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ORBITS IN A REAL REDUCTIVE LIE ALGEBRA

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Abstract. The purpose of this paper is to give a classification of the orbits in a real reductive Lie algebra under the adjoint action of a corresponding connected Lie group. The classification is obtained by examining the intersection of the Lie algebra with the orbits in its complexification. An algebraic characterization of the minimal points in the closed orbits is also given.

0. **Introduction.** Let $\mathfrak{g}_c$ be a complex semisimple Lie algebra, and $\mathfrak{g}$ a real form of $\mathfrak{g}_c$. $G_c = \text{Ad}\ \mathfrak{g}_c$ will denote the adjoint group of $\mathfrak{g}_c$, and $G \subset G_c$ the connected subgroup corresponding to $\mathfrak{g}$. If $a \in G_c$, we write $a \cdot x$ for the image of $x \in \mathfrak{g}_c$ under the action of the automorphism a. We shall classify the G-orbits in $\mathfrak{g}$ by giving representatives for the G-orbits in the set

$$G_c \cdot x \cap \mathfrak{g} = \{y \in \mathfrak{g} : y = a \cdot x \text{ for some } a \in G_c\}$$

for $x \in \mathfrak{g}_c$.

The orbits of maximal dimension for the action of G_c on $\mathfrak{g}_c$ have been classified by Kostant [5, Theorem 8, p. 382] who has constructed a cross-section for them, using the result that such orbits are completely determined by polynomials on $\mathfrak{g}_c$ which are invariant under G_c. Analogous results for conjugacy classes in algebraic groups have been obtained independently by Steinberg [14], using the group characters in place of the invariant polynomials.

It is known that $G_c \cdot x \cap \mathfrak{g}$ is actually a finite union of G-orbits [1, 2.3 Proposition]. The number of such G-orbits for certain $x \in \mathfrak{g}$ (Corollary 2.6) is the same as that obtained by Wolf [16, 4.7 Corollary] for the number of open G-orbits in the space G_c/B_c where B_c is a Borel subgroup; this suggests close connection between these two actions of G.

The plan of this paper is as follows. The first section introduces notation and preliminary results, mainly on the conjugacy classes of Cartan subalgebras. In the second section a decomposition of $G_c \cdot x \cap \mathfrak{g}$ into G-orbits is given for x semisimple, using Weyl groups. A characterization of the elements closest to the origin in the closed orbits is given in the next section, as well as an interpretation of $G_c \cdot x \cap \mathfrak{g}$ in terms of algebraic geometry. In the third section it is shown that the set of all

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nilpotents in $\mathfrak{g}$ whose orbits have maximal possible dimension form a single orbit under the automorphisms of $\mathfrak{g}$. Furthermore, the number of G-orbits in this set is counted and is shown to be the same as the number of connected orbits in the set of nilpotents of maximal orbit dimension in a complex symmetric space under the action of a complex reductive group. (See Kostant and Rallis [8].) The last section deals with finding representatives for the G-orbits in $G_c\cdot x \cap \mathfrak{g}$ when x is an arbitrary element whose orbit has maximal dimension.

1. **Notation and preliminary results.** If $G'\subseteq G_c$ is any reductive subgroup we write $G'\cdot x=\{a\cdot x : a\in G'\}$, the *$G'$-orbit of* $x\in\mathfrak{g}_c$. $x, y\in\mathfrak{g}_c$ will be called *G'-conjugate* if $G'\cdot x=G'\cdot y$. Now let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition for $\mathfrak{g}$. All Cartan decompositions are G-conjugate and there is a corresponding decomposition $G=K\cdot(\exp\mathfrak{p})$, where K is the connected (compact) subgroup of G corresponding to $\mathfrak{k}$ and exp denotes the exponential map.

The real form $\mathfrak{g}$ is defined by a conjugate-linear involution σ of $\mathfrak{g}_c$ defined by

$$\sigma|\mathfrak{g} = 1, \qquad \sigma|i\mathfrak{g} = -1,$$

where "$|$" denotes restriction and $i=(-1)^{1/2}$. The Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ defines a conjugate linear automorphism τ of $\mathfrak{g}_c$ defined by $\tau|\mathfrak{k}=1$, $\tau|\mathfrak{p}=-1$ and $\tau(ix)=-i\tau(x)$ for $x\in\mathfrak{g}$. Then $\mathfrak{g}_u=\{x\in\mathfrak{g}_c : \tau(x)=x\}$ is a compact real form. We write G_u for the connected subgroup of G_c corresponding to $\mathfrak{g}_u$.

The following simple result, which will be important for proofs involving conjugation by G_c, shows that $G_c\cdot x\cap\mathfrak{g}_u$ is always connected for $x\in\mathfrak{g}_u$. Essentially it says that a similarity transformation of two skew hermitian matrices by a positive definite matrix must be trivial.

PROPOSITION 1.1. *$G_u\cdot x=G_c\cdot x\cap\mathfrak{g}_u$ for any $x\in\mathfrak{g}_u$. More generally, $x_1, x_2\in\mathfrak{k}$ are G-conjugate iff they are K-conjugate; $y_1, y_2\in\mathfrak{p}$ are G-conjugate iff they are K-conjugate.*

Proof. Suppose that $a\cdot x=y$ with $a\in\exp i\mathfrak{g}_u$ and $x, y\in\mathfrak{g}_u$. We claim that $x=y$. Indeed, $a\cdot x=y=\tau(y)=\tau(a\cdot x)=a^{-1}\cdot x$, which shows that $a^2\cdot x=x$. If $x\neq y$, this implies that a has at least one negative or nonreal eigenvalue, which contradicts the fact that a is a positive definite transformation. Now since $G_c=\exp i\mathfrak{g}_u\cdot G_u$, the first assertion follows immediately. The first part of the second assertion follows also since $G=K\cdot(\exp\mathfrak{p})$ and $\exp\mathfrak{p}\subset\exp i\mathfrak{g}_u$. Finally, note that if $y_1, y_2\in\mathfrak{p}$ are G-conjugate, then $iy_1, iy_2\in i\mathfrak{p}\subset\mathfrak{g}_u$ are G-conjugate and hence are K-conjugate by the above. This proves the proposition.

$x\in\mathfrak{g}_c$ is called *semisimple* if ad x is a diagonalizable matrix. If x is semisimple, x is called *symmetric* if ad x has all real eigenvalues; x is called *elliptic* if ad x has all pure imaginary eigenvalues. An arbitrary $x\in\mathfrak{g}$ has a unique decomposition $x=x_k+x_p$, where x_k is elliptic, x_p is symmetric, and $[x_k, x_p]=0$. Since conjugation by G_c does not change the eigenvalues these definitions are invariant. Hence, if $a\cdot x=x'$, where $x'=x'_k+x'_p$, $a\in G_c$, then $a\cdot x_k=x'_k$ and $a\cdot x_p=x'_p$.

We now state some known results concerning Cartan subalgebras which we shall use implicitly in what follows. Detailed proofs can be found in [15].

Two Cartan subalgebras of $\mathfrak{g}$ are said to be *conjugate* if one can be transformed into the other by an element of G. (We shall show (Corollary 2.4) that this is equivalent to conjugacy by G_c.) It is well known that every real reductive Lie algebra has only a finite number of conjugacy classes of Cartan subalgebras. These were first classified for all real forms by Kostant [7]; a complete list is given in Sugiura [15].

There are two distinguished classes for each algebra, defined as follows. Any Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ has a unique decomposition $\mathfrak{h}=\mathfrak{h}_k+\mathfrak{h}_p$, where

$$\mathfrak{h}_k = \{x\in\mathfrak{h} \mid x \text{ elliptic}\} \quad\text{and}\quad \mathfrak{h}_p = \{x\in\mathfrak{h} \mid x \text{ symmetric}\}.$$

$\mathfrak{h}_k$ is called the *toroidal part* of $\mathfrak{h}$, and $\mathfrak{h}_p$ is called the *vector part* of $\mathfrak{h}$. $\mathfrak{h}$ is called *maximally toroidal* (resp. *maximally vector*) if

$$\dim \mathfrak{h}_k = \max_{\mathfrak{h}' \text{ Cartan of } \mathfrak{g}} \{\dim \mathfrak{h}'_k\} \qquad \left(\text{resp.} \dim \mathfrak{h}_p = \max_{h' \text{ Cartan of } \mathfrak{g}} \{\dim \mathfrak{h}'_p\}\right).$$

PROPOSITION 1.2 *All maximally toroidal Cartan subalgebras of* $\mathfrak{g}$ *are G-conjugate. All maximally vector Cartan subalgebras of* $\mathfrak{g}$ *are G-conjugate.*

Proof. (See Sugiura [15, pp. 380–381].)

A Cartan subalgebra $\mathfrak{h}$ is called *standard* (with respect to the Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$) if $\mathfrak{h}=\mathfrak{h}\cap\mathfrak{k}+\mathfrak{h}\cap\mathfrak{p}$. Note that in this case $\mathfrak{h}_k=\mathfrak{h}\cap\mathfrak{k}$ is the toroidal part of $\mathfrak{h}$, and $\mathfrak{h}_p=\mathfrak{h}\cap\mathfrak{p}$ is the vector part of $\mathfrak{h}$.

PROPOSITION 1.3. *Every Cartan subalgebra of* $\mathfrak{g}$ *is G-conjugate to a standard Cartan subalgebra.*

Proof. (See Suguira [15, Theorem 2].)

It follows from this that any semisimple $x\in\mathfrak{g}$ is G-conjugate to some $x'\in\mathfrak{g}$ with $x'_k\in\mathfrak{k}$ and $x'_p\in\mathfrak{p}$. x' satisfying the above condition is called *normal.*

It will often be convenient to use the centralizer of a semisimple element, which is again a reductive algebra. If $\mathfrak{s}\subseteq\mathfrak{g}_c$ is any commutative subset of semisimple elements, and $\mathfrak{g}'$ is any reductive subalgebra of $\mathfrak{g}_c$, let $\mathfrak{g}'^{\mathfrak{s}}=\{x\in\mathfrak{g}' : [x, y]=0$ for all $y\in\mathfrak{s}\}$ the *centralizer of* $\mathfrak{s}$ *in* $\mathfrak{g}'$. If G' is the connected subgroup of G_c corresponding to $\mathfrak{g}'$, let $G'^{\mathfrak{s}}=\{a\in G' : a\cdot x=x$ for all $x\in\mathfrak{s}\}$.

PROPOSITION 1.4. *Two Cartan subalgebras of* $\mathfrak{g}$ *are G-conjugate iff their vector parts are G-conjugate iff their toroidal parts are.*

Proof. The first equivalence is contained in [15, Theorem 3, p. 385]. For the second, suppose that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are arbitrary Cartan subalgebras of $\mathfrak{g}$ with conjugate toroidal parts. Then we may assume that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are standard and that $\mathfrak{h}_1\cap\mathfrak{k}=\mathfrak{h}_2\cap\mathfrak{k}$. But then $\mathfrak{h}_1\cap\mathfrak{p}$ and $\mathfrak{h}_2\cap\mathfrak{p}$ contain maximally vector subalgebras of $\mathfrak{g}^{\mathfrak{h}_1\cap\mathfrak{k}}$. Hence they are $G^{\mathfrak{h}_1\cap\mathfrak{k}}$-conjugate.

We shall be concerned mainly with elements, both semisimple and nonsemisimple, whose orbits are of maximal possible dimension, i.e. whose centralizers are of minimal possible dimension. $x \in \mathfrak{g}_c$ is called *regular* if $\dim G_c \cdot x = \max_{y \in \mathfrak{g}} \{\dim G_c \cdot y\}$, where $\dim X$ is the topological dimension of X. If $x \in \mathfrak{g}$, this is equivalent to $\dim G \cdot x = \max_{y \in \mathfrak{g}} \{\dim G \cdot y\}$. x is not assumed to be semisimple in this definition. For example, if $\mathfrak{g} = \mathrm{sl}\,(n, \boldsymbol{R})$, a matrix $A \in \mathfrak{g}$ is regular iff its characteristic polynomial equals its minimal polynomial. If $\mathfrak{g}'$ is any reductive subalgebra of $\mathfrak{g}_c$, $x \in \mathfrak{g}'$ is called *regular-in-*$\mathfrak{g}'$ if x is regular as an element of $\mathfrak{g}'$.

LEMMA 1.5. *If $x = x_k + x_p$ is semisimple, then x is regular iff x_k is regular-in-$\mathfrak{g}^{x_p}$ iff x_p is regular-in-$\mathfrak{g}^{x_k}$.*

Proof. It is easy to show that x is regular iff $\mathfrak{g}^x$ is commutative, from which the lemma follows immediately.

PROPOSITION 1.6. *If $\mathfrak{s}$ is any commutative subset of $\mathfrak{g}_c$, then $G_c^{\mathfrak{s}}$ is the connected subgroup of G_c corresponding to $\mathfrak{g}_c^{\mathfrak{s}}$.*

Proof. It suffices to prove that $G_c^{\mathfrak{s}}$ is connected, which is proved in [5, Lemma 5, p. 353].

Proposition 1.6 is false if G_c is replaced by G and $\mathfrak{g}_c$ by $\mathfrak{g}$, since $G^{\mathfrak{s}}$ is disconnected, in general.

We conclude this preliminary section with some results which will be needed in studying orbits of nilpotents. For any reductive algebra $\mathfrak{g}'$, let $\mathfrak{g}'_c = \mathfrak{g}' + i\mathfrak{g}'$ and G'_c the adjoint group of $\mathfrak{g}'_c$. Put $N(\mathfrak{g}') = \{a \in G'_c : a \cdot \mathfrak{g}' = \mathfrak{g}'\}$, *the normalizer of* $\mathfrak{g}'$ *in* G'_c. If $\mathfrak{g}' = \mathfrak{g}$ then we will simply write N instead of $N(\mathfrak{g})$. (In terms of algebraic geometry, N is the set of real points in G_c.)

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace, and let $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be a set of simple restricted roots for $\mathfrak{a}$. There exist ε_j, $j = 1, 2, \ldots, r$, such that $\alpha_t(\varepsilon_j) = \delta_{tj}$ for all j; ε_j is called the *dual* of α_j. $\{\varepsilon_j\}$ forms a linear basis for the vector space $\mathfrak{a}$. (The reader is referred to Helgason [3, Chapter VII] for results on restricted root systems.)

Any $x \in \mathfrak{a}$ can be written $x = \sum_1^r r_j \varepsilon_j$. Since the ε_j commute, $\exp ix = \prod_j \exp (ir_j \varepsilon_j)$. Let F be the group generated by $\{\exp ix : x \in \mathfrak{a}$ and $(\exp ix)^2 = 1\}$. From the above remarks it is easy to show that F is generated by $\{\exp \pi i \varepsilon_j, j = 1, 2, \ldots, r\}$.

PROPOSITION 1.7. *G is the connected component of the identity in N. Furthermore, $N = G \cdot F$.*

A proof of the above, as well as calculations for all real forms, may be found in Matsumoto [10].

Now let θ be the complex linear automorphism of $\mathfrak{g}_c$ defined by $\theta|\mathfrak{k} = 1$, $\theta|\mathfrak{p} = -1$. θ extends uniquely to an automorphism of G_c, which we shall also denote by θ. Let $K_\theta = \{a \in G_c : \theta(a) = a\}$. Kostant and Rallis [8] have classified the orbits of K_θ acting on $\mathfrak{p}_c = \mathfrak{p} + i\mathfrak{p}$. It is also shown that $K_\theta = K_c \cdot F$.

PROPOSITION 1.8. *Let $F_1 \subseteq F$ be any subgroup. Then $K_\theta / K_c \cdot F_1 \approx N / G \cdot F_1$.*

Proof. It suffices to show that if $f \in F$, then $f \in K_c \cdot F_1$ iff $f \in G \cdot F_1$. Suppose $f \in K_c \cdot F_1$. Then

$$f = (\exp ix_1)(\exp ix_2)$$

with $\exp ix_1 \in K_c = K \cdot (\exp i\mathfrak{k})$ and $\exp ix_2 \in F_1$, $x_1, x_2 \in \mathfrak{a}$. Since exp is 1-1 on $i\mathfrak{g}_u$, $\exp ix_1 \in K$. Hence $f \in K \cdot F_1 \subset G \cdot F$. Conversely, if $f \in G \cdot F_1$, then

$$f = (\exp iy_1)(\exp iy_2)$$

with $\exp iy_1 \in G$, $\exp iy_2 \in F_1$, $y_1, y_2 \in \mathfrak{a}$. Since $G = K \cdot (\exp \mathfrak{p})$, it follows that $\exp iy_1 \in K$ so that $f \in K \cdot F_1 \subseteq K_c \cdot F_1$.

2. **Orbits of semisimple elements.** Our first result shows that if ad x has all real eigenvalues, then the orbit $G_c \cdot x \cap \mathfrak{g}$ is actually connected.

THEOREM 2.1 *Let $x, y \in \mathfrak{p}$. Then x and y are G_c-conjugate iff they are G-conjugate.*

Proof. Clearly G-conjugacy implies G_c-conjugacy. Conversely, suppose x and y are G_c-conjugate. ix and iy are in $i\mathfrak{p} \subseteq \mathfrak{g}_u = \mathfrak{k} + i\mathfrak{p}$. By Proposition 1.1, two elements of $\mathfrak{g}_u$ are G_c-conjugate iff they are G_u-conjugate, where G_u is the subgroup of G_c corresponding to $\mathfrak{g}_u$. Hence ix and iy are G_u-conjugate which implies x and y are G_u-conjugate. Since $G_u = K(\exp i\mathfrak{a})K$ we have $c_1 a c_2 \cdot x = y$, where $c_1, c_2 \in K$, $a \in \exp i\mathfrak{a}$. Then $a \cdot (c_2 \cdot x) = c_1^{-1} \cdot y$. Since $c_2 \cdot x$ and $c_1 \cdot y$ are in $\mathfrak{p}$ and are K-conjugate to x and y respectively, it suffices to replace them by x and y and to assume $a \cdot x = y$ for $a \in \exp i\mathfrak{a}$. Let σ be as in §1. Then $y = \sigma(y) = \sigma(a \cdot x) = \sigma(a)\sigma(x) = a^{-1} \cdot y$ since $a \in \exp i\mathfrak{a}$. Hence $a^2 \cdot x = x$ and $a^2 \cdot y = y$. The reductive subalgebra $\mathfrak{g}^{a^2}$ has the Cartan decomposition $\mathfrak{g}^{a^2} = \mathfrak{k}^{a^2} + \mathfrak{p}^{a^2}$ with $x, y \in \mathfrak{p}^{a^2}$. We claim $\mathfrak{k}^{a^2}$ and $\mathfrak{p}^{a^2}$ are invariant under a. Indeed, $\theta(a \cdot z) = a^{-1} \cdot z = a \cdot k$ for $z \in \mathfrak{k}$, and $\theta(a \cdot q) = a^{-1} \cdot q = a \cdot q$ for $q \in \mathfrak{p}$.

Let u be any polynomial on $\mathfrak{p}^{a^2}$ which is G^{a^2}-invariant; i.e. $u(g \cdot x) = u(x)$ for any $x \in \mathfrak{p}^{a^2}$, $g \in G^{a^2}$. The function $a \cdot u$, defined by $a \cdot u(x) = u(a^{-1} \cdot x)$, is again a polynomial on $\mathfrak{p}^{a^2}$, since a leaves $\mathfrak{p}^{a^2}$ stable. In fact, $a^{-1} \cdot u$ is K^{a^2}-invariant, since $(k \cdot (a^{-1} \cdot u))x = (a^{-1} \cdot u)(k^{-1} \cdot x) = u((a \cdot k^{-1}) \cdot x)$, for any $k \in K^{a^2}$. But $aka^{-1} \in K^{a^2}$ since a leaves k^{a^2} invariant. Hence $u((ak^{-1}) \cdot x) = u((aka^{-1})(ak^{-1}) \cdot x) = u(a \cdot x)$ $= a^{-1} \cdot u(x)$ so that $a^{-1} \cdot u$ is K^{a^2}-invariant. We claim that $a^{-1} \cdot u = u$. Since every element of $\mathfrak{p}^{a^2}$ is K^{a^2}-conjugate to an element of $\mathfrak{a}^{a^2}$, u is completely determined by its values on $\mathfrak{a}^{a^2}$. But a^{-1} leaves $\mathfrak{a}^{a^2}$ pointwise fixed since $a^{-1} \in \exp i\mathfrak{a}$, so that $a^{-1} \cdot u = u$ for all invariant u. Hence $u(y) = u(a \cdot x) = a^{-1} \cdot u(x) = u(x)$ for all such u. This shows that every K^{a^2}-invariant polynomial on $\mathfrak{p}^{a^2}$ agrees on x and y, which implies that x and y are K^{a^2}-conjugate [3, Chapter X, Lemma 6.20]. This completes the proof of Theorem 2.1.

In general no theorem like the above is true for semisimple elements which have complex eigenvalues.

COROLLARY 2.2. *If x, y are symmetric, then $G \cdot x = G \cdot y$ iff $G_c \cdot x = G_c \cdot y$.*

Proof. This is immediate from Theorem 2.1 since x and y are G-conjugate to elements in $\mathfrak{p}$.

COROLLARY 2.3. *If $z \in \mathfrak{g}$ is semisimple and $y \in G_c \cdot z \cap \mathfrak{g}$, then there exists $y' \in G \cdot y$ such that $z \in \mathfrak{h}$ and $y' \in \mathfrak{h}$ for some Cartan subalgebra $\mathfrak{h}$, and $y'_p = x_p$.*

Proof. By Corollary 2.2 we may assume $y_p = x_p$, up to G-conjugacy. Then y_k and z_k are elliptic elements in the subalgebra $\mathfrak{g}^{x_p}$. If $\mathfrak{h}$ is a maximally toroidal subalgebra of $\mathfrak{g}^{x_p}$ containing x_k, then z_k is G^{x_p}-conjugate to some element of $\mathfrak{h}$. Let y' be that element.

COROLLARY 2.4. *Let $\mathfrak{h}$, $\mathfrak{h}'$ be any two Cartan subalgebras of $\mathfrak{g}$. Then $\mathfrak{h}$ and $\mathfrak{h}'$ are G-conjugate iff they are G_c-conjugate.*

Proof. It suffices to prove that if $\mathfrak{h}$ and $\mathfrak{h}'$ are G_c-conjugate, then they are G-conjugate. Suppose $g \cdot \mathfrak{h} = \mathfrak{h}'$ for some $g \in G_c$. Choose some regular element $x \in \mathfrak{h}$, and let $y = g \cdot x$. Then x and y satisfy the conditions of Corollary 2.3 so that there exists $g' \in G$ such that $g' \cdot x \in \mathfrak{h}'$. Hence $g' \cdot \mathfrak{h} = \mathfrak{h}'$.

If $\mathfrak{h} = \mathfrak{h}_k + \mathfrak{h}_p$ is any Cartan subalgebra of $\mathfrak{g}$, let $\mathfrak{h}_c = \mathfrak{h} + (-1)^{1/2}\mathfrak{h}$ be the complexification of $\mathfrak{h}$. $\mathfrak{h}_c$ is a Cartan subalgebra of $\mathfrak{g}_c$. Let W_c be the Weyl group of $\mathfrak{g}_c$. It is well known that $x, y \in \mathfrak{h}_c$ are G_c-conjugate iff there exists $w \in W_c$ such that $w \cdot x = y$. Let $(W_c)_{\mathfrak{h}} = \{w \in W_c : w \cdot \mathfrak{h} = \mathfrak{h}\}$. Now let $W_{\mathfrak{h}} = N_K(\mathfrak{h})/Z_K(\mathfrak{h})$, where $N_K(\mathfrak{h})$ is the normalizer of $\mathfrak{h}$ in K and $Z_K(\mathfrak{h})$ is the centralizer of $\mathfrak{h}$ in K. Let $W^{\mathfrak{h}_p} = N_{K^{\mathfrak{h}_p}}(\mathfrak{h})/Z_{K^{\mathfrak{h}_p}}(\mathfrak{h})$, where $N_{K^{\mathfrak{h}_p}}(\mathfrak{h})$ and $Z_{K^{\mathfrak{h}_p}}(\mathfrak{h})$ are the normalizer and centralizer, respectively, taken in the group $K^{\mathfrak{h}_p}$.

LEMMA 2.5. *Let $\mathfrak{h}$ be a maximally toroidal subalgebra of $\mathfrak{g}$. Then $x, y \in \mathfrak{h}$ are G_c-conjugate iff they are $(W_c)_{\mathfrak{h}}$-conjugate.*

Proof. It suffices to assume that $\mathfrak{h}$ is standard and to show that if x, y are G_c-conjugate then they are $(W_c)_{\mathfrak{h}}$-conjugate. By Proposition 1.1 if x, y are G_c-conjugate they are G_u-conjugate, so there exists $a \in G_u$ such that $a \cdot x = y$. Since $\mathfrak{h}$ and $a \cdot \mathfrak{h}$ are both maximally toroidal standard subalgebras of $\mathfrak{g}^y$, there exists $b \in G_u^y$ such that $ba \cdot \mathfrak{h} = \mathfrak{h}$. Then $ba \cdot x = y$ and $ba \cdot \mathfrak{h} = \mathfrak{h}$. Hence x and y are $(W_c)_{\mathfrak{h}}$-conjugate.

We may now state our result for the decomposition of $G_c \cdot x \cap \mathfrak{g}$ into G-orbits for any regular semisimple $x \in \mathfrak{g}$.

THEOREM 2.6. *Let $x \in \mathfrak{g}$ be regular semisimple and $\mathfrak{h} = \mathfrak{h}_k + \mathfrak{h}_p$ the unique Cartan subalgebra containing x. Then*

$$G_c \cdot x \cap \mathfrak{g} = \bigcup_{j \in I} G \cdot (w_j \cdot x)$$

is the decomposition of $G_c \cdot x \cap \mathfrak{g}$ into distinct G-orbits, where $\{w_j\}_{j \in I}$ is a set of representatives in G_c for the coset space $W_c^{\mathfrak{h}_p}/W^{\mathfrak{h}_p}$. Furthermore, the result is still true when $W_c^{\mathfrak{h}_p}/W^{\mathfrak{h}_p}$ is replaced by the quotient space $(W_c)_{\mathfrak{h}}/W_{\mathfrak{h}}$.

Before proving Theorem 2.6, we need some lemmas.

LEMMA 2.7. *Let $\mathfrak{h}=\mathfrak{h}_k+\mathfrak{h}_p$ be a Cartan subalgebra of $\mathfrak{g}$. Then there exists $x \in \mathfrak{h}$, $x=x_k+x_p$, $x_k \in \mathfrak{h}_k$, $x_p \in \mathfrak{h}_p$, having the property that $\mathfrak{g}^{x_k}=\mathfrak{g}^{\mathfrak{h}_k}$ and $\mathfrak{g}^{x_p}=\mathfrak{g}^{\mathfrak{h}_p}$.*

Proof. Since $\mathfrak{g}^{x_k}=\mathfrak{g}_c^{x_k} \cap \mathfrak{g}$, we need only show that $\mathfrak{g}_c^{x_k}=\mathfrak{g}_c^{\mathfrak{h}_k}$, and similarly for x_p. Suppose no such $x_k \in \mathfrak{h}_k$ exists. Then choose $x_k' \in \mathfrak{h}_k$ such that

$$\dim \mathfrak{g}_c^{x_k} = \min \{\dim \mathfrak{g}_c^{y} \mid y \in \mathfrak{h}_k\}.$$

Now let Δ be a set of roots for $\mathfrak{h}_c=\mathfrak{h}+i\mathfrak{h}$. Then $\mathfrak{g}_c^x=\mathfrak{h}_c+\sum_{\phi\in\Delta:\phi(x)=0} \mathfrak{g}_c^{\phi}$. If $\mathfrak{g}_c^x \neq \mathfrak{g}_c^{\mathfrak{h}_k}$, as assumed, then there exists $y \in \mathfrak{h}_k$ and $\psi \in \Delta$ such that $\psi(y)\neq 0$, but $\psi(x)=0$. We claim there exists a real number r such that $\dim \mathfrak{g}_c^{x+ry}<\dim \mathfrak{g}_c^x$. Indeed, choose r sufficiently large so that for any $\phi \in \Delta$, if $\phi(y)\neq 0$, then $|\phi(ry)|>\phi(x)$. Then for an arbitrary $\eta \in \Delta$, $\eta(x+ry)=0$ iff $\eta(x)=0$ and $\eta(y)=0$. But then $\psi(x+ry)\neq 0$ whenever $\psi(x)\neq 0$. Hence $\dim \mathfrak{g}_c^{x+ry}<\dim \mathfrak{g}_c^x$. This contradiction shows that the desired $x_k \in \mathfrak{h}_k$ exists. Similarly we may find an x_p satisfying $\mathfrak{g}^{x_p}=\mathfrak{g}^{\mathfrak{h}_p}$, which proves the lemma.

If $x=x_k+x_p$ satisfies the conditions of Lemma 2.7, then x will be called *generic*.

LEMMA 2.8. *Let $\mathfrak{h}=\mathfrak{h}_k+\mathfrak{h}_p$ be a maximally toroidal Cartan subalgebra, and let x and $y \in \mathfrak{h}$ be regular. Then if x and y are G_c-conjugate they are $(W_c)_{\mathfrak{h}}$-conjugate.*

Proof. If x and y are G_c-conjugate, there exists $w \in W_c$ such that $w\cdot x=y$. Let θ be the linear automorphism of $\mathfrak{g}_c$ defined in §1. Then $\theta|\mathfrak{h}$ is an element W of the Cartan group of $\mathfrak{h}$. We show that $w \in W_{\mathfrak{h}}$ by proving that $\theta w\theta^{-1}=w$, or equivalently, $\theta w\theta^{-1}\theta w=1$. Since W_c is a normal subgroup of the Cartan group, $\theta w^{-1}\theta w \in W_c$. Since x is regular, it suffices to prove that $\theta w^{-1}\theta w\cdot x=x$. Let $x=x_k+x_p$, $y=y_k+y_p$ By Corollary 2.3 we may assume that $x_p=y_p$, so that $w\cdot x_p=x_p$. Then we have

$$\begin{aligned}\theta w^{-1}\theta w\cdot(x_k+x_p) &= \theta w^{-1}\theta(y_k+y_p) = \theta w^{-1}(y_k-y_p)\\ &= \theta(x_k-x_p) = x_k+x_p=x,\end{aligned}$$

which proves Lemma 2.8.

PROPOSITION 2.9. *Let $x=x_k+x_p$, $y=y_k+y_p$, $x, y \in \mathfrak{h}$ such that $G_c\cdot x=G_c\cdot y$. Then x and y are G-conjugate iff x_k and y_k are G-conjugate iff x and y are $W_{\mathfrak{h}}$-conjugate.*

Proof. Suppose x_k and y_k are G-conjugate, so that $a\cdot x_k=y_k$ for some $a \in G$. Since $a\cdot x$ and y are G_c-conjugate, $a\cdot x_p$ and y_p are $G_c^{y_k}$-conjugate. Since $G_c^{y_k}$ is connected by Proposition 1.6, we may apply Corollary 2.2 to show that $a\cdot x_p$ and y_p are G^{x_k}-conjugate, which proves that x and y are G-conjugate. The opposite implication is obvious, so the first equivalence is proved. To prove the last equivalence it suffices to show that if x and y are G-conjugate they are $W_{\mathfrak{h}}$-conjugate. By Proposition 1.1 applied to x_k, y_k and x_p, y_p separately, x and y are G-conjugate iff they are K-conjugate. Since x is regular, the conjugation by K leaves $\mathfrak{h}$ invariant since $y \in \mathfrak{h}$. Hence x and y are $W_{\mathfrak{h}}$-conjugate.

LEMMA 2.10. *If x, y are any two regular elements of $\mathfrak{h}=\mathfrak{h}_k+\mathfrak{h}_p$, then $G_c\cdot x \cap \mathfrak{g}$ and $G_c\cdot y \cap \mathfrak{g}$ consist of the same number of connected components. More precisely,* *for any $w \in (W_c)_{\mathfrak{h}}$, $w \cdot x$ represents the same G-orbit as x iff $w \cdot y$ represents the same G-orbit as y.*

Proof. Let $x = x_k + x_p$, $y = y_k + y_p$. Then $w \cdot x = w \cdot x_k + w \cdot x_p$ with $w \cdot x_k \in \mathfrak{h}_k$. If x and $w \cdot x$ are G-conjugate, then $w \cdot x = a \cdot x$ for some $a \in G$. Since x is regular, this means that $w | \mathfrak{h}_c = a | \mathfrak{h}_c$, so that y and $w \cdot y$ are also G-conjugate.

By interchanging the roles of x and y, the proof of Lemma 2.10 is completed.

Proof of Theorem 2.6. By Lemma 2.10 it suffices to assume that $x = x_k + x_p$ is generic. We show first that, for any $y \in \mathfrak{g}$ such that x and y are G_c-conjugate, there exists $w \in G_c$, a representative of some coset in $W_c^{\mathfrak{h}_p}/W^{\mathfrak{h}_p}$ such that $w \cdot x$ and y are G-conjugate. By Corollary 2.3 we may assume, up to G-conjugacy, that $y \in \mathfrak{h}$ and $y_p = x_p$. Hence x_k and y_k are $G_c^{x_p}$-conjugate. Since x is generic, $\mathfrak{g}_c^{x_p} = \mathfrak{g}_c^{\mathfrak{h}_p}$ and, since by Proposition 1.6, $G_c^{\mathfrak{h}_p}$ is connected, this proves that $G_c^{x_p} = G_c^{\mathfrak{h}_p}$. By Lemma 1.5, x_k and y_k are regular-in-$\mathfrak{g}^{x_p}$. Hence by Proposition 2.9 it follows that x_k and y_k are $(W_c^{\mathfrak{h}_p})_{\mathfrak{h}}$-conjugate, where $(W_c^{\mathfrak{h}_p})_{\mathfrak{h}}$ is the subgroup of $W_c^{\mathfrak{h}_p}$ which leaves $\mathfrak{h}_k$ stable. Since $\mathfrak{h}_k$ contains a compact Cartan subalgebra of $\mathfrak{g}^{\mathfrak{h}_p}$, all of $W_c^{\mathfrak{h}_p}$ leaves $\mathfrak{h}_k$ stable. Therefore, x and y are $W_c^{\mathfrak{h}_p}$-conjugate, which proves the above assertion.

To complete the proof of the assertion about $W_c^{\mathfrak{h}_p}/W^{\mathfrak{h}_p}$ we must show that if $w \in W_c^{\mathfrak{h}_p}$, then x and $w \cdot x$ are G-conjugate if and only if $w \in W^{\mathfrak{h}_p}$. As before, $w \cdot x_k$ and x_k are regular-in-$\mathfrak{g}^{x_p} = \mathfrak{g}^{\mathfrak{h}_p}$. By Proposition 2.10, x and $w \cdot x$ are G-conjugate iff they are W-conjugate iff they are $W^{\mathfrak{h}_p}$-conjugate. To prove the assertion about $(W_c)_{\mathfrak{h}}/W_{\mathfrak{h}}$, it suffices to show that if $w \cdot x \in \mathfrak{g}$, then $w \in (W_c)_{\mathfrak{h}}$. By the above, up to G-conjugacy, x and $w \cdot x$ are conjugate by an element of $W_c^{\mathfrak{h}_p} \subseteq (W_c)_{\mathfrak{h}}$. Since G-conjugacy of regular elements in $\mathfrak{h}$ always leaves $\mathfrak{h}$ invariant, the above assertion is proved. This completes the proof of Theorem 2.6.

If s is a set, we shall let $\#(s)$ denote the cardinality of s.

COROLLARY 2.11. *If $x \in \mathfrak{g}$ is semisimple, then the G_c-orbit of x in $\mathfrak{g}$, $G_c \cdot x \cap \mathfrak{g}$, is the union of at most $\#(W_c)/\#(W)$ G-orbits.*

A real form $\mathfrak{g}$ is said to be of *split rank* if all its Cartan subalgebras are conjugate.

COROLLARY 2.12. *If $\mathfrak{g}$ is of split rank, then $G_c \cdot x \cap \mathfrak{g}$ is a single G-orbit for any semisimple $x \in \mathfrak{g}$.*

Proof. In this case the semisimple part of $\mathfrak{g}^{\mathfrak{h}_p}$ is a compact Lie algebra so that $W_c^{\mathfrak{h}_p} = W^{\mathfrak{h}_p}$ by Proposition 1.1.

COROLLARY 2.13. *If $\mathfrak{h}$ is a compact Cartan subalgebra, then $G_c \cdot x \cap \mathfrak{g}$ is the union of $\#(W_c)/\#(W)$ G-orbits for any regular $x \in \mathfrak{h}$.*

Proof. In this case $\mathfrak{h}_p = 0$.

REMARK. Theorem 2.1 and its corollaries are used to prove a one-to-one correspondence between the conjugacy classes of Cartan subalgebras of $\mathfrak{g}$ and the connected components of the image of the regular semisimple elements under the invariant map $u: \mathfrak{g} \to \mathbf{C}^l$ defined in [5]. (See [11] for details.)

3. **Geometric properties of the closed orbits.** It is well known that an orbit $G\cdot x$ is closed iff x is semisimple. (See e.g. [1, 10.1. Proposition].) We shall show that for x regular semisimple, $G\cdot x$ is Zariski closed only when $G_c\cdot x \cap \mathfrak{g} = G\cdot x$. For all topologically closed orbits we shall show that the points of minimal distance from the origin in an appropriate metric are exactly the normal points defined in §1.

If $x \in \mathfrak{g}_c$, then $G_c\cdot x \subseteq \{y \in \mathfrak{g}_c : f(x)=f(y)$ for all G_c-invariant polynomials f on $\mathfrak{g}_c\}$, with equality holding iff x is regular and semisimple. The Zariski closed orbits in $\mathfrak{g}_c$ are exactly those of semisimple elements. For any semisimple $x, y \in \mathfrak{g}_c, f(x)=f(y)$ for all G_c-invariant polynomials f iff $G_c\cdot x = G_c\cdot y$. Hence the invariant polynomials separate the closed orbits in $\mathfrak{g}_c$. The results of the previous section show that in general this is not true for the closed G-orbits. (Note that every G-invariant polynomial on $\mathfrak{g}$ is obtained as the restriction of a G_c-invariant polynomial on $\mathfrak{g}_c$ since G is a Zariski dense subset of G.)

The separation of orbits by invariant polynomials can be explained by considering the Zariski topology on $\mathfrak{g}$ as a real algebraic variety. Since G is Zariski dense in G_c, the Zariski topology on $\mathfrak{g}$ is induced from that of $\mathfrak{g}_c$.

PROPOSITION 3.1. *If $x \in \mathfrak{g}$ is regular semisimple, then $G_c\cdot x \cap \mathfrak{g}$ is the closure of $G\cdot x$ in the Zariski topology on $\mathfrak{g}$. In particular, $G\cdot x = G_c\cdot x \cap \mathfrak{g}$ iff $G\cdot x$ is Zariski closed in $\mathfrak{g}$.*

Proof. Since G is Zariski dense in G_c, $G_c\cdot x$ is the smallest Zariski closed subset of $\mathfrak{g}_c$ containing $G\cdot x$. Hence $G_c\cdot x \cap \mathfrak{g}$ is the smallest Zariski closed subset containing $G\cdot x$ in $\mathfrak{g}$.

We return to the problem of characterizing points of minimal distance in the topologically closed orbits, i.e. those of semisimple elements. These will be the normal elements, i.e. those which are contained in standard Cartan subalgebras for a fixed Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. By Proposition 1.1 it follows that two normal elements in $\mathfrak{g}$ are G-conjugate iff they are K-conjugate.

Let σ be as defined in §1. Let B be the Killing form of $\mathfrak{g}_c$. Then $|x| = B(x, \sigma x)$ is a K-invariant metric on $\mathfrak{g}$.

THEOREM 3.2. *Let $x \in \mathfrak{g}$ be semisimple; x is normal iff $|x| \leqq |a\cdot x|$ for all $a \in G$.*

Proof. Since the regular elements of $\mathfrak{g}$ are dense, and the limit of normal elements is normal, it suffices to assume x is regular. We claim that it suffices to prove that

$$(1) \qquad d\,|\exp ty\cdot x|/dt = 0 \quad \text{for all } y \in \mathfrak{g}$$

iff x is normal. Indeed, let r be any number $> |x|$, and let S_r be the closed ball of radius r in $\mathfrak{g}$. Then $V = G\cdot x \cap S_r$ is a closed compact set since $G\cdot x$ is closed and the function $z \to |z|$ is differentiable on V. Hence $|z|$ must have a minimum on V and since $r > |x|$, the minimum cannot occur on the boundary of S_r. Hence the derivative must be 0 at the minimum, so that (1) holds if x is such a point. If (1) holds only if x is normal, then no nonnormal point can have minimal norm. Hence there exists $x_1 \in V$ *normal* such that $|x_1| \leqq |a \cdot x_1|$ for all $a \in G$. Since $|\ |$ is K-invariant and all normal elements in V are K-conjugate, every normal element must have minimal norm.

Since

$$\frac{d}{dt}|\exp ty \cdot x|^2 = \frac{d}{dt} B(\exp ty \cdot x, \sigma(\exp ty \cdot x))$$

$$= B([y, x], \sigma x) + B(x, \sigma[y, x]) = 2B([\sigma y, \sigma x], x),$$

it suffices to prove the following.

LEMMA 3.3. *$B([z, \sigma x], x) = 0$ for all $z \in \mathfrak{g}$ iff x is normal.*

Proof. Let $\mathfrak{h}$ be the unique Cartan subalgebra containing x. Then x is normal iff $\mathfrak{h} = \sigma(\mathfrak{h})$, since $\sigma x \in \mathfrak{h}$ iff $[\sigma x, x] = 0$. Therefore, it suffices to show that $B([z, y], x) \neq 0$ for some $z \in \mathfrak{g}$ iff $y \notin \mathfrak{h}$. We may prove this by assuming that $z \in \mathfrak{g}_c$. Then $z = z^+ + z^0 + z^-$, where z^+, z^0 and z^- are positive, zero and negative root vectors, respectively, for some choice of roots on $\mathfrak{h}_c$. Then if $y \in \mathfrak{h}$, $t = [z, y]$ has the decomposition $t = t^+ + t^-$, so that $B(t, x) = 0$. If $y \in \mathfrak{h}$, then $y = y^+ + y^0 + y^-$, where either y^+ or $y^- \neq 0$. Suppose, therefore, that $y^+ = \sum_{\phi \in \Delta^+} r_\phi e_\phi$, where the e_ϕ are positive root vectors, with $r_{\phi_1} \neq 0$. Taking $z = e_{-\phi_1}$, we get

$$B([z, y], x) = B(r_{\phi_1}[e_{-\phi_1}, e_{\phi_1}], x) \neq 0$$

since x is regular. This proves the lemma, so that Theorem 3.2 is proved.

In the previous sections it has been shown that two semisimple elements in $\mathfrak{g}$ are G_c-conjugate (resp. G-conjugate) iff any normal elements in their respective G-orbits are G_u-conjugate (resp. K-conjugate). Theorem 3.2 shows that the normal elements in each orbit are exactly the elements of smallest norm. Hence the decomposition of $G_c \cdot x \cap \mathfrak{g}$ into G-orbits for x semisimple reduces to the decomposition of $G_u \cdot y \cap \mathfrak{g}$ into K-orbits, where y is any element in $G \cdot x$ of lowest norm. Then Theorem 3.2 shows that if x is symmetric, the elements of smallest norm in $G_c \cdot x \cap \mathfrak{g}$ form a single K-orbit. In general, the elements of smallest norm in $G_c \cdot x \cap \mathfrak{g}$ form a finite number of K-orbits, the exact number of which can be obtained from Theorem 2.6.

4. **Orbits of nilpotent elements.** A *nilpotent* is an element $e \in \mathfrak{g}$ such that ad e is a nilpotent operator.

e will be called $\mathfrak{g}$-*regular* if the dimension of its orbit is maximal among all nilpotent orbits in $\mathfrak{g}$; i.e. $\dim G \cdot e \geqq \dim G \cdot f$ for any *nilpotent* $f \in \mathfrak{g}$. Let $N = \{a \in G_c : a \cdot \mathfrak{g} = \mathfrak{g}\}$, the normalizer of $\mathfrak{g}$ in G_c. Note that in general $\mathfrak{g}$ does not contain a regular nilpotent. We shall prove the set of all $\mathfrak{g}$-regular nilpotents in $\mathfrak{g}$ forms a single N-orbit. We also determine $N \cdot e$ as the union of G-orbits.

A subalgebra $\mathfrak{u} \subset \mathfrak{g}$ will be called a three-dimensional simple subalgebra (TDS) if $\mathfrak{u}$ is isomorphic to $\mathfrak{sl}(2, \boldsymbol{R})$, the Lie algebra of all 2×2 real matrices of trace 0.

A TDS is spanned by $\{f, x, e\}$ satisfying the relations $[e, f]=x$, $[x, e]=e$, and $[x, f]=-f$. The set $\{f, x, e\}$ is called an *S-triple*. x is called the *monosemisimple* part, and e and f are called the nilpositive and nilnegative parts of $\{f, x, e\}$, respectively. The importance of TDS's in the theory of nilpotent orbits in $\mathfrak{g}$ comes from the following embedding theorem due to Morosov.

THEOREM 4.1 (MOROSOV). *Let e be any nonzero nilpotent in $\mathfrak{g}$. Then e is the nilpositive part of some TDS in $\mathfrak{g}$.*

Proof. See Jacobson [4, p. 100, Theorem 17] or Kostant [6, Theorem 3.4].

A subalgebra $\mathfrak{u}_c \subset \mathfrak{g}_c$ isomorphic to $\mathfrak{sl}(2, \boldsymbol{C})$ will be called a *complex* TDS. Complex TDS's have been studied extensively by Kostant in [6]. In particular, the following result will be crucial in reducing the conjugacy of nilpotents to that of semisimple elements.

THEOREM 4.2 (KOSTANT) [6, THEOREM 4.2]. *Let e_1, e_2 be any nonzero nilpotents in $\mathfrak{g}_c$, and $\{f_1, x_1, e_1\}, \{f_2, x_2, e_2\}$ S-triples containing e_1 and e_2 respectively as nilpositive part. Then e_1 and e_2 are G_c-conjugate iff x_1 and x_2 are G_c-conjugate.*

Let $\mathfrak{p}_c = \mathfrak{p} + i\mathfrak{p}$. Rallis and Kostant have studied the orbits of maximal dimension of the action of K_θ on $\mathfrak{p}_c$. In particular, it is shown that there exists a nilpotent $e_0 \in \mathfrak{p}_c$ whose orbit has maximal dimension. More precisely, the following holds.

THEOREM 4.3 (KOSTANT-RALLIS) [8, THEOREM 2]. *There exists a nilpotent $e_r \in \mathfrak{p}_c$ such that* $\dim K_\theta \cdot e_r \geqq \dim K_\theta \cdot y$ *for all $y \in \mathfrak{p}_c$. The set of all such nilpotents is $K_\theta \cdot e_r$.*

We shall write $\mathfrak{m} = \{y \in \mathfrak{k} : [y, z]=0 \text{ for all } z \in \mathfrak{a}\}$, $M = K^{\mathfrak{a}}$, and $M_c = K_c^{\mathfrak{a}}$. Note that M and M_c are not connected in general.

The correspondence between nilpotents in $\mathfrak{p}_c$ and nilpotents in $\mathfrak{g}$ is established through the *Cayley transform*. The nilpotent $e \in \mathfrak{p}_c$ may be embedded in a complex TDS, $\mathfrak{u}_c$, spanned by $\{f_r, x_r, e_r\}$ with $e_r, f_r \in \mathfrak{p}_c$ and $x_r \in i\mathfrak{k}$ [8, Proposition 4]. Then the Cayley transform is defined as $v = \exp (\pi/4)^{1/2}(e_r - f_r)$. Then $v \cdot e_r \in \mathfrak{g}$ and $v \cdot x_r \in \mathfrak{p}$.

PROPOSITION 4.4. *Let $e_r \in \mathfrak{p}_c$ be as above. Then $v \cdot e_r \in \mathfrak{g}$ is a $\mathfrak{g}$-regular nilpotent. Furthermore, any $\mathfrak{g}$-regular nilpotent is obtained this way, up to G-conjugacy.*

Proof. Kostant and Rallis have shown that, up to K_θ-conjugacy, one may assume that $v(x_r) = \sum \varepsilon_i$, where the ε_i are the duals of the simple roots. Then $\mathfrak{g}_c$ as a $\mathfrak{u}_c$-module has all odd-dimensional irreducible components. Hence $\dim \mathfrak{g}_c^{v(e_r)} = \dim \mathfrak{g}_c^{v(x_r)}$. Now let $e \in \mathfrak{g}$ be any $\mathfrak{g}$-regular nilpotent, and $\{f, x, e\}$ an S-triple spanning TDS, $\mathfrak{u}'$. We may assume, up to conjugacy, that $x \in \mathfrak{g}$. Then $\dim \mathfrak{g}_c^e \geqq \dim \mathfrak{g}_c^x$; more precisely, $\dim \mathfrak{g}_c^e = \dim \mathfrak{g}_c^x$ iff $\mathfrak{g}_c$ has all odd-dimensional irreducible components as a $\mathfrak{u}'$-module. It is easy to show that $\mathfrak{g}^{v(x_r)} = \mathfrak{m} + \mathfrak{a}$, where

$$\mathfrak{m} = \{y \in \mathfrak{k} : [y, z] = 0 \text{ for all } z \in \mathfrak{a}\}.$$

Since $\mathfrak{g}^x \supseteq \mathfrak{m}+\mathfrak{a}$, $\dim \mathfrak{g}_c^{v(x_r)} \leqq \dim \mathfrak{g}_c^x$, or $\dim G\cdot v(x_r) \geqq \dim G\cdot x$. Since e is $\mathfrak{g}$-regular, $\dim G\cdot e \geqq \dim G\cdot v(e_r)$. Since $\dim \mathfrak{g}^e \geqq \dim \mathfrak{g}^x \geqq \dim \mathfrak{g}^{v(x_r)} = \dim \mathfrak{g}^{v(e_r)}$ it follows that

$$\dim G\cdot v(e_r) \geqq \dim G\cdot e,$$

with equality holding iff all the inequalities are equalities. In particular, $v(e_r)$ is a $\mathfrak{g}$-regular nilpotent. Furthermore, since $\dim \mathfrak{g}^e = \dim \mathfrak{g}^x$, it follows that $\mathfrak{g}_c$ is the sum of odd-dimensional irreducible $\mathfrak{u}'$-modules. Up to G-conjugacy we may assume $x=\sum n_i\varepsilon_i$. Since x is the monosemisimple part of an S-triple, it follows that $n_i=0, \frac{1}{2}$, or 1 for each i. (For proof of this see Kostant [5, Lemma 5.1] or Dynkin [2].) Since all the irreducible components are odd-dimensional, it follows that $n_i \neq \frac{1}{2}$ for all i. For if some $n_{i_0}=\frac{1}{2}$, then there is an irreducible component of the $\mathfrak{u}'$-module $\mathfrak{g}_c$ having $\frac{1}{2}$ as a weight of x. However, an odd-dimensional representation has only integral weights. We claim that $n_i=1$ for all i. If some $n_i=0$, then $\mathfrak{g}^x \supsetneq \mathfrak{m}+\mathfrak{a}$, so that $\dim \mathfrak{g}_c^{v(x_r)} < \dim \mathfrak{g}_c^x$. Hence $n_i=1$ for all i, so that $x=v(x_r)$. By Theorem 4.2 above, this implies that e is G_c-conjugate to $v(e_r)$, which shows that every G_c-conjugacy class of $\mathfrak{g}$-regular nilpotents is obtained from nilpotents of maximal dimensional orbit in $\mathfrak{p}_c$, proving Proposition 4.4.

We may now prove the following:

THEOREM 4.5. *The set of* $\mathfrak{g}$*-regular nilpotents forms a single N-orbit; i.e. any two* $\mathfrak{g}$*-regular nilpotents are N-conjugate.*

REMARK. This theorem was proved by Springer [12] in the case where $\mathfrak{g}$ contains a nilpotent which is actually regular. Orbits of regular nilpotents will be discussed in §5.

Proof. Let Π be a set of simple restricted roots for $\mathfrak{a}$. Kostant and Rallis [8] have shown that $e=\sum_{\alpha_i\in\Pi} e_{\alpha_i}$ is a $\mathfrak{g}$-regular nilpotent in $\mathfrak{g}$, where e_{α_i} is a root vector for α_i. $x=\sum_{i=1}^r \varepsilon_i$ is a monosemisimple part of a TDS containing e, and any other $\mathfrak{g}$-regular nilpotent in $\mathfrak{g}$ is G_c-conjugate to e. Let e_0 be any $\mathfrak{g}$-regular nilpotent in $\mathfrak{g}$ and let $x_0 \in \mathfrak{g}$ be the monosemisimple part of an S-triple containing x as nilpositive. We shall show that e_0 is N-conjugate to e. Since e and e_0 are G_c-conjugate, so are x and x_0. By Theorem 4.2 x and x_0 are G-conjugate. Hence we may assume that $x=x_0$. Since $[x, e]=2e$, it now follows that $e_0=\tilde{e}_{\alpha_1}+\tilde{e}_{\alpha_2}+\cdots+\tilde{e}_{\alpha_r}$ where the $\tilde{e}_\alpha$ are simple restricted root vectors. (The $\tilde{e}_{\alpha_i}$ may not be the same as the e_{α_i} since the restricted root spaces are not, in general, one-dimensional.) By Kostant's result [6, Theorem 4.2] e and e_0 are G_c^x-conjugate. Since

$$G_c^x = \exp \mathfrak{m} \exp i\mathfrak{m} \exp (\mathfrak{a}+i\mathfrak{a}),$$

it follows that

$$\exp m_1 (\exp im_2)(\exp y)\cdot e_0 = e,$$

for some $m_1, m_2 \in \mathfrak{m}$, and some $y \in \mathfrak{a}+i\mathfrak{a}$. Since $\exp m_1 \in G$,

$$\exp m_1(\exp im_2)(\exp y)\cdot e_0 \in \mathfrak{g} \quad \text{iff} \quad (\exp im_2)(\exp y)\cdot e_0 \in \mathfrak{g}.$$

$(\exp y)\cdot e_0=\sum(\exp y)\cdot e_\alpha=\sum y_\alpha e_\alpha$, where y_α is a constant depending on α. Since $\exp im_2\cdot e_\alpha$ is again a root vector for the simple root α, it follows that

$$(\exp im_2)(\exp y)\cdot e_0 \in \mathfrak{g} \quad \text{iff} \quad (\exp im_2)y_\alpha\cdot e_\alpha \in \mathfrak{g},$$

for all α.

Let $e_\alpha=e_{\alpha_k}+e_{\alpha_p}$ with $e_{\alpha_k}\in\mathfrak{k}$ and $e_{\alpha_p}\in\mathfrak{p}$. Since $m_2\in\mathfrak{k}$, $\exp im_2\cdot\mathfrak{k}_c=\mathfrak{k}_c$ and $\exp im_2\cdot\mathfrak{p}_c=\mathfrak{p}_c$, where $\mathfrak{k}_c=\mathfrak{k}+i\mathfrak{k}$ and $\mathfrak{p}_c=\mathfrak{p}+i\mathfrak{p}$. $\exp im_2\cdot y_\alpha e_\alpha$ is in $\mathfrak{g}$ iff $(\exp im_2)\cdot y_\alpha e_{\alpha_k}$ is in $\mathfrak{k}$ and $(\exp im_2)\cdot(y_\alpha e_{\alpha_p})$ is in $\mathfrak{p}$. However, $y_\alpha e_{\alpha_k}$ cannot be conjugate to an element of $\mathfrak{k}$ unless y_α is a real number, since $y_\alpha e_{\alpha_k}$ has eigenvalues which are not pure imaginary unless y_α is real, and every element of $\mathfrak{k}$ has pure imaginary eigenvalues. Hence $y_\alpha e_\alpha\in\mathfrak{k}$. By Proposition 1.1, since $\exp im_2\in\exp i\mathfrak{g}_u$, the conjugation by $\exp im_2$ must be trivial. That is, $\exp im_2\cdot y_\alpha e_{\alpha_k}=y_\alpha e_{\alpha_k}$. By a similar argument, $\exp im_2\cdot y_\alpha e_{\alpha_p}=y_\alpha e_{\alpha_p}$, where y_α is real for all simple restricted roots α, it follows that $\exp y\in N$. Since $\exp(m_1)\in G$, this completes the proof of the theorem.

We now express $N\cdot e$ as the union of G-orbits.

THEOREM 4.6. *Let $F'\subset N$ be the subgroup generated by $\{\varepsilon_i \mid \dim\mathfrak{g}^{\alpha_i}>1\}$, and let $\{n_1, n_2, \ldots, n_s\}$ be a set of representatives in N for the quotient group $N/G\cdot F'$. Then for any $\mathfrak{g}$-regular nilpotent $e\in\mathfrak{g}$, $\{n_1\cdot e, n_2\cdot e, \ldots, n_s\cdot e\}$ is a complete set of representatives for the distinct G-orbits of regular nilpotents.*

Proof. Let $e=e_{\alpha_1}+e_{\alpha_2}+\cdots+e_{\alpha_r}$, where $0\neq e_{\alpha_i}\in\mathfrak{g}^{\alpha_i}$ and $\dim\mathfrak{g}^{\alpha_j}=1$, $j=1, 2, \ldots, s$, and $\dim\mathfrak{g}^{\alpha_t}>1$, $t=s+1, s+2, \ldots, r$. To each coset representative of $N/G\cdot F'$ we associate a $\mathfrak{g}$-regular nilpotent of $\mathfrak{g}$, unique up to G-conjugacy, as follows. For any subset $I\subseteq\{1, 2, \ldots, s\}$ let $w_I\in N$ be the element $w_I=\prod_{j\in I} w_j$, where $w_j=\exp\pi i\varepsilon_j$, and let

$$e_I = \sum_{j=1}^{s} \operatorname{sgn} I_j e_{\alpha_j} + \sum_{t=s+1}^{r} e_{\alpha_t},$$

where

$$\operatorname{sgn} I_j = -1 \quad \text{if } j\in I,$$
$$\qquad = +1 \quad \text{if } j\notin I.$$

Every element of $N/G\cdot F'$ has a representative of the form w_I for some I, and every w_I represents some element of $N/G\cdot F'$.

We define the map $p\colon N/G\cdot F'\to\{\mathfrak{g}\text{-regular nilpotent } G\text{-orbits}\}$ by $p(\bar{w}_I)=(G\cdot e_I)$, where $\bar{w}_I$ is the class of w_I in $N/G\cdot F'$. We claim first that this is well defined. For, suppose $\bar{w}_I=1$ in $G\cdot F'$, i.e., $w_I=rq$ for some $r\in G$, $q\in F'$. We want to show that $G\cdot e=G\cdot e_I$, i.e. there exists $r'\in G$ such that $r'\cdot e=e_I$. We know $(rq)\cdot e=e_I$, so it suffices to show that there exists $m\in M$ such that $m\cdot e=q\cdot e$. An element $q\in F'$ acts as scalar multiplication by ±1 on those e_{α_t} where $\dim\mathfrak{g}^{\alpha_t}>1$. For any $e'\in\mathfrak{g}^{\alpha_t}$, $e'\neq0$, one has

$$[\mathfrak{m}+\mathfrak{a}, e'] = \mathfrak{g}^{\alpha_t} \quad \text{and} \quad \dim[\mathfrak{a}, e'] = 1,$$

so that $[\mathfrak{m}, e']$ is of codimension 1 in $\mathfrak{g}^{\alpha_i}$. Hence $\exp \mathfrak{m}=M^0$ is transitive on the unit sphere of $\mathfrak{g}^{\alpha_i}$, $i \geqq s+1$, since $\dim \mathfrak{g}^{\alpha_i} \geqq 2$ for $i=s+1, s+2, \ldots, r$. It follows that M^0 is transitive on the products of the unit spheres of the components of $\mathfrak{g}^{\alpha_{s+1}} \times \mathfrak{g}^{\alpha_{s+2}} \times \cdots \times \mathfrak{g}^{\alpha_r}$. Hence the map p is well defined.

We now show p is 1-1. Suppose $G\cdot e_I=G\cdot e_J$. We want to show that $\bar{w}_I=\bar{w}_J$ in $N/G\cdot F'$. But $e_J=w_J\cdot e$ and $e_I=r\cdot e_J=rw_J\cdot e$ for some $r \in G$. Then $w_J r^{-1}\cdot e_I=e$ $=r'(w_J\cdot e_I)$, for some $r' \in G$ since $G\cdot F'=F'\cdot G$, and $w_J\cdot e_{I'}=e_{I'}$, for some $I' \subseteq \{1, 2, \ldots, s\}$. Hence it suffices to prove that $G\cdot e_I=G\cdot e$ implies $w_I \in G\cdot F'$. So assume $r\cdot e=e_I$ for some $r \in G$. Then e and e_I are nilpositive elements of S-triples $\{f, x, e\}$ and $\{f_I, x, e_I\}$, respectively, for some nilpotents f, f_I, where $x=\sum_{i=1}^r \varepsilon_i$. We claim we can choose $r' \in G$ such that $r'\cdot e=e_I$, $r'\cdot x=x$. Indeed, if $r\cdot x=x'$, then x and x' are G^{e_I}-conjugate by the argument given by Kostant [6, p. 986] applied to G instead of G_c. Hence there exists $r' \in MA$ such that $r'\cdot e=e_I$. Since $M=K^{\mathfrak{a}}$, it follows that any element $r' \in M$ can be written

$$r' = \left(\exp \pi i\left(\sum_{j=1}^{s} \varepsilon_{i_j}\right)\right)(\exp m)a,$$

where ε_{i_j}, $j=1, 2, \ldots, r$, can be chosen so that $\exp \pi i(\sum \varepsilon_{i_j}) \in M$, $m \in \mathfrak{m}$ and $a \in A$. a acts as a scalar, $c_{\alpha_i}>0$, on each $\mathfrak{g}^{\alpha_i}$. Hence

$$r'\cdot e = \left(\exp \pi i\left(\sum_{j=1}^{s} \varepsilon_{i_j}\right)\right)\cdot\left(\sum_{i=1}^{r} c_{\alpha_i} e_{\alpha_i}\right)$$

since $\exp m$ acts trivially on the 1-dimensional root spaces. The above expression equals

$$\left(\exp \pi i\left(\sum_{j=1}^{t} \varepsilon_{i_j}\right)\right)\left(\exp \pi i\left(\sum_{k=t+1}^{r} \varepsilon_{i_k}\right)\cdot\left(\sum_{i=1}^{r} (c_{\alpha_i} e_{\alpha_i})\right)\right.$$

where $i_t \leqq s$ and $i_{t+1} \geqq s+1$.

$$\left(\exp \pi i\left(\sum_{j=1}^{t} \varepsilon_{i_j}\right)\right)\cdot\left(\sum_{j=1}^{s} c_{\alpha_j} e_{\alpha_j}+\sum_{k=s+1}^{r} c'_{\alpha_k} e_{\alpha_k}\right)$$

where the c'_{α_k} are new coefficients >0. But since $r'\cdot e=e_I$, this implies that $\{i_1, i_2, \ldots, i_t\}$ is the same as the set I. Hence $\exp \pi i(\sum_{j=1}^t \varepsilon_{i_j})=w_I$. Since

$$\exp \pi i\left(\sum_{j=t+1}^{s} \varepsilon_{i_j}\right) \in F',$$

$\exp m \in M$, $a \in A$, and $r'^{-1} \in G$, and the equation above shows that w_I equals their product, it follows that $w_I \in F'\cdot G=G\cdot F'$. Hence w_I equals the identity coset in $N/G\cdot F'$, which proves that p is 1-1. Since every $\mathfrak{g}$-regular nilpotent G-orbit is N-conjugate to one of the form $G\cdot e_I$ for some I by Theorem 4.5, it is obvious that p is surjective. Hence p is bijective, which proves Theorem 4.6.

We shall now apply this result to express $K_\theta\cdot e$ as the union of K_c-orbits.

COROLLARY 4.7. *Let* $\{n_1', n_2', \ldots, n_{s'}'\}$ *be a set of representatives in* K_θ *for the quotient group* $K_\theta/K_c\cdot F'$. *Then* $\{n_1'\cdot e, n_2'\cdot e, \ldots, n_{s'}'\cdot e\}$ *is a complete set of representatives for the distinct K-orbits of* $K_\theta\cdot e$. *Furthermore*, $K_\theta/K_c\cdot F' \approx N/G\cdot F'$.

Proof. By Proposition 1.8, if $a_1, a_2, \ldots, a_s \in \exp i\mathfrak{a}$ are representatives for the quotient group $N/G\cdot F'$, then they are also representatives for the quotient group $K_\theta/K\cdot F'$. Now let e_1 and e_2 be any two nilpotents in $\mathfrak{p}_c$ whose K_θ-orbits have maximal dimension, i.e. by Theorem 4.3 this means $K_\theta\cdot e_1 = K_\theta\cdot e_2 = K_\theta\cdot e$. Let v_1 and v_2 be Cayley transforms associated to e_1 and e_2 respectively, where $\{f_1, x_1, e_1\}$ and $\{f_2, x_2, e_2\}$ are TDS's with $x_1, x_2 \in i\mathfrak{k}$. Since e_1 and e_2 are conjugate we may assume that $v_1(x_1) = v_2(x_2)$. The action of the Cayley transforms is given as follows:

$$\begin{aligned} v_i(x_i) &= (-1/2^{1/2})(e_i+f_i) \in \mathfrak{p}_c, \\ v_i(e_i) &= (i/2^{1/2})x_i + (i/2^{1/2})(e_i - f_i), \\ v_i(f_i) &= (i/2^{1/2})x_i - (i/2^{1/2})(e_i - f_i), \qquad i = 1, 2. \end{aligned}$$

A simple computation shows that $b\cdot v(e_1) = v(e_2)$ iff $b\cdot e_1 = e_2$, for $b \in G_c$. It suffices to show that e_1 and e_2 are $K_c\cdot F'$-conjugate iff $v_1(e_1)$ and $v_2(e_2)$ are $G\cdot F'$-conjugate. Suppose first that $a\cdot e_1 = e_2$ for $a \in K_c\cdot F'$. By the above, we may assume $a \in M_cA_c$. Now $(K_c\cdot F' \cap M_cA_c) = (\exp \mathfrak{k})(\exp i\mathfrak{k})\cdot F'$. As in the proof of Theorem 4.6, $\exp i\mathfrak{k}$ acts trivially on $\mathfrak{g}$ by Proposition 1.1, so that $a'\cdot v_1(e_1) = v_2(e_2)$ for some $a' \in \exp \mathfrak{k}\cdot F'$. Conversely, if $b'\cdot v_1(e_1) = v_2(e_2)$, for $b' \in G\cdot F'$, then since $(G\cdot F' \cap M_c\cdot A_c) = M\cdot F' \subset K_c\cdot F'$, it follows that e_1 and e_2 are $K_c\cdot F'$-conjugate.

5. **Orbits of regular elements.** The study of orbits of arbitrary regular elements can be reduced to that of semisimple and nilpotent elements by the following. Any $x \in \mathfrak{g}$ can be uniquely written $x = x_s + x_n$, where x_s is semisimple, x_n nilpotent, and $[x_s, x_n] = 0\cdot \mathfrak{g}^{x_s}$ is again reductive since x_s is semisimple. Furthermore, x is regular iff x_n is regular-in-$\mathfrak{g}^{x_s}$. (A proof of this latter fact is given in [5, Proposition 13].)

In view of the above, we shall be concerned with determining when $\mathfrak{g}^{x_s}$ contains a nilpotent regular-in-$\mathfrak{g}^{x_s}$. (If $\mathfrak{g}$ is actually complex, this is always the case.) In the real case $\mathfrak{g}^{x_s}$ is characterized as follows.

A real form $\mathfrak{g}_0$ of $\mathfrak{g}_c$ is called *quasi-split* if there is a subalgebra $\mathfrak{b}_0 \subset \mathfrak{g}_0$ such that $\mathfrak{b} = \mathfrak{b}_0 + i\mathfrak{b}_0$ is a Borel subalgebra of $\mathfrak{g}_c$.

PROPOSITION 5.1. $\mathfrak{g}$ *contains a regular nilpotent iff* $\mathfrak{g}$ *is quasi-split iff* $\mathfrak{g}$ *contains a regular semisimple element x such that* ad x *has all real eigenvalues.*

Proof. Suppose that $\mathfrak{g}$ contains a regular nilpotent e. By Theorem 4.1 there is an S-triple $\{f, x, e\}$ with e as a nilpositive part. Kostant has shown that e regular implies that x is also regular, so that $\mathfrak{g}$ contains the desired regular semisimple element, since all the eigenvalues of ad x are real. To show that $\mathfrak{g}$ is quasi-split, it suffices to show that the involution σ defining $\mathfrak{g}$ leaves invariant a set of simple positive roots. Let $\mathfrak{h}_c$ be the Cartan subalgebra of $\mathfrak{g}_c$ containing x, and let $\mathfrak{h}^0$ be the real subspace of $\mathfrak{h}_c$ spanned by the roots of $\mathfrak{h}_c$. Then since ad x has all real eigenvalues, we may assume $x \in \mathfrak{h}^0$ and therefore since x is regular it is contained in an open Weyl chamber C of $\mathfrak{h}^0$. But since $\sigma(x)=x$, $\sigma(C)=C$ also, so that σ leaves invariant a set of positive simple roots.

Conversely, suppose $\mathfrak{g}$ is quasi-split. Then the involution σ defining $\mathfrak{g}$ leaves invariant a set of positive roots Π. We may assume up to conjugacy of σ that there is a set of simple root vectors $\{e_\alpha\}$ where $e_\alpha \in \mathfrak{g}_c^\alpha$ such that $\sigma(e_\alpha)=e_{\sigma(\alpha)}$. Then the element $\sum e_\alpha$ is fixed by α and therefore is in $\mathfrak{g}$. Kostant [5, Theorem 5.3] has shown that this element is always regular. It is obvious by the above that $\mathfrak{g}$ is quasi-split iff $\mathfrak{g}$ contains a regular semisimple element x with ad x having all its eigenvalues real, so that Proposition 5.1 is proved.

COROLLARY 5.2. *x_s is the semisimple part of a regular $x \in \mathfrak{g}$ iff $\mathfrak{g}^{x_s}$ is quasi-split iff there exists $y \in \mathfrak{g}$ symmetric, x_s+y is regular, and $[y, x_s]=0$.*

Proof. Apply Proposition 5.1 to the reductive algebra $\mathfrak{g}^{x_s}$ to find a nilpotent e regular-in-$\mathfrak{g}^{x_s}$. y can be chosen as the monosemisimple part of a TDS in $\mathfrak{g}^{x_s}$ containing e (Theorem 4.1).

We now turn to the problem of finding representatives for the orbits of $\mathfrak{g}$-regular nilpotents in $\mathfrak{g}$.

THEOREM 5.3. *Let $\mathfrak{g}$ be quasi-split. The set of all regular nilpotents in $\mathfrak{g}$ forms a single N-orbit; i.e. any two regular nilpotents in $\mathfrak{g}$ are N-conjugate. Furthermore, the N-orbit of regular nilpotents is the union of $\#(N/G)$ G-orbits.*

Proof. (Note that if $\mathfrak{g}$ is a normal form then this follows immediately from Theorem 4.6 since $F'=1$ because dim $\mathfrak{g}^\alpha=1$ for all restricted roots α.) We define the map $p: N/G \to \{G\text{-orbits of regular nilpotents}\}$ by $p(\overline{w}_I)=e_I$, where $I \subseteq (1, 2, \ldots, r)$, $w_I=\Pi \exp \pi i \varepsilon_j$, and

$$e_I = \sum_{j=1}^{r} \operatorname{sgn} I_j e_{\alpha_j},$$

where

$$\begin{aligned} \operatorname{sgn} I_j &= -1 \quad \text{if } j \in I, \\ &= +1 \quad \text{if } j \notin I. \end{aligned}$$

As in the proof of Theorem 4.6, it suffices to show that p is 1-1. In fact, it suffices to show that $w_I \cdot e=e$ implies $w_I=1$ in N/G. Let $\{f, x, e\}$ be an S-triple containing e. Since $w_I \cdot x$ is G_c-conjugate to x by Theorem 4.2 there exists $r \in G$ such that $r \cdot x=w_I \cdot x$. But by the argument given by Kostant [6, pp. 986–987] we can choose r satisfying $r \cdot e=e$. Therefore $r^{-1}w_I$ leaves $\{f, x, e\}$ pointwise fixed, which proves that $r^{-1}w_I$ is the identity, since $\{f, x, e\}$ spans a "principal" TDS in the sense of Kostant [6]. This proves Theorem 5.3.

We shall now give the decomposition of $G_c \cdot x \cap \mathfrak{g}$ into G orbits for any regular semisimple $x \in \mathfrak{g}$. Our main result is the following.

THEOREM 5.4. *Let $x=x_s+x_n \in \mathfrak{g}$ be regular and let $\mathfrak{h}\subset\mathfrak{g}$ be a maximally vector Cartan subalgebra of $\mathfrak{g}^{x_s}$. A set of representatives for the distinct G-orbits in $G_c\cdot x \cap \mathfrak{g}$ is given by $\{w_j\cdot x_s+g_{j_r}\cdot x_n'\}$ where $w_1, w_2, \ldots, w_{m_0}$ is a set of representatives for the coset space $W_c^{\mathfrak{h}_p}/W^{\mathfrak{h}_p}$ and $\{g_{j_r}\}$, $r=1, 2, \ldots, t$, is a set of representatives for the quotient group $N(\mathfrak{g}^{w_j\cdot x_s})/G^{w_j\cdot x_s}$.*

Proof. Let $S_x=\{z \in \mathfrak{g} \mid z=y_s$ for some $y \in G_c\cdot x \cap \mathfrak{g}\}$, and for any $z \in S_x$, let $R_z=\{e \in \mathfrak{g}^z \mid z=q_s$ and $e=q_n$ for some $q \in G_c\cdot x \cap \mathfrak{g}\}$. We shall first prove the following two statements.

(a) $S_x=\bigcup_{j=1}^{m_0} G\cdot(w_j\cdot x_s)$ as a disjoint union. That is, every $z \in S_x$ is G-conjugate to exactly one $w_j\cdot x_s$.

(b) For each j, $0\leqq j\leqq m_0$, there exists an e_j nilpotent and regular-in-$\mathfrak{g}^{w_j\cdot x_s}$.

We claim that (a) and (b) will prove Theorem 5.4. We must first show that if $y \in G_c\cdot x \cap \mathfrak{g}$, then there exist $w_j \in W_c^{\mathfrak{h}_p}/W^{\mathfrak{h}_p}$ and $g_{j_r} \in N(\mathfrak{g}^{x_s})/G^{x_s}$ such that $y \in G\cdot(w_j\cdot x_s+g_{j_r}\cdot e_j)$. By definition, $y_s \in S_x$. Therefore, by (a) there exists $w_j \in W_c^{\mathfrak{h}_p}/W^{\mathfrak{h}_p}$ such that $w_j\cdot x_s=g\cdot y_s$ for some $g \in G$. Since $w_j\cdot x_s \in S_k$, by (b) there exists e_j a nilpotent regular-in-$\mathfrak{g}^{w_j\cdot x_s}$. Since e_j and $g\cdot y_n$ are both nilpotents regular-in-$\mathfrak{g}^{w_j\cdot x_s}$, Theorem 5.3 implies e_j is $N(\mathfrak{g}^{x_s})$-conjugate to $g\cdot y_n$. It follows immediately that $y \in G\cdot(w_j\cdot x_s+g_{j_r}\cdot e_j)$.

To prove that the orbits are distinct, note that $w_j\cdot x_s+g_{j_r}\cdot e_j=w_d\cdot x_s+g_{d_r}\cdot e$ implies $w_j\cdot x_s=w_d\cdot x_s$ which means that $j=d$, by (a). Since the above also implies $g_{j_r}\cdot e_j$ is $G^{w_j\cdot x_s}$-conjugate to $g_{j_r}\cdot e$, we have $g_{j_r}=g_{j_{r'}}$ by Theorem 5.3.

To prove (a) we shall need the following.

LEMMA 5.5. *There exists $q \in \mathfrak{h}$ such that x_s is regular-in-$\mathfrak{g}^q$ and $S_x=G\cdot(S_x^q)$.*

Proof. By Theorem 4.1 we may embed x_n as the nilpositive element of an S-triple with monosemisimple q. By conjugacy we may assume that $q \in \mathfrak{h}$, since $[q, x_s]=0$.

It suffices to show that for any $y \in S_k$ there exists $y' \in S_x^q$ such that y and y' are G-conjugate. Let $z=y+z_n \in G_c\cdot x \cap \mathfrak{g}$ and let q' be the monosemisimple element of an S-triple in $\mathfrak{g}^y$ containing z_n as nilpositive. Since z_n and x_n are G_c-conjugate so are q' and q since the nilpositive parts of an S-triple are G_c-conjugate iff the monosemisimple parts are by Kostant's result [5, Theorem 3.6]. By Corollary 2.2 we have q' and q are G-conjugate since q and q' are symmetric elements. This proves $y_s \in G\cdot S_x^q$.

Finally, to prove that x_s is regular-in-$\mathfrak{g}^q$, it suffices to show that q is regular-in-$\mathfrak{g}^{x_s}$. Since the TDS is principal-in-$\mathfrak{g}^{x_s}$, its monosemisimple elements are regular-in-$\mathfrak{g}^{x_s}$ by Kostant's result [6, Lemma 5.1]. This proves Lemma 5.5.

LEMMA 5.6. *Choose q as in Lemma 5.5. If $y \in S_x^q$, then x_s and y_s are G_c^q-conjugate. Furthermore, $S_x^q=G_c^q\cdot x_s \cap \mathfrak{g}^q$.*

Proof. By definition of S_x, y_s and x_s are G_c-conjugate, so that there exists $g_c \in G_c$ such that $g_c\cdot y_s=x_s$. Since q is the monosemisimple element of a principal S-triple in $\mathfrak{g}^{y_s}$, $g_c\cdot q$ is the monosemisimple element of a principal S-triple in $\mathfrak{g}_c^{x_s}$. Since q is also such an element in $\mathfrak{g}_c^{x_s}$, it follows that q and q' are G-conjugate, by the result cited above [6, Theorem 3.6]. If $g_c' \in G_c^{x_s}$ satisfies $g_c' \cdot (g_c \cdot q) = q$, then $g_c'(g_c \cdot y_s) = x_s$ with $g_c' g_c \in G_c^q$, which proves the first assertion. This also shows immediately that

$$S_x^q \subseteq G_c^q \cdot x_s \cap \mathfrak{g}^q.$$

To show the opposite inclusion, let $z_s \in G_c^q \cdot x_s \cap \mathfrak{g}^q$. To show that $z_s \in S_x^q$ it suffices to find z_n, a nilpotent regular-in-$\mathfrak{g}^{z_s}$, such that $z_s + z_n$ is in $G_c \cdot x \cap \mathfrak{g}$. Since q is regular-in-$\mathfrak{g}^{z_s}$ and q is symmetric, it follows from Corollary 5.2 that $\mathfrak{g}^{z_s}$ is quasi-split, so that there exists z_n nilpotent reguiar-in-$\mathfrak{g}^{z_s}$. Since x and $z_s + z_n$ are both regular and x_s and z_s are G_c-conjugate, it follows that $x = z_s + z_n$, which proves that $z_s \in S_x^q$, completing the proof of Lemma 5.6.

Since q is symmetric and $w_j \cdot x_s \in \mathfrak{g}^q$ for all q, (b) follows immediately from Proposition 5.1 and its corollary.

LEMMA 5.7. *Let q be as before, and let $\mathfrak{h}$ be the unique Cartan subalgebra of $\mathfrak{g}^q$ containing x. Then*

$$S_x^q = \bigcup_{j=1}^{m_0} G^q \cdot (w_j \cdot x_s)$$

is the decomposition of S_x^q into G^q-orbits.

Proof. By Lemma 5.6 and Theorem 2.6, we have $S_x^q = G_c^q \cdot x_s \cap \mathfrak{g}^q = \bigcup_{j=1}^{m_0} G^q \cdot (w_j' \cdot x_s)$, where $w_1', w_2', \ldots, w_{m_0}'$ is a set of representatives for the coset space $(W_c^q)^{\mathfrak{h}_p}/(W^q)^{\mathfrak{h}_p}$. However, since $q \in \mathfrak{h}$, this latter coset space is the same as $W_c^{\mathfrak{h}_p}/W^{\mathfrak{h}_p}$, proving Lemma 5.7.

Theorem 5.4 now follows immediately from Lemmas 5.5 and 5.7.

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