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THE VARIATION OF SINGULAR CYCLES IN AN ALGEBRAIC FAMILY OF MORPHISMS

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Abstract. (1) Let $g: V^r \to W^m$ $(m \geqq r)$ be a morphism of nonsingular varieties over an algebraically closed field. Under certain conditions, one can define a cycle S_i on V with Supp $(S_i) = \{x \mid \dim_{k(x)} (\Omega^1_{X/Y})(x) \geqq i\}$.

The multiplicity of a component of S_i can be computed directly from local equations for g. If $V^r \subset \boldsymbol{P}^n$, and if $g: V \to \boldsymbol{P}^m$ is induced by projection from a suitable linear subspace of $\boldsymbol{P}^n$, then S_1 is $c_{m-r+1}(N \otimes \mathcal{O}(-1))$, up to rational equivalence, where N is the normal bundle of V in $\boldsymbol{P}^n$.

(2) Let $f: X \to S$ be a smooth projective morphism of noetherian schemes, where S is connected, and the fibres of f are absolutely irreducible r-dimensional varieties. For a geometric point η: Spec $(k) \to S$, and a locally free sheaf E on X, let X_η be the corresponding geometric fibre, and E_η the sheaf induced on X_η. If $E_1, \ldots, E_m$ are locally free sheaves on X, and if $i_1 + \cdots + i_m = r$, then the degree of the zero-cycle $c_{i_1}(E_{1\eta}) \cdots c_{i_m}(E_{m\eta})$ is independent of the choice of η.

(3) The results of (1) and (2) are used to study the behavior under specialization of a closed subvariety $V' \subset \boldsymbol{P}^{2r-1}$ which is the image under generic projection of a nonsingular $V^r \subset \boldsymbol{P}^n$.

1. **Introduction.** Let V^r be a nonsingular r-dimensional projective variety over an algebraically closed field k. If V is projected generically onto $V' \subset \boldsymbol{P}^{2r-1}$, then V' has a singular curve with finitely many points of a type known as *pinch points* (cf. §5). Suppose that char $(k)=0$ and that V can be specialized (along with its projective embedding) to a nonsingular variety V_1, defined over k_1, which can be projected generically onto $V_1' \subset \boldsymbol{P}^{2r-1}$. We can ask whether V_1' has the same number of pinch points as V'. The answer is "yes" if char $(k_1) \neq 2$; if char $(k_1)=2$, then V_1' has half as many pinch points as V'.

In this paper we develop some techniques which enable one to answer this and other enumerative questions of a similar nature. In §2, we prove a result which says roughly that a Chern polynomial of weight r is constant in a connected family of nonsingular projective varieties of dimension r (cf. Theorem 1). In §3 we recall the definition of the dependency cycle of a set of sections of a locally free sheaf on V. Mattuck [6] has shown how to express the rational equivalence class of this cycle as a Chern class of E. We express the multiplicities of its components as the

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lengths of certain Fitting ideals (cf. Proposition 4). In the case that the set of sections is part of a Serre sequence, these results follow from Theorem 2.7 of [3]. In §4, we define, under suitable assumptions, the singular cycles S_i of a morphism $f: V^r \to W^m$ of nonsingular varieties ($m \geqq r$). Intuitively, Supp (S_i) is the set of points where the kernel of the tangent map has dimension $\geqq i$. Our definition is stated in a form which gives immediately the multiplicities of the components of S_i. Let $V \subset \boldsymbol{P}^n$, and let $\pi: V \to \boldsymbol{P}^m$ be induced by generic projection. (See Lemma 3 for the meaning of "generic" in this context.) Theorem 2 says that the rational equivalence class of $S_1(\pi)$ is $c_{m-r+1}(N \otimes \mathcal{O}(-1))$, where N is the normal bundle of V in $\boldsymbol{P}^n$. Finally, §5 gives the application of the results of §§2, 3, and 4 to the problem stated in the first paragraph. We also give a concrete example to illustrate our result.

We will deal with Chern classes constructed in the rational equivalence ring $\mathscr{A}(V)$. Our references for this topic are Grothendieck's appendix to the Borel-Serre paper [4], and Séminaire Chevalley 1958, "Anneaux de Chow et applications." As usual, Grothendieck's *Eléments de géométrie algébrique* is denoted EGA.

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2. **Chern classes and algebraic families.** Let $f: X \to S$ be a smooth projective morphism of noetherian schemes. Assume that S is connected and that all fibres of f are absolutely connected. In particular, f is flat, and the fibre $X_s = f^{-1}(s)$ is an absolutely nonsingular irreducible projective variety over $\boldsymbol{k}(s)$ for all $s \in S$, where $\boldsymbol{k}(s)$ is the residue field of $\mathcal{O}_{S,s}$.

Let $E_1, \ldots, E_m$ be locally free sheaves on X. For each geometric point η: Spec $(k) \to S$ (k is an algebraically closed field), let X_η be the corresponding geometric fibre, and let $E_{1\eta}, \ldots, E_{m\eta}$ be the sheaves induced on X_η by $E_1, \ldots, E_m$. We will consider the Chern classes $c_i(E_{j\eta})$, which are elements of $\mathscr{A}(X_\eta)$, the Chow ring of X_η. For a rational equivalence class, z, of zero-cycles on X_η, we will denote by $\deg_\eta(z)$ the degree of z. Finally, we note that the dimension of X_η is independent of η.

THEOREM 1. *Let $f: X \to S$ and $E_1, \ldots, E_m$ be as above. If $i_1, \ldots, i_m$ is a sequence of positive integers such that $i_1 + \cdots + i_m = \dim(X_\eta)$, and $i_j \leqq \operatorname{rk}(E_j)$, for $j = 1, \ldots, m$, then the value of*

$$\deg_\eta (c_{i_1}(E_{1\eta}) \cdots c_{i_m}(E_{m\eta}))$$

is independent of the choice of the geometric point, η, of S.

REMARK. The E_i need not be distinct.

Proof. We first consider the case where the E_j are all invertible, so that $c_i(E_{j\eta}) = 0$ for $i > 1$. Thus let $r = \dim(X_\eta)$, and let $L_1, \ldots, L_r$ be invertible sheaves on X (not necessarily distinct). Then

$$\deg (c_1(L_{1\eta}) \cdots c_1(L_{r\eta})) = (L_{1\eta} \cdots L_{r\eta}),$$

which is the intersection number, computed on X_η. Using the techniques of [7, Lecture 12], one shows

$$(L_{1\eta}\cdots L_{r\eta}) = \chi(\mathcal{O}_{X_\eta})-\sum_{i=1}^{r}\chi(L_{i\eta}^{-1})+\sum_{i<j}\chi(L_{i\eta}^{-1}\otimes L_{j\eta}^{-1}) - \cdots +(-1)^r\chi(L_{1\eta}^{-1}\otimes\cdots\otimes L_{r\eta}^{-1}).$$

(By induction on the dimension, one shows that our expression is correct when $L_1 \cong \mathcal{O}_V(D)$, with D very ample, and is linear in each variable.)

The fact that this intersection number is independent of η is a consequence of the following lemma.

LEMMA 1. *Let $f: X \to S$ be a projective morphism, where S is noetherian, and let E be a coherent sheaf on X which is flat over S. For a field, k, and a k-valued point η:* Spec $(k) \to S$, *let*

$$\chi(E, \eta) = \sum_{i\geq 0} (-1)^i \dim_k H^i(X_\eta, E_\eta),$$

where E_η is the sheaf induced on the fibre X_η. If S is connected, then $\chi(E, \eta)$ is independent of the choice of k and η.

This lemma is a consequence of EGA III.7.9.11; cf. also 7.7.4, 7.7.12(i), and 7.9.3 of the same chapter.

We will now reduce the general case to the case just considered. We claim that there is a smooth projective morphism $g: Y \to X$ with connected fibres such that

(1) For each j, g^*E_j has a filtration of locally free subsheaves

$$E_{j1} \subset E_{j2} \subset \cdots \subset E_{j,p_j} = g^*E_j,$$

where $p_j = \operatorname{rk}(E_j)$, such that the quotients $L_{j\nu} = E_{j\nu}/E_{j,\nu-1}$ are invertible.

(2) There are invertible sheaves, $\Lambda_1, \ldots, \Lambda_s$, on Y and positive integers, $d_1, \ldots, d_s$, such that

(α) $d_1 + \cdots + d_s = d =$ dimension of any fibre of g,

(β) $g_{\eta *}(c_1(\Lambda_{1\eta})^{d_1}\cdots c_1(\Lambda_{s\eta})^{d_s}) = 1_{X_\eta}$,

for all geometric points η: Spec $(k) \to S$, where g_η is obtained by base extension.

We will first use (1) and (2) to achieve the desired reduction. From (1) we have

$$(*) \qquad c((g^*E_j)_\eta) = \prod_\nu (1+c_1(L_{j\nu\eta})).$$

Next, let $z \in \mathscr{A}(X_\eta)$, and set $y_\eta = (c_1(\Lambda_{1\eta})^{d_1}\cdots c_1(\Lambda_{s\eta})^{d_s})$. Using the projection formula (cf. [1, p. 3–17]), and (2), we obtain

$$z = z\cdot 1_{X_\eta} = z\cdot g_{\eta *}(y_\eta) = g_{\eta *}(g_\eta^*(z)\cdot y_\eta).$$

In particular, this implies that

$$(**) \qquad \deg_\eta(z) = \deg_\eta(g_\eta^*(z)\cdot y_\eta)$$

if z is the class of a zero-cycle on X_η. Note that the expression on the right is the degree of a zero-cycle on Y_η.

We apply (**) in the case $z=c_{i_1}(E_{1\eta})\cdots c_{i_m}(E_{m\eta})$ and use (*) to express $g_\eta^*(z)\cdot y_\eta$ as a polynomial of degree $r+d$ in the $c_1(\Lambda_{i\eta})$ and the $c_1(L_{j\nu\eta})$. In this way, we reduce the question to the case of the theorem already proved.

We now prove the existence of Y. In the case $m=1$, write $E=E_1$, and take $Y=\text{Flag}(E)$, the flag bundle of E over X (cf. [1, pp. 4–18 and 4–19.]). Thus, if $p=\text{rk}(E)$, we have a sequence of morphisms

$$Y = P_{p-1} \to P_{p-2} \to \cdots \to P_1 \to P_0 = X,$$

such that $P_1=\boldsymbol{P}(E)$, and $P_{i+1}=\boldsymbol{P}(F_i)$, for $i\geqq 1$, F_i being defined by the exactness of $0\to F_i\to q_i^*(F_{i-1})\to \mathcal{O}_{P_i}(1)\to 0$, where $q_i\colon P_i\to P_{i-1}$, and $F_0=E$. Now, it is known that

$$(***) \qquad q_{i\eta *}((c_1(\mathcal{O}_{P_i\eta *}(1))_\eta)^e) = 0, \qquad e < p-i,$$
$$= 1, \qquad e = p-i$$

(cf. [1, p. 4–13]). Letting $g_i\colon Y\to P_i$ be the composition $P_{p-1}\to\cdots\to P_i$, we set $\Lambda_i=g_i^*(\mathcal{O}_{P_i}(1))$ and $d_i=p-i$. Writing $g=g_0$ and using (***) and the projection formula repeatedly, we find

$$g_{\eta *}(c_1(\Lambda_{1\eta})^{p-1}\cdots c_1(\Lambda_{p-1,\eta})) = 1_{X_\eta}.$$

In the case $m>1$, we proceed by induction on m. Suppose that $q\colon Z\to X$ and the invertible sheaves $\Lambda_1,\ldots,\Lambda_s$ on Z have properties (1) and (2) relative to $E_1,\ldots,E_{m-1}$. Let $Y=\text{Flag}(q^*E_m)$, and let $r\colon Y\to Z$. We pull back $\Lambda_1,\ldots,\Lambda_s$ to $r^*\Lambda_1,\ldots,r^*\Lambda_s$ on Y and form $p-1$ other invertible sheaves on Y by the process used in the case $m=1$, where $p=\text{rk}(E_m)$. Setting $g=q\circ r$, one uses the projection formula and the fact that $g_{\eta *}=q_{\eta *}\circ r_{\eta *}$ to check that the sheaves constructed on Y have property (2).

3. **Dependency cycles.** In this section, V will be a nonsingular quasi-projective variety defined over an algebraically closed field k, and E will be a locally free sheaf on V, of rank $p\leqq\dim(V)$. We will recall the definition of the dependency cycle of a set of sections of E (cf. Mattuck [6, §4]), and we will give an expression for the multiplicities of the components of this cycle.

Let $\hat{E}=\boldsymbol{P}(\mathcal{O}_V\oplus E^*)$ (cf. EGA II, §4), where $E^*=\mathcal{H}om(E,\mathcal{O}_V)$, and let $\pi\colon\hat{E}\to V$ be the natural projection. Let $\sigma_0\colon V\to\hat{E}$ correspond to the surjection $\mathcal{O}_V\oplus E^*\to\mathcal{O}_V$ which restricts to the identity on $\mathcal{O}_V$ and to the zero map on E^*. Thus $V\cong\sigma_0(V)$. Let $s\in\Gamma(V,E)$, and let $\sigma\colon V\to\hat{E}$ correspond to the surjection $\mathcal{O}_V\oplus E^*\to\mathcal{O}_V$ which restricts to the identity on $\mathcal{O}_V$ and to the map dual to $s\colon\mathcal{O}_V\to E$ on E^*. Under the assumption that $\sigma^{-1}(\sigma_0(V))=\{x\in V\mid s(x)=0\}$ is of pure codimension p in V, we say that *the cycle of zeros of s is defined*; this cycle is defined to be $\sigma^*(\sigma_0(V))$ and is denoted $s^*(0)$.

Let us write $s^*(0)=\sum \mu_Z Z$; the sum ranges over the irreducible components of $\sigma^{-1}(\sigma_0(X))$. For a fixed Z, let x be the generic point of Z, and let U be a neighborhood of x on which E is free. On U, we can write $s=\sum_{i=1}^{p} f_i e_i$, where $f_i \in \Gamma(U, \mathcal{O}_V)$, and $\{e_1, \ldots, e_p\}$ is a basis of $\Gamma(U, E)$.

PROPOSITION 1. *With the above notation,* $\mu_Z=\ell_A(A/(f_1, \ldots, f_p))$, *where* $A=\mathcal{O}_{V,x}$, *and* $\ell_A(M)$ *denotes the length of the A-module M.*

Proof. Since $\sigma_0(V)$ and $\sigma(V)$ are locally complete intersections on $\hat{E}$, one can check that $\ell_A(A/(f_1, \ldots, f_p))$ is the multiplicity of the corresponding component of $\sigma_0(V)\cdot\sigma(V)$. Q.E.D.

PROPOSITION 2. *If* $s \in \Gamma(V, E)$ *is such that* $s^*(0)$ *is defined, then* $s^*(0)=c_p(E)$ *in* $\mathcal{A}(V)$.

The proof can be modeled directly after one given by Grothendieck [4, Theorem 2].

Let $q \leqq p$, and let $s_1, \ldots, s_q \in \Gamma(V, E)$. Now, $s_1, \ldots, s_q$ define a map of $\mathcal{O}_V$-modules, $\phi\colon \mathcal{O}_V^q \to E$; by duality we get $\phi^*\colon E^* \to \mathcal{O}_V^q$. Let $P=V\times \boldsymbol{P}^{q-1}$, and let $\pi\colon P \to V$ and $\rho\colon P \to \boldsymbol{P}^{q-1}$ be the two projections. Then ϕ^* gives rise to a map of $\mathcal{O}_P$-modules: $\pi^*E^* \to \rho^*\mathcal{O}(1)$. Tensoring with $\mathcal{O}(-1)$ and dualizing, we obtain $s \in \Gamma(P, \pi^*E \otimes \mathcal{O}(1))$. We will make two assumptions:

(1) $s^*(0)$ is defined;

(2) π sends every irreducible component of $s^*(0)$ to a subvariety of codimension $p-q+1$ in V.

When these assumptions are satisfied, we define the *dependency cycle*, $D(\Sigma)$ of the set $\Sigma=\{s_1, \ldots, s_q\}$ to be $\pi_*(s^*(0))$.

PROPOSITION 3. *Let* $\Sigma\subset\Gamma(V, E)$ *and* $s \in \Gamma(P, \pi^*E \otimes \mathcal{O}(1))$ *be as above. If* $D(\Sigma)$ *is defined, then* $D(\Sigma)=c_{p-q+1}(E)$ *in* $\mathcal{A}(V)$.

For a proof, see Mattuck [6, Theorem 2].

We will now give a local description of the section $s \in \Gamma(P, \pi^*E \otimes \mathcal{O}(1))$. Let $U=\operatorname{Spec}(A)\subset V$ be an affine open set on which E is free, and let $\{e_1, \ldots, e_p\}$ be a basis of $\Gamma(U, E)$. Thus, $s_i=\sum_{j=1}^{p} f_{ij}e_j$, where $f_{ij} \in \Gamma(U, \mathcal{O}_V)$, for $i=1, \ldots, q$. Let $T_1, \ldots, T_q$ be a basis of $\Gamma(\pi^{-1}U, \mathcal{O}(1))$, and let U_m be the open subset where $T_m\neq 0$, for $1\leqq m\leqq q$. It is easy to verify:

$$(*)\qquad s|U_m = \left(\sum_{i=1}^{q}\sum_{j=1}^{p} (t_i/t_m)f_{ij}e_j\right)\otimes T_m,$$

where $T_i|U_m=(t_i/t_m)\cdot(T_m|U_m)$. This implies

$$(**)\qquad s|(\pi^{-1}U) = \sum_{i,j} f_{ij}(e_j\otimes T_i).$$

Relation $(*)$ also implies that $x \in \operatorname{Supp}(D(\Sigma))$ iff $s_1, \ldots, s_q$ become dependent in $E\otimes \boldsymbol{k}(x)$.

Assume that $D(\Sigma)$ is defined, and let $D(\Sigma)=\sum \mu_Z \cdot Z$. For a fixed Z, let x be a generic point of Z, and choose $U \subset V$, as above, so that $x \in U$.

PROPOSITION 4. *With the above notation and assumptions,* $\mu_Z = \ell_A(A/I)$, *where* $A = \mathcal{O}_{V,x}$, *and* I *is the ideal in* A *generated by the* $q \times q$ *minors of the* $q \times p$ *matrix* (f_{ij}). *Thus,* I *is the* 0*th Fitting ideal of* Coker (ϕ_x), *where* $\phi\colon \mathcal{O}_V^q \to E$ *is defined by* $s_1, \ldots, s_q$.

Proof. Every $q \times q$ minor of the matrix (f_{ij}) vanishes along $W = Z \cap U$. Assumption (2) implies that some $(q-1)\times(q-1)$ minor of (f_{ij}) is nonzero at x. We may assume that this minor is nonzero at all points of U. Thus, some subset of Σ, say $\{s_1, \ldots, s_{q-1}\}$, is a subset of a basis of $\Gamma(U, E)$.

Let $J \subset \Gamma(U, \mathcal{O}_V)$ be generated by the $q \times q$ minors of the matrix (f_{ij}). Since $E|U$ is free, J is the $(p-q)$th Fitting ideal of Coker $(\Gamma(U, \mathcal{O}_V)^q \to \Gamma(U, E))$. Hence J is independent of the choice of basis of $\Gamma(U, E)$ (cf. Fitting [2, Hauptsatz]), and we may assume that $e_i = s_i$, for $i = 1, \ldots, q-1$. Therefore J is generated by the $p-q+1$ elements $f_{qq}, \ldots, f_{qp}$. Moreover, the relation (**) becomes

$$s|(\pi^{-1}U) = \sum_{j=1}^{q-1} (e_j \otimes T_j + f_{qj}(e_j \otimes T_q)) + \sum_{j=q}^{p} f_{qj}(e_j \otimes T_q).$$

Thus, Supp $(s^*(0) \cap U) \subset U_q$. Now, (*) becomes

$$s|U_q = \sum_{j=1}^{q-1} (t_j + f_{qj})(e_j \otimes T_q) + \sum_{j=q}^{p} f_{qj}(e_j \otimes T_q),$$

where we have set $t_q = 1$. Let Z' be the unique component of $s^*(0)$ lying above Z, let y be the generic point of Z', and let $B = \mathcal{O}_{\hat{E},y}$. If $\mu_{Z'}$ is the multiplicity of Z' in $s^*(0)$, Proposition 1 implies that $\mu_{Z'} = \ell_B(B/I^*)$, where

$$I^* = (t_1 + f_{q1}, \ldots, t_{q-1} + f_{q,q-1}, f_{qq}, \ldots, f_{qp})B.$$

Further, we have isomorphisms

$$B/I^* \simeq (A/I)[T_1, \ldots, T_{q-1}]/(T_1 + \bar{f}_{q1}, \ldots, T_{q-1} + \bar{f}_{q,q-1}) \simeq A/I.$$

This implies that (i) A and B have the same residue field, and (ii) $\ell_A(A/I) = \ell_B(B/I^*)$. Now, (i) implies that $\pi_*(Z') = Z$. Using (ii), we obtain $\mu_Z = \ell_A(A/I)$. Q.E.D.

4. **The cycles S_i.** Let $f\colon V^r \to W^m$ be a morphism of nonsingular varieties over the algebraically closed field k, where $m \geq r$. Let $\Omega^1_{V/W}$ be the sheaf of relative differentials, and let $S_i \subset V$ be the closed subset $\{x \mid \dim_{k(x)} (\Omega^1_{V/W} \otimes k(x)) \geq i\}$, for each $i \geq 1$. We will say that S_i *has the proper codimension* iff every irreducible component has codimension $i(m-r+i)$. If S_i has the proper codimension, we define the *cycle* S_i by

$$S_i = \sum \nu_Z \cdot Z;$$

the sum ranges over all components, and $\nu_Z=\ell_{\mathcal{O}_x}(\mathcal{O}_x/J)$, where x is the generic point of Z, and J is the $(i-1)$st Fitting ideal of $(\Omega^1_{V/W})_x$. We must check that ν_Z is finite. We have an exact sequence of $\mathcal{O}_x$-modules

$$\Omega^1_{\mathcal{O}_y/k}\otimes\mathcal{O}_x\xrightarrow{\phi}\Omega^1_{\mathcal{O}_x/k}\longrightarrow\Omega^1_{\mathcal{O}_x/\mathcal{O}_y}\longrightarrow 0,$$

where $y=f(x)$. The first two terms in this sequence are free of rank m and r respectively. If we choose suitable bases for these modules, then ϕ is described by an $m\times r$ matrix, and J is generated by its minors of rank $r-i+1$. Since Z is the only component of S_i containing x, the maximal ideal $\mathfrak{m}_x$ is the only associated prime of J. Therefore ν_Z is finite. We will call the cycles S_i the *singular cycles* of f.

LEMMA 2. *Suppose that S_1 is of the proper codimension. Let Z be a component of S_1, with $Z\not\subset S_2$, and with generic point x. Then $\nu_Z=\ell_{\mathcal{O}_x}((\Omega^1_{V/W})_x)$.*

Proof. Since $(\Omega^1_{V/W})_x$ is generated by one element, $(\Omega^1_{V/W})_x\simeq\mathcal{O}_x/J$, where J is the 0th Fitting ideal. Q.E.D.

If x is a closed point of V, then $x\in S_i$ iff $\dim_k(\mathfrak{m}_x/(\mathfrak{m}_y\mathcal{O}_x+\mathfrak{m}_x^2))\geqq i$, where $y=f(x)$, and $\mathfrak{m}_x$ and $\mathfrak{m}_y$ are the maximal ideals of $\mathcal{O}_x$ and $\mathcal{O}_y$. In particular, suppose that $V\subset\boldsymbol{P}^n$ and that $f=\pi\colon V\to\boldsymbol{P}^m$ is induced by projection from an $(n-m-1)$-subspace $L\subset\boldsymbol{P}^n$ such that $L\cap V=\varnothing$. For a closed point $x\in V$, it follows that $x\in S_i$ iff $\dim(L\cap t_{V,x})\geqq i-1$, where $t_{V,x}$ is the r-subspace of $\boldsymbol{P}^n$ tangent to V at x. (This can be checked using the techniques of the proof of Proposition 3 of [8].)

LEMMA 3. *Let $r\leqq m<n$, and let $V^r\subset\boldsymbol{P}^n$ be nonsingular. Then there is a dense open subset of the Grassmann variety $G=G(n,n-m-1)$ consisting of linear subspaces $L\subset\boldsymbol{P}^n$ such that $L\cap V=\varnothing$, and $S_i(\pi)$ is purely of codimension $i(m-r+i)$ for all i, where $\pi\colon V\to\boldsymbol{P}^m$ is induced by projection from L.*

Proof. For each i, consider the correspondence $Z_i\subset V\times G$ consisting of all (x,L) such that $\dim(L\cap t_{V,x})\geqq i-1$. By a counting of constants which uses standard facts about Schubert cycles on G (cf. [5, Chapter XIV, §2]), one finds that $\dim(Z_i)=\dim(G)+r-i(m-r+i)$. Q.E.D.

Henceforth we will fix an $(n-m-1)$-subspace $L\subset P=\boldsymbol{P}^n$, such that $L\cap V=\varnothing$, and $S_i=S_i(\pi)$ is of the proper codimension for all i, where $\pi\colon V\to\boldsymbol{P}^m$ is induced by projection from L. We will also fix a basis, $\{T_0,\dots,T_n\}$ of $\Gamma(P,\mathcal{O}(1))$, such that L is given by $T_0=\cdots=T_m=0$.

THEOREM 2. *Let V, L, and π be as above. Then*

$$S_1=c_{m-r+1}(N\otimes\mathcal{O}_V(-1))\quad in\ \mathscr{A}(V),$$

where N is the normal bundle of V in $\boldsymbol{P}^n$, viz., $N=(I/I^2)^*=\mathscr{H}om_{\mathcal{O}_V}(I/I^2,\mathcal{O}_V)$, where $I=\operatorname{Ker}(\mathcal{O}_P\to\mathcal{O}_V)$.

Proof. For $j=0,\dots,n$, let T_j be as above, and let $U_j\subset P$ be the open set $\{x\mid T_j(x)\neq 0\}$. Choose $t_0=1,t_1,\dots,t_n\in k(P)$ such that $T_i=(t_i/t_j)T_j$ on U_j. Thus,

$\Gamma(U_j, \mathcal{O}_P)$ is the polynomial ring $k[t_0/t_j, \ldots, t_n/t_j]$. For $0 \leqq i \leqq n$ and $i \neq j$, let D_{ij} be the derivation of $\Gamma(U_j, \mathcal{O}_P)$ given by $D_{ij}(t_v/t_j) = \delta_{iv}$ (Kronecker delta), and let $D_{jj} = -\sum_{i \neq j} (t_i/t_j) D_{ij}$. We can extend these derivations to derivations of the function field $k(P) = k(t_1, \ldots, t_n)$ and check that $D_{ij} = (t_j/t_h) D_{ih}$ for all i, j, h.

For each (i, j), D_{ij} induces an element $\bar{D}_{ij} \in \Gamma(U_j, (I/I^2)^*)$. We define

$$s_{ij} = \bar{D}_{ij} \otimes T_j^{-1} \in \Gamma(U_j, N \otimes \mathcal{O}(-1)).$$

Since $\bar{D}_{ij} | U_j \cap U_h = (t_j/t_h) \bar{D}_{ih} | U_j \cap U_h$, the various s_{ij} (for each i) fit together to give sections $s_0, \ldots, s_n \in \Gamma(V, N \otimes \mathcal{O}(-1))$.

Let x be any closed point of V. Since $L \cap V = \varnothing$, we have $T_q(x) \neq 0$ for some $q \leqq m$. We may assume $q = 0$; thus $x \in U_0$. Let $g_1, \ldots, g_{n-r}$ generate I in a neighborhood of x, and let $\sigma_i \colon I/I^2 \to \mathcal{O}_V$ be given locally by $\sigma_i([g_j]) = \delta_{ij}$. With these notations, we find that

$$s_i = \sum_{j=1}^{n-r} (\partial g_j / \partial t_i)(\sigma_j \otimes T_0^{-1})$$

in some neighborhood of x. (The g_j are polynomials in $t_1, \ldots, t_n$.) In particular, it follows that $s_{m+1}, \ldots, s_n$ become dependent in $(N \otimes \mathcal{O}(-1)) \otimes \boldsymbol{k}(x)$ iff $x \in S_1$. To see this, we note that $L \cap t_{V,x} \neq \varnothing$ iff there are elements $b_{m+1}, \ldots, b_n \in k$, not all zero, such that $\sum_{j=m+1}^{n} b_j \cdot (\partial g_i / \partial t_j)(x) = 0$, for $i = 1, \ldots, n-r$. Since S_1 is purely of codimension $m - r + 1 = (n-r) - (n-m) + 1$, it follows that $D(\Sigma)$ is defined ($\Sigma = \{s_{m+1}, \ldots, s_n\}$), and $\operatorname{Supp}(D(\Sigma)) = \operatorname{Supp}(S_1)$. Proposition 4 implies that $D(\Sigma) = \sum \mu_Z Z$, where $\mu_Z = \ell_{\mathcal{O}_x}(\mathcal{O}_x / J')$, with x the generic point of Z, and J' the ideal in $\mathcal{O}_x$ generated by the minors of order $(n-m-1)$ of the $(n-r) \times (n-m-1)$ matrix $(\partial g_i / \partial t_j)_{1 \leqq i \leqq n-r, m+1 \leqq j \leqq n}$. Finally, Proposition 3 implies that $D(\Sigma) = c_{m-r+1}(N \otimes \mathcal{O}_V(-1))$ in $\mathscr{A}(V)$.

Since S_1 and $D(\Sigma)$ have the same irreducible components, it will follow that $S_1 = D(\Sigma)$ if we can show that the ideal J' defined above is the 0th Fitting ideal of $\Omega^1_{\mathcal{O}_x/\mathcal{O}_y}$ ($y = \pi(x)$). This will complete the proof. Thus, let $U = \boldsymbol{P}^n - L = U_0 \cup \cdots \cup U_m$. The projection $\pi \colon \boldsymbol{P}^n - L \to \boldsymbol{P}^m$ has the property that $\pi | U_i$ looks like the projection $\boldsymbol{A}^m \times \boldsymbol{A}^{n-m} \to \boldsymbol{A}^m$. Hence in the exact sequence of $\mathcal{O}_V$-modules

$$I/I^2 \xrightarrow{\delta} \Omega^1_{U/\boldsymbol{P}^m} \otimes \mathcal{O}_V \longrightarrow \Omega^1_{V/\boldsymbol{P}^m} \longrightarrow 0,$$

the first two terms are free. Moreover, if $x \in U_0$, and if $g_1, \ldots, g_{n-r}$ generate I_x, then δ is given by

$$\delta([g_i]) = \sum_{j=m+1}^{n} (\partial g_i / \partial t_j)(dt_j \otimes 1),$$

for $i = 1, \ldots, n-r$. We conclude that J' is the 0th Fitting ideal of $\Omega^1_{V/\boldsymbol{P}^m}$. Q.E.D.

COROLLARY 1. *With the assumptions of Theorem* 2,

$$S_1 = \sum_{j=0}^{m-r+1} \binom{m+1}{j} \gamma_{q-j} h^j \quad \textit{in } \mathscr{A}(V),$$

where h is the divisor class of a hyperplane section, $(1-\gamma_1+\gamma_2-\cdots+(-1)^r\gamma_r)=c(\Omega_V^1)^{-1}$, *and* $q=m-r+1$.

Proof. With $P=\boldsymbol{P}^n$, let D_{ij} be the derivations of $k(P)$ defined above. Let $\Theta=(\Omega_P^1)^*$, and let $s_0,\ldots,s_n\in\Gamma(P,\Theta\otimes\mathcal{O}_P(-1))$ satisfy $s_i|U_j=D_{ij}\otimes T_j^{-1}$. The s_i give a map $\psi\colon\mathcal{O}_P^{n+1}\to\Theta\otimes\mathcal{O}_P(-1)$ and thus an exact sequence

$$0\to\Omega_P^1\to\mathcal{O}_P(-1)^{n+1}\to\mathcal{O}_P\to 0.$$

This gives $c(\Omega_P^1)=(1-h)^{n+1}$.

On V we have an exact sequence

$$0\to(I/I^2)\to\Omega_P^1\otimes\mathcal{O}_V\to\Omega_V^1\to 0.$$

Thus $c(N^*)=j^*(c(\Omega_P^1))\cdot c(\Omega_V^1)^{-1}$, where $j\colon V\to P$. The rest of the computation will be omitted.

COROLLARY 2. *If* $m=r=1$, *and if the curve* $V\subset\boldsymbol{P}^n$ *has genus g and degree d, then*

$$2g-2 = -2d+\sum_{x\in S_1(\pi)}\ell_{\mathcal{O}_x}((\Omega^1_{V/\boldsymbol{P}^1})_x).$$

This is a special case of Hurwitz' formula for the genus change under a morphism of curves. For a proof, use the exact sequence

$$0\to I/I^2\to j^*(\Omega^1_{\boldsymbol{P}^n/k})\to\Omega^1_{V/k}\to 0$$

(with $j\colon V\to\boldsymbol{P}^n$) to show that deg $(N\otimes\mathcal{O}(-1))=2g+2d-2$. Lemma 2 shows that the summation on the right side of the formula also is deg $(N\otimes\mathcal{O}(-1))$.

GENERALIZATION. It is also possible to express the cycles S_i, $i>1$, in terms of Chern classes. Thus let $G'=G'(n,n-r-1)$ be the Grassmannian which parameterizes $(n-r)$-quotients of rank $n+1$ free sheaves, and let $\Phi\colon\mathcal{O}_{G'}^{n+1}\to E$ be the universal surjection. There is a morphism $u\colon V\to G'$ such that Φ pulls back to $\psi\colon\mathcal{O}_V^{n+1}\to N\otimes\mathcal{O}_V(-1)$. The Chern classes of E can be expressed in terms of Schubert cycles on G'. On the other hand, it seems clear that the Schubert cycles which pull back to the cycles S_i can be expressed in terms of the Fitting ideals of Coker (Φ). Thus, one should obtain formulas which are similar to formula (10) on p. 357 of [5].

5. **Enumeration of pinch points.** Let V^r be a nonsingular projective variety over an algebraically closed field k. One can find a projective embedding $V\subset\boldsymbol{P}^n$ such that there is a finite morphism $\pi\colon V\to\boldsymbol{P}^{2r-1}$ induced by projection from an $(n-2r)$-subspace $L\subset\boldsymbol{P}^n$ satisfying

(I) $S_1(\pi)$ is purely 0-dimensional, and $\pi|S_1(\pi)$ is injective. Moreover, if $r\geqq 2$, then $\pi^{-1}(\pi(x))=\{x\}$ for all $x\in S_1(\pi)$.

(II) If $x\in S_1(\pi)$ and $y=\pi(x)$, then $\hat{\mathcal{O}}_{V,x}$ and $\hat{\mathcal{O}}_{\boldsymbol{P}^{2r-1},y}$ can be identified with formal power series rings $B=k[[t_1,\ldots,t_r]]$ and $A=k[[t_1,\ldots,t_{2r-1}]]$ so that π induces the homomorphism $f\colon A \to B$ given by

$$\begin{aligned} f(t_i) &= t_i && \text{for } i = 1, \ldots, r-1, \\ f(t_i) &= t_{i-r+1}t_r && \text{for } i = r, \ldots, 2r-2, \\ f(t_{2r-1}) &= t_r^2 + t_r^3. \end{aligned}$$

(If char $(k) \neq 2$ we can replace $t_r^2+t_r^3$ by t_r^2.)

In fact, if $r \geqq 2$, a suitable embedding may be found by replacing any given embedding by the embedding determined by hypersurface sections of degree $d \geqq 2$, and Theorem 3 of [8] states that if L is chosen generically, then π has the following properties which imply (I) and (II). *If $V' = \pi(V)$, then V' is birational to V*, Sing (V') *is purely of dimension* 1, *and V' has singular branches at only finitely many closed points $y \in V'$. If V' has a singular branch at y, then $\hat{\mathcal{O}}_{V',y}$ is isomorphic to $f(A)$, where $f\colon A \to B$ is as above.* (Recall that if $\{x\} = \pi^{-1}(y)$, then $x \in S_1(\pi)$ iff y has a singular branch at y.)

If $r=1$, and V is of genus g, one embeds V by using a complete linear system of degree $\geqq 2g+3$ and uses techniques like those used in the proof of Theorem 3 of [8] to obtain (I) and (II).

PROPOSITION 5. *Let $\pi\colon V \to \boldsymbol{P}^{2r-1}$ be as above, and let $S_1 = \sum_x \nu_x \cdot x$, where the summation extends over all points of* Supp $(\Omega^1_{V/\boldsymbol{P}^{2r-1}})$.

If char $(k) \neq 2$, *then $\nu_x = 1$ for all x.*

If char $(k) = 2$, *then $\nu_x = 2$ for all x.*

Proof. Let $x \in S_1(\pi)$ and $y = \pi(x)$. Since π is finite, $\mathcal{O}_x \cong R_{\mathfrak{m}}$, where R is a semi-local ring which is a finite $\mathcal{O}_y$-module, and $\mathfrak{m}$ is maximal. Hence, $\hat{R} \cong R \otimes_{\mathcal{O}_y} \hat{\mathcal{O}}_y$, and $\hat{\mathcal{O}}_x \cong R/\mathfrak{a}$, where $\mathfrak{a}$ is generated by idempotent elements. Therefore, $\Omega^1{}_{\hat{\mathcal{O}}_x/\hat{\mathcal{O}}_y} \cong \Omega^1_{\mathcal{O}_x/\mathcal{O}_y} \otimes \hat{\mathcal{O}}_x$, so that the 0th Fitting ideal of $\Omega^1{}_{\hat{\mathcal{O}}_x/\hat{\mathcal{O}}_y}$ is $I \cdot \hat{\mathcal{O}}_x$, where I is the 0th Fitting ideal of $\Omega^1_{\mathcal{O}_x/\mathcal{O}_y}$. Let $A = k[[t_1, \ldots, t_{2r-1}]]$, $B = k[[t_1, \ldots, t_r]]$, and let $f\colon A \to B$ be given as above. Then $\ell_{\mathcal{O}_x}(\mathcal{O}_x/I) = \ell_B(B/J)$, where J is the 0th Fitting ideal of $\Omega^1_{B/A}$. We have an exact sequence of B-modules

$$\hat{\Omega}^1_{A/k} \hat{\otimes}_A B \xrightarrow{\ u\ } \hat{\Omega}^1_{B/k} \longrightarrow \Omega^1_{B/A} \longrightarrow 0.$$

(Cf. EGA 0_{IV}, 20.7.17.3 and 0_{I}, 7.3.5.) The first two terms are free, and u is given by

$$\begin{aligned} u(dt_i \otimes 1) &= dt_i && \text{for } i = 1, \ldots, r-1, \\ u(dt_i \otimes 1) &= t_r\, dt_{i-r+1} + t_{i-r+1}\, dt_r && \text{for } i = r, \ldots, 2r-2, \\ u(dt_{2r-1} \otimes 1) &= (2t_r + 3t_r^2)\, dt_r. \end{aligned}$$

The 0th Fitting ideal is thus $J = (t_1, \ldots, t_{r-1}, t_r)$ if char $(k) \neq 2$, and

$$J = (t_1, \ldots, t_{r-1}, t_r^2)$$

if char $(k) = 2$. The length of B/J is 1 (resp. $=2$) if char $(k) \neq 2$ (resp. $=2$). Q.E.D.

REMARK. As a consequence of Proposition 5, the number of points in $S_1(\pi)$ is independent of the choice of projection center, L, provided (I) and (II) are satisfied.

We will now see how the number of points varies as V is specialized. Thus, let A be a noetherian ring, and X a closed subscheme of $\boldsymbol{P}_A^n = \operatorname{Proj} A[T_0, \ldots, T_n]$; assume that $p: X \to \operatorname{Spec}(A)$ is smooth and has absolutely irreducible fibres of dimension r. Assume that $n \geqq 2r-1$ and that X does not meet the closed subscheme given by $T_0 = \cdots = T_{2r-1} = 0$. We define N to be the normal bundle of X in $P = \boldsymbol{P}_A^n$; thus $N = (I/I^2)^*$, $I = \operatorname{Ker}(\mathcal{O}_P \to \mathcal{O}_X)$.

We consider geometric points $\eta: \operatorname{Spec}(k) \to \operatorname{Spec}(A)$ $(k = \bar{k})$ such that the projection π_η of the geometric fibre $X_\eta \subset \boldsymbol{P}_k^n$ from the linear subspace $T_0 = \cdots = T_{2r-1} = 0$ satisfies (I) and (II) above. Now, the bundle N_η induced on X_η by N is just the normal bundle of X_η in $\boldsymbol{P}_k^n$. Thus, by Theorem 2, the degree of the cycle $S_1(\pi_\eta)$ is $\deg(c_r((N \otimes \mathcal{O}_X(-1))_\eta))$. By Theorem 1, this is independent of η. Using Proposition 5, we obtain the following conclusion.

PROPOSITION 6. *Let A and $X \subset \boldsymbol{P}_A^n$ be as above. Assume that* $\operatorname{Spec}(A)$ *is connected. For $i = 1, 2$, let* $\eta_i: \operatorname{Spec}(k_i) \to \operatorname{Spec}(A)$ *be geometric points such that the corresponding projections π_{η_i} both satisfy* (I) *and* (II). *If* $\operatorname{char}(k_1)$ *and* $\operatorname{char}(k_2)$ *are both $\neq 2$ or both $= 2$, then #(points in $S_1(\pi_{\eta_1})$) = #(points in $S_1(\pi_{\eta_2})$). If* $\operatorname{char}(k_1) = 2$ *and* $\operatorname{char}(k_2) \neq 2$, *then #(points in $S_1(\pi_{\eta_1})$) = $\frac{1}{2}$(#(points in $S_1(\pi_{\eta_2})$)).*

If $r \geqq 2$, we substitute #(pinch-points of $\pi_{\eta_i}(X_{\eta_i})$) for #(points in $S_1(\pi_{\eta_i})$) to obtain a statement about the behavior under specialization of the number of pinch-points. If $r = 1$, we obtain a similar statement about the behavior under specialization of the number of ramification points of the covering $V \to \boldsymbol{P}^1$.

EXAMPLE. Let $V^2 \subset \boldsymbol{P}^5$ be the Veronese surface, i.e. the image of $\boldsymbol{P}^2$ embedded by the complete linear system of conics. Explicitly, let points of $\boldsymbol{P}^5$ have homogeneous coordinates (y_{ij}) with $0 \leqq i \leqq j \leqq 2$. Then $(x_0, x_1, x_2) \in \boldsymbol{P}^2$ is sent to the point of $\boldsymbol{P}^5$ with $y_{ij} = x_i x_j$. Let $\pi: V \to \boldsymbol{P}^3$ be induced by projection from the line $y_{01} = y_{02} = y_{12} = y_{00} + y_{11} + y_{22} = 0$. Thus, the composed map $\boldsymbol{P}^2 \to \boldsymbol{P}^3$ sends (x_0, x_1, x_2) to $(x_1x_2, x_0x_2, x_0x_1, x_0^2 + x_1^2 + x_2^2)$, and the image is the surface $V' \subset \boldsymbol{P}^3$ whose equation is $t_0^2t_1^2 + t_0^2t_2^2 + t_1^2t_2^2 - t_0t_1t_2t_3 = 0$. The singular locus of V' consists of the three lines Λ_0, Λ_1, and Λ_2 given respectively by $t_1 = t_2 = 0$, $t_0 = t_2 = 0$, and $t_0 = t_1 = 0$. If $\operatorname{char}(k) \neq 2$, then V' has 6 pinch-points, two of which lie on each of the lines Λ_i; if $\operatorname{char}(k) = 2$, then V' has 3 pinch-points, one on each Λ_i.

To see this, note that $C_i = \pi^{-1}(\Lambda_i)$ is a plane conic for each i. The projection center meets the plane of C_i in a point which lies on two tangent lines of C_i if $\operatorname{char}(k) \neq 2$, but on just one tangent line of C_i if $\operatorname{char}(k) = 2$. (The thing to note is that if $\operatorname{char}(k) = 2$, there is a point, not on the projection center, which lies on every tangent line of C_i.) It might also be noted that the plane conic provides the simplest example of the case $r = 1$ of Proposition 6.

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