R_{n+1}(j_{n+1}))\}_{m=n+1}^{\infty}$ is a B-defining sequence. Thus, $X=\bigcap_{m=n+1}^{\infty} R_m(j_m)$ is homeomorphic to the universal curve. Note that X is contained in $g^{-1}(y)$. Since $x\in g^{-1}(y)$, we know by the construction that $R_{n+1}(j_{n+1}) \cap \mathrm{St}^2\,(x, H_n)^*\neq\varnothing$. Therefore, $X\cap \mathrm{St}^2\,(x, H_n)^*\neq\varnothing$ and thus $X\cap U\neq\varnothing$. Since the universal curve is locally not imbeddable in the plane, U is not imbeddable in the plane.

Proposition 5. *If Y is any nondegenerate continuous curve, then there exist two sequences of finite collections of compacta $J=\{J_n\}_{n=1}^{\infty}$ and $K=\{K_n\}_{n=1}^{\infty}$ with the following properties*:

1. *$J_1=\{Y\}$ and $J_n^*=Y$ for all n.*
2. *$K_1=\{I^3\}$ and $K_n^*\subseteq K_{n-1}^*$ for all n.*
3. *$\mu(J_n)<1/n$.*
4. *J_n and the collection of components of members of K_n are simple collections.*
5. *There exists an integer $L_n>1$ with the property that if $j_n\in J_n$ and $j_{n-1}^1,\ldots,j_{n-1}^r$ are all the members of J_{n-1} which meet $\mathrm{St}^{L_n+1}\,(j_n, J_n)^*$, then $j_{n-1}^1\cap\cdots\cap j_{n-1}^r\neq\varnothing$.*
6. *There exists a one-to-one and onto correspondence between J_n and K_n given by R_n such that $R_n(j_n)\cap R_n(j_n')\neq\varnothing$ if and only if $j_n\cap j_n'\neq\varnothing$. If $j_n\cap j_n'\neq\varnothing$, then $R_n(j_n)\subseteq N_{8/2^{n-1}}(R_n(j_n'))$.*
7. *If $j_n\in J_n$ and $j_{n-1}\in J_{n-1}$, then $R_n(j_n)\cap R_{n-1}(j_{n-1})\neq\varnothing$ if and only if j_{n-1} meets $\mathrm{St}^{L_n}\,(j_n, J_n)^*$. Also if $k_n\in K_n$, $k_{n-1}\in K_{n-1}$, and $k_n\cap k_{n-1}\neq\varnothing$, then $k_n \cap \mathrm{Int}\,(k_{n-1})\neq\varnothing$.*
8. *If $j_n\in J_n$, $j_{n-1}\in J_{n-1}$ and $j_{n-1}\cap j_n\neq\varnothing$, then $R_n(j_n)$ meets every member of $H_{n-1}(R_{n-1}(j_{n-1}))$, and $R_{n-1}(j_{n-1})\subseteq N_{8/2^n}(R_n(j_n))$.*

9. *There exists a finite simple collection of polyhedral 3-cells H_n such that H_n refines K_n, $K_n^* = H_n^*$, and $H = \{H_n\}_{n=1}^{\infty}$ is a B-defining sequence. Also, $\mu(H_n) < 4/2^n$. Distinct members of H_n meet in the empty set or a 2-cell.*

10. *For each $j_n \in J_n$ there exists a collection $A_{j_n} \subseteq H_n$ such that $A_{j_n}^* \subseteq R_n(j_n)$ and such that if $j_n' \in J_n$ and $j_n \cap j_n' \neq \varnothing$, then $R_n(j_n')$ meets each member of A_{j_n}. Each component of $R_n(j_n)$ will contain exactly one member of A_{j_n} and each component of $R_n(j_n')$ will meet some member of A_{j_n}. No member of H_n will meet two members of the collection $\{a \in A_{j_n} : j_n \in J_n\}$.*

11. *If $j_n^1 \cap \cdots \cap j_n^r \neq \varnothing$, then the diameter of each component of $R_n(j_n^1) \cup \cdots \cup R_n(j_n^r)$ is less than $20/2^{n-1}$.*

Proof. We can choose the metric on Y so that $d[Y] < 1$. Define R_1 between J_1 and K_1 by $R_1(Y) = I^3$. Since $d[Y] < 1$, $\mu(J_1) < 1$.

Let $H_1 = \{I^3\}$. Since $d[I^3] = 3^{1/2} < 2$, $\mu(H_1) < 4/2^1$. All of the other conditions in the first stage of the induction are trivially satisfied. Assume the theorem for the integer n.

Let $y \in Y$. Since $y \in \mathrm{Int}(\mathrm{St}(y, J_n)^*)$, there exists $\varepsilon_y > 0$ such that $N_{\varepsilon_y}(y) \subseteq \mathrm{St}(y, J_n)^*$. The collection $\{N_{\varepsilon_y}(y) : y \in Y\}$ covers Y. Since Y is compact, there exists a number $\varepsilon' > 0$ such that every subset of Y of diameter less than ε' will be contained in some member of this cover.

Let $L_{n+1}' = 1 + |\{h \cap h' : h, h' \in H_n \text{ and } h \cap h' \neq \varnothing\}|$. Let $L_{n+1} = 3L_{n+1}'$. Let y_1 and y_2 be two distinct points in Y. Let ε be any positive real number such that

$$\varepsilon < \min\{(n+1)^{-1}, \varepsilon'/(4L_{n+1}+3), \tfrac{1}{3}d(y_1, y_2)\}.$$

Let J_{n+1}' be an ε-partitioning of Y and $J_{n+1} = \{\mathrm{Cl}_Y(j') : j' \in J_{n+1}'\}$.

Let $j_n \in J_n$ and let A_{j_n} be the collection given in Property 10 of the induction. If $j_{n+1} \in J_{n+1}$ and $j_n \cap \mathrm{St}^k(j_{n+1}, J_{n+1})^* \neq \varnothing$, but $j_n \cap \mathrm{St}^{k-1}(j_{n+1}, J_{n+1})^* = \varnothing$, then $R_{n+1}(j_{n+1})$ is to meet exactly those members of $H_n(R_n(j_n))$ in $\mathrm{St}^{L_{n+1}-k}(A_{j_n}^*)$.

Since we will need stronger versions of Properties 5 and 8 as well as Property 7 to define the construction of H_{n+1}, we mention them now. In the same way that we proved Property 5 in Proposition 4, we can prove that if $j_n^1, \ldots, j_n^r$ are all the members of J_n which meet $\mathrm{St}^{2 \cdot L_{n+1}}(j_{n+1}, J_{n+1})^*$, then $j_n^1 \cap \cdots \cap j_n^r \neq \varnothing$. Similarly, we can prove that if $j_n \cap \mathrm{St}^{2 \cdot L_{n+1}'}(j_{n+1}, J_{n+1})^* \neq \varnothing$, then $R_{n+1}(j_{n+1})$ meets every member of $H_n(R_n(j_n))$. Property 7 is the same as before.

Let h and h' be members of H_n such that $h \cap h' \neq \varnothing$. For each such pair we want to define a subcollection of J_{n+1}. If $h = h'$, then let $J(h) = \{j_{n+1} : R_{n+1}(j_{n+1}) \text{ is to meet } h\}$. If $h \neq h'$, then $J(h \cap h')$ will be a subcollection of $J(h) \cap J(h')$ defined by the two cases below.

Case 1. Let G_n be defined exactly as G_n is defined in the proof of Proposition 3. If $g \in G_n$ and $g = j_n^1 \cup \cdots \cup j_n^r$, then let

$$\lambda(g) = \{h \cap h' : h \cap h' \neq \varnothing, h \neq h', h \cup h' \subseteq R_n(j_n^1) \cup \cdots \cup R_n(j_n^r)\}.$$

Note that $|\lambda(g)| < L_{n+1}'$.

We want to define a collection of subsets ξ with the following properties:

1. The set ξ is contained in J_{n+1}.

2. The members of ξ are pairwise disjoint.

3. If $j'_{n+1} \in \xi$ and $j''_{n+1} \in \mathrm{St}^2(j'_{n+1}, J_{n+1})$, then $R_{n+1}(j''_{n+1})$ meets exactly those members of H_n in $H_n(R_n(j_n^1) \cup \cdots \cup R_n(j_n^r))$.

4. $|\xi| = |\lambda(g)|$.

If $\lambda(g)$ is the empty collection, then let ξ also be the empty collection. If $\lambda(g)$ is nonempty, then let j_{n+1} be a member of J_{n+1} such that $j_n^1 \cap \cdots \cap j_n^r \cap j_{n+1} \neq \varnothing$. Let $\xi' = \{j_{n+1}^0, \ldots, j_{n+1}^k\}$ be a simple chain in J_{n+1} such that $j_{n+1}^0 = j_{n+1}$ and $j_{n+1}^k \cap \mathrm{Bd}(g) \neq \varnothing$. The collection $\xi = \{j_{n+1}^0, j_{n+1}^2, \ldots, j_{n+1}^{2\cdot|\lambda(g)|}\}$ will have the desired properties. Let ϕ be any one-to-one correspondence from ξ onto $\lambda(g)$. If $j_{n+1} \in \xi$, then let j_{n+1} be a member of $J(\phi(j_{n+1}))$. Note that if $\phi(j_{n+1}) = h \cap h'$, then $j_{n+1} \in J(h) \cap J(h')$.

Case 2a. Let $h \in A_{j_n}$, where $j_n \in J_n$. If $j_{n+1} \in J(h)$, but there exists $j'_{n+1} \in \mathrm{St}(j_{n+1}, J_{n+1}) - J(h)$, then choose $j'_n \in J_n$ so that $j'_n \cap j_{n+1} \neq \varnothing$. Since $j'_n \cap j_n \neq \varnothing$, $R_n(j'_n) \cap h \neq \varnothing$. Let h' be a member of $H_n(R_n(j'_n))$ which meets h. Let j_{n+1} be a member of $J(h \cap h')$. Since $R_{n+1}(j_{n+1})$ meets h', j_{n+1} is also a member of $J(h) \cap J(h')$.

Case 2b. Let $h \in H_n(R_n(j_n))$, but $h \notin A_{j_n}$. Let h_1 be the member of A_{j_n} in the same component of $R_n(j_n)$ as h. Let $\eta = \{h_1, \ldots, h_{i-1}, h_i\}$ be a simple chain of minimal length in $H_n(R_n(j_n))$ such that $h_i = h$. Note that η has at least two members. If $j_{n+1} \in J(h)$, but there exists $j'_{n+1} \in \mathrm{St}(j_{n+1}, J_{n+1}) - J(h)$, then let j_{n+1} be a member of $J(h_{i-1} \cap h)$. The chain η will be considered fixed for all members of $J(h)$. Since η is of minimal length, we know that $j_{n+1} \in J(h_{i-1})$.

Let Γ_n be a simple 1-dimensional collection of 3-cells with the same properties that Γ_n had in Proposition 4. Also, let $\gamma(h \cap h')$ be defined as before.

By the same methods as those used in Proposition 4, we can define a simple 1-dimensional collection C of polyhedral 3-cells in Γ_n^* with the following properties:

1. If h and h' are members of H_n with a point in common, then there exist two one-to-one functions of $J(h \cap h')$ into C, denoted by $\theta_1^{h,h'} = \theta_1^{h',h}$ and $\theta_2^{h,h'} = \theta_2^{h',h}$. If $C_i(h, h')$ denotes the range of $\theta_i^{h,h'}$, then the members of $C_i(h, h')$ are pairwise disjoint.

2. The set $C_i(h, h')^*$ is contained in $\mathrm{Int}(\Gamma_n(h \cap h')^*)$.

3. Each member of $C_i(h, h')$ meets each member of $\Gamma_n(h \cap h')$ in a 3-cell.

4. If $j_{n+1} \in J(h)$, then $\theta_1^h(j_{n+1}) \cap \theta_2^h(j_{n+1}) \neq \varnothing$.

5. Let $C' = \bigcup_{h \cap h' \neq \varnothing} C_1(h, h') \cup C_2(h, h')$. If $h \neq h'$, then $\theta_i^{h,h'}(j_{n+1})$ meets exactly four members of C'. In particular, $\theta_i^{h,h'}(j_{n+1})$ meets each $\theta_k^h(j_{n+1})$ and $\theta_k^{h'}(j_{n+1})$, where $k = 1, 2$. Since $J(h \cap h') \subseteq J(h) \cap J(h')$, $\theta_k^h(j_{n+1})$ and $\theta_k^{h'}(j_{n+1})$ will exist.

6. If j_{n+1} and j'_{n+1} are distinct members of $J(h)$ such that $j_{n+1} \cap j'_{n+1} \neq \varnothing$, then there exist four simple collections of polyhedral 3-cells, $\alpha_i^h(j_{n+1})$ and $\alpha_i^h(j'_{n+1})$ where $i = 1, 2$, with the following properties:

(a) The set $\alpha_i^h(j_{n+1})^*$ is contained in $\gamma(h)$ and $|\alpha_i^h(j_{n+1})|=2$.

(b) One member of $\alpha_i^h(j_{n+1})$ meets $\theta_i^h(j_{n+1})$ in a 2-cell, and the other meets $\theta_i^h(j'_{n+1})$ in a 2-cell. Moreover, $\alpha_i^h(j_{n+1})^*$ meets no other members of C'.

(c) Distinct chains of the form α_i^h have disjoint underlying point sets.

Let $\alpha(j_{n+1})=$ the collection of all 3-cells which are members of some $\alpha_i^h(j_{n+1})$. Let $C=C' \cup (\bigcup_{j_{n+1}\in J_{n+1}} \alpha(j_{n+1}))$.

Let H_{n+1} and K_{n+1} be defined as in Proposition 4. Let

$$R_{n+1}(j_{n+1}) = \bigcup_{j_{n+1}\in J(h\cap h')} (\theta_1^{h,h'}(j_{n+1}) \cup \theta_2^{h,h'}(j_{n+1})) \cup \alpha(j_{n+1})^*.$$

We now have to check that the eleven properties in the induction statement hold for the integer $n+1$.

The first nine properties can be checked in the same way that the first nine properties in Proposition 4 were checked.

Property 10. Let K be a component of $R_{n+1}(j_{n+1})$. Let h be any member of H_n such that $K \cap \operatorname{Int}(h)\neq\varnothing$. Note that $\theta_1^h(j_{n+1})\subseteq K$. If $\operatorname{St}(j_{n+1}, J_{n+1})\subseteq J(h)$, then $\theta_1^h(j'_{n+1})$ exists for all $j'_{n+1}\in \operatorname{St}_{J_{n+1}}(j_{n+1})$. But if this is the case, then $R_{n+1}(j'_{n+1}) \cap \theta_1^h(j_{n+1})\neq\varnothing$.

If $\operatorname{St}(j_{n+1}, J_{n+1})\nsubseteq J(h)$ and $h\in A_{j_n}$ for some $j_n\in J_n$, then by Case 2a, we know that there exists $j'_n\in J_n$ and $h'\in H_n(R_n(j'_n))$ such that $h'\cap h\neq\varnothing$, $j'_n\cap j_{n+1}\neq\varnothing$, and $j_{n+1}\in J(h\cap h')$. Since $j_{n+1}\in J(h\cap h')$, $\theta_1^{h,h'}(j_{n+1})$ exists. Moreover, $\theta_1^h(j_{n+1}) \cup \theta_1^{h,h'}(j_{n+1}) \cup \theta_1^{h'}(j_{n+1})$ is a connected subset of $R_{n+1}(j_{n+1})$ so that $\theta_1^{h'}(j_{n+1})\subseteq K$. Since $\operatorname{St}(j_{n+1}, J_{n+1})\subseteq J(h')$, $\theta_1^{h'}(j_{n+1})$ will be the desired member of $H_{n+1}(K)$.

If $\operatorname{St}(j_{n+1}, J_{n+1})\nsubseteq J(h)$ and $h\subseteq R_n(j_n)$, but $h\notin A_{j_n}$, then by Case 2b, we found a simple chain $\eta=\{h_1,\dots,h_{i-1},h_i=h\}$ in $H_n(R_n(j_n))$ of minimal length where $h_1\in A_{j_n}$. By the starring rules we know that $\operatorname{St}(j_{n+1}, J_{n+1})\subseteq J(h_{i-1})$. As above, $\theta_1^{h_{i-1}}(j_{n+1})$ will be the desired member of $H_{n+1}(K)$. Pick one such member from each component of $R_{n+1}(j_{n+1})$ and denote this collection by $A_{j_{n+1}}$. Note that we can assume that $\bigcup_{j_{n+1}\in J_{n+1}} A_{j_{n+1}}$ is a subcollection of $\theta_1=\{\theta_1^h(j_{n+1}) : h\in H_n, j_{n+1}\in J(h)\}$. Since no member of H_{n+1} meets two members of θ_1, no member of H_{n+1} will meet two members of $\{a\in A_{j_{n+1}} : j_{n+1}\in J_{n+1}\}$.

Property 11. Let $j^1_{n+1},\dots,j^s_{n+1}$ be distinct members of J_{n+1} with a point in common. We want to show that if K is a component of $R_{n+1}(j^1_{n+1})\cup\cdots\cup R_{n+1}(j^s_{n+1})$, then $d[K]<5\cdot 4/2^n$.

Let $D_1=\{j_{n+1}\in J_{n+1} : j_{n+1}\in J(h\cap h')$ under the rules of Case 1$\}$. Let $D_2=\{j_{n+1}\in J_{n+1} : j_{n+1}\in J(h\cap h')$ under rules of Cases 2a and 2b$\}$. We want to show that $D_1^*\cap D_2^*=\varnothing$. If $j''_{n+1}\in \operatorname{St}^2(D_1^*, J_{n+1})$, then there exists $g\in G_n$ such that $g=j^1_n\cup\cdots\cup j^r_n$ and $R_{n+1}(j''_{n+1})$ meets exactly those members of H_n contained in $R_n(j^1_n)\cup\cdots\cup R_n(j^r_n)$. Therefore, no member of $\operatorname{St}(D_1^*, J_{n+1})$ can be a member of D_2 and $D_1^*\cap D_2^*=\varnothing$. Therefore, the collection $\{j^1_{n+1},\dots,j^s_{n+1}\}$ does not contain members from both D_1 and D_2. Hence, we can consider the two cases separately.

Case 1. Since the members of D_1 are pairwise disjoint, only one member of $\{j_{n+1}^1, \ldots, j_{n+1}^s\}$ can possibly belong to D_1. If $j_{n+1}^i \in D_1$, then exactly one component of $R_{n+1}(j_{n+1}^1) \cup \cdots \cup R_{n+1}(j_{n+1}^s)$ will meet more than one member of H_n. Namely, the component containing $\theta_1^h(j_{n+1}^i) \cup \theta_1^{h,h'}(j_{n+1}^i) \cup \theta_1^{h'}(j_{n+1}^i)$, where $h \cap h' = \phi(j_{n+1}^i)$. In this case $d[K] \leqq d[h \cup h'] < 2 \cdot 4/2^n$.

Case 2. Suppose K is a component of $R_{n+1}(j_{n+1}^1) \cup \cdots \cup R_{n+1}(j_{n+1}^s)$ and some member of $\{j_{n+1}^1, \ldots, j_{n+1}^s\}$ is in D_2. If $K \cap R_n(j_n) \neq \varnothing$, then let h_0 be a member of $H_n(R_n(j_n))$ such that $h_0 \in \mathrm{St}^k(A_{j_n}^*)$ and $K \cap \mathrm{St}^{k-1}(A_{j_n}^*)^* = \varnothing$.

Note that there does not exist any simple chain of four members $\{h_0, h_1, h_2, h_3\}$ in $H_n(R_n(j_n))$ such that $K \cap h_i \neq \varnothing$ for $i = 0, 1, 2, 3$ and $K \cap (h_i \cup h_{i+1})$ is connected. Therefore, if C is a component of $R_n(j_n)$, then $K \cap C \subseteq \mathrm{St}^2(h_0, H_n)^*$. If $R_{n+1}(j_{n+1}^i) \cap K \nsubseteq C$, then by Case 2a we know that there exists j_{n+1}^i such that $j_n^i \cap j_{n+1}^i \neq \varnothing$ and $j_{n+1}^i \in J(h_0 \cap h')$, where $h' \in H_n(R_n(j_n^i))$. Since $R_{n+1}(j_{n+1}^i)$ meets every member of $H_n(R_n(j_n^i))$, j_{n+1}^i is not a member of any $J(h^\alpha \cap h^\beta)$, where $h^\alpha \cup h^\beta \subseteq R_n(j_n^i)$ and $h^\alpha \neq h^\beta$. Therefore, $K \cap R_n(j_n^i) \subseteq \mathrm{St}(h_0, H_n)^*$. Since no member of H_n meets two members of $A = \bigcup_{j_n \in J_n} A_{j_n}$, each member of $\mathrm{St}(h_0, H_n)$ meets no member of A other than h_0. Therefore, $K \subseteq \mathrm{St}^2(h_0, H_n)^*$ and $d[K] < 20/2^n$. We have now established our theorem.

Definition. A map is called *light* if each point-inverse set is totally disconnected.

Theorem 2. *There exists a light open map of the universal curve onto any nondegenerate continuous curve such that each point-inverse set is a Cantor set.*

Remark. The above theorem is not true if we replace the universal curve by the plane universal curve. L. F. McAuley [17] showed that there is no light open map from the plane universal curve onto a 2-cell.

Proof. Let Y be any nondegenerate continuous curve. By Proposition 5, we can find sequences of compacta $\{J_n\}_{n=1}^\infty$, $\{K_n\}_{n=1}^\infty$, and $\{H_n\}_{n=1}^\infty$ with the eleven properties stated there. Let $M = \bigcap_{n=1}^\infty H_n^* = \bigcap_{n=1}^\infty K_n^*$. Proposition 3 and Proposition 5 now combine to give an open map g of M onto Y. Since $\{H_n\}_{n=1}^\infty$ is a B-defining sequence, M is homeomorphic to the universal curve. Since the diameters of the components of $R_n(j_n^1) \cup \cdots \cup R_n(j_n^r)$, where $j_n^1 \cap \cdots \cap j_n^r \neq \varnothing$, are less than $20/2^n$, g will be light.

Each point-inverse set is a Cantor set. Let $y \in Y$ and $x \in g^{-1}(y)$. Since g is light, it is sufficient to show that every neighborhood of x contains two points of $g^{-1}(y)$. Let V be any neighborhood of x in $g^{-1}(y)$. Let $\{f_n\}_{n=1}^\infty$ be any nested sequence given by the conclusion of Proposition 3 such that $g(\bigcap_{n=1}^\infty f_n) = y$. For each n, pick $j_n \in J_n$ such that $y \in j_n$ and $R_n(j_n) \subseteq f_n$. Choose n large enough that $\mathrm{St}^2(x, H_n)^* \cap g^{-1}(y) \subseteq V$.

Let h be a member of $\mathrm{St}^2(x, H_n)$ such that $R_{n+1}(j_{n+1})$ meets h. Since $R_{n+2}(j_{n+2})$ meets every member of $R_{n+1}(j_{n+1})$, $R_{n+2}(j_{n+2})$ meets both $\theta_1^h(j_{n+1}) = h_1$ and $\theta_2^h(j_{n+1}) = h_2$. Hence, $\theta_1^{h_1}(j_{n+2})$ and $\theta_1^{h_2}(j_{n+2})$ exist and are disjoint. Since $R_{n+i}(j_{n+i})$ meets every member of $R_{n+2}(j_{n+2})$, $\bigcap_{i=2}^\infty R_{n+i}(j_{n+i}) \cap \theta_1^{h_1}(j_{n+2}) \neq \varnothing$. Similarly,

$\bigcap_{i=2}^{\infty} R_{n+i}(j_{n+i}) \cap \theta_1^{h_2}(j_{n+2}) \neq \varnothing$. Therefore, V contains at least two points, and we have established Theorem 2.

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