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A CHARACTERIZATION OF M-SPACES IN THE CLASS OF SEPARABLE SIMPLEX SPACES

BY

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ABSTRACT. We show that a separable simplex space is an M-space iff the arbitrary intersection of closed ideals is always an ideal.

Edward Effros in [5, end of §3] was unable to determine if an arbitrary intersection of closed ideals in a simplex space was necessarily an ideal. That this was not true in general was shown by J. Bunce [2] and F. Perdrizet. Here we shall show that for separable simplex spaces this property is equivalent to the simplex space being a (Kakutani) M-space.

In §1 we extend some of the results of [7]. We show that certain subspaces of certain simplex spaces are again simplex spaces. In §2 we give the aforementioned characterization.

I should like to thank the referee for his comments. The much simplified proof of Theorem 1.2 and its extension to the nonstandard case are due to him.

0. **Conventions.** All vector spaces are assumed to have nonzero elements. The term measure will always denote a regular bounded Borel measure. We use $\delta(q)$ for the point measure at q.

1. **An existence theorem.** An ordered Banach space V with closed positive cone is a simplex space if its dual is a (Kakutani) L-space. If Y is a compact Hausdorff space, we let $C(Y)$ be the space of (real) continuous functions on Y with the natural pointwise order and the supremum norm. Obviously, $C(Y)$ is a simplex space. Its dual, $C^*(Y)$, is the space of all measures on Y. More generally, if X is a Borel subset of Y, we let $C^*(Y; X)$ be the space of all measures on Y whose total variation on X is zero. Then $C^*(Y; X)$ is an L-space and the extreme points of the positive part of its unit ball are $\{\delta(y) | y \in Y - X\} \cup \{0\}$ [7, Proposition 1.1].

If V is a simplex space, we let

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$$P_1(V) = \{f \in V^* \mid \|f\| \leq 1,\ f(x) \geq 0 \text{ for each } x \geq 0\}$$

and $EP_1(V)$ be its extreme points. We take

$$EP_1(V)^+ = EP_1(V) - \{0\}, \qquad Z(V) = \text{weak* closure of } EP_1(V)^+.$$

Since $P_1(V)$ is a simplex, each $q \in P_1(V)$ is the resultant of a unique maximal probability measure μ_q. We take $\pi_q = \mu_q - \mu_q(\{0\})\delta(0)$. If V is separable, μ_q is supported by $EP_1(V)$ and π_q by $EP_1(V)^+$.

We may characterize V as the set of affine continuous functions on $Z(V)$ vanishing at zero. Hence

$$V = \{f \in C(Z(V)) \mid f(q) = \pi_q(f) \text{ for each } q \in Z(V) - EP_1(V)^+\}.$$

Let

$$X(V) = \{\delta(q) - \pi_q \mid q \in Z(V) - EP_1(V)^+\}.$$

We shall always assume that $X(V)$ is given the weak* topology relative to $C(Z(V))$.

We will be using the following well-known results repeatedly and include them for completeness. First, a map $x \longrightarrow \eta_x$ of a topological space E into the measures on a locally compact space F is *weak* Borel measurable* iff $x \longrightarrow \eta_x(f)$ is a Borel measurable function for each $f \in C(F)$. It is *bounded* if $\sup \|\eta_x\| < \infty$. We then have [1, V, §3, Proposition 2, Definition 3, Corollary to Proposition 12]

Lemma 1.1. *Let E and F be locally compact Hausdorff spaces. Suppose $x \longrightarrow \eta_x$ is a weak* bounded Borel measurable map of E into the positive measures on F. Let μ be a positive measure on E.*

1. *Then there is a measure ν on E defined by*

$$\nu = \int \eta_x \, d\mu(x).$$

2. *Suppose f is a bounded universally measurable function on F. Then*

$$x \longrightarrow \int f(y)\, d\eta_x(y)$$

is universally measurable and

$$\int f(y)\, d\nu = \int d\mu(x) \int f(y) d\eta_x(y).$$

Our first result identifies $V^\perp$ for V a simplex space. Throughout, we always consider V as a subset of $C(Z(V))$ and not as a subset of $C(P_1(V))$.

Theorem 1.2. *Let V be a simplex space. Let $Z = Z(V)$. Then $V^\perp \subseteq C^*(Z)$ may be identified as follows:*

$$V^{\perp} = \left\{ w \in C^*(Z) \,\middle|\, \textit{there is a measure } \nu \textit{ on } Z \textit{ such that } w(f) = \int_Z (\delta(z) - \pi_z)(f)\, d\nu \textit{ for each } f \in C(Z) \right\}.$$

Proof. Let $w \in V^{\perp}$. We may assume that $\|w\| \leq 1$. Writing $w = w^+ - w^-$, we obviously have $\|w^+\| \leq 1$, $\|w^-\| \leq 1$. Let

$$w_1 = w^+ + (1 - \|w^+\|)\delta(0), \qquad w_2 = w^- + (1 - \|w^-\|)\delta(0).$$

Then w_1, w_2 are probability measures on $P_1(V)$. Let h be a continuous affine function on $P_1(V)$. Then $h = f + h(0)1$ for some $f \in V$. Since $w^{\pm}(1) = \|w^{\pm}\|$ and $w^+(f) - w^-(f) = w(f) = 0$ for $f \in V$, we have that $w_1(h) = w_2(h)$. Hence, w_1 and w_2 have the same resultant. Let n be the unique maximal measure dominating both w_1 and w_2. Then [8, Theorem 30, p. 232]

$$n = \int_{P_1(V)} \mu_q \, dw_1(q) = \int_{P_1(V)} \mu_q \, dw_2(q).$$

Since w_1, w_2 are supported by $Z \cup \{0\}$ and recalling that $\pi_q = \mu_q - \mu_q(\{0\})\delta(0)$ we get

$$w_i - n = \int_Z (\delta(q) - \pi_q)\, dw_i(q) + c_i \delta(0), \qquad i = 1, 2,$$

for suitable constants c_i. But then

$$w = \int_Z (\delta(q) - \pi_q)\, d(w_1 - w_2) + c\delta(0)$$

for a suitable constant c. Noting that $\pi_0 = 0$, we may write $\nu = w_1 - w_2 + c\delta(0)$ to get the required representation.

Conversely, any $w \in C^*(Z)$ which has the representation

$$w = \int_Z (\delta(q) - \pi_q)\, d\nu$$

for some measure ν obviously annihilates V.

Corollary 1.3. *Let V be a simplex space. Let $Z = Z(V)$, $E = EP_1(V)^+$, and $X = X(V)$. Suppose E is a universally measurable subset of Z. Then $V^{\perp} \subseteq C^*(Z)$ may be identified as follows*:

$$\begin{aligned} V^{\perp} &= \text{linear span}\,(\overline{\text{co}}\,(X)) \\ &= \left\{ w \in C^*(Z) \,\middle|\, \textit{there is a measure } \nu \textit{ on } Z \textit{ such that } w = \int_{Z-E} (\delta(z) - \pi_z)\, d\nu \right\}. \end{aligned}$$

Proof. For each $z \in E$, we have $\delta(z) = \pi_z$ and so the second representation is clear. The first follows from the second by approximating ν by atomic measures.

A simplex K is *standard* if it satisfies

1. The extreme points of K, denoted by $E(K)$, is a universally measurable subset of K.

2. Each maximal measure is supported by $E(K)$.

By Lemma 1.1 above, condition (2) is equivalent to

2′. The maximal measures representing each point in the closure of $E(K)$ are supported by $E(K)$.

We call a simplex space V a *standard simplex space* if $P_1(V)$ is a standard simplex. Note that all separable simplex spaces are standard.

We are now prepared to show that certain subspaces of standard simplex spaces are again simplex spaces.

Theorem 1.4. *Let V be a standard simplex space. Let $Z = Z(V)$ and $E = EP_1(V)^+$. Let $q_0 \in Z - E$. Suppose Y is a closed set in Z satisfying*

1. $Y \cap (Z - E) = \{q_0\}$.

2. $\pi_{q_0}(E - Y) \neq 0$.

Then $A = \{f \in V | f(y) = \pi_{q_0}(f)$ for each $y \in Y\}$ is a nontrivial simplex space with the relative order and norm. Further, A^* is isometrically order isomorphic to $C^*(Z; (Z - E) \cup Y)$ and so $EP_1(A)^+ = E - Y$.

Proof. We divide the proof into several stages. Throughout, we take π_0 to be π_{q_0}. Let $\alpha = \pi_0(Y)$. Then

$$0 < \pi_0(E - Y) \leq 1 - \alpha. \tag{1.1}$$

A. First, let us consider the space D defined by

$$D = \{f \in C(Z) | f(y) = \pi_0(f) \text{ for each } y \in Y\}.$$

Let $X(D) = \{\delta(Y) - \pi_0 | y \in Y\}$. As $y \longrightarrow \delta(y)$ is continuous and Y is compact, Y is homeomorphic to $X(D)$. Let m be a measure on $X(D)$. Then there is a measure λ on Y induced by m. So, for $f \in C(Z)$,

$$\int_{X(D)} x(f)\, dm(x) = \int f(y)\, d\lambda - \pi_0(f)\lambda(Y). \tag{1.2}$$

Let

$$F = \left\{ w \in C^*(Z) \,\middle|\, \begin{array}{l} \text{there exists a measure } \lambda \text{ on } Y \text{ such that} \\ w(f) = \int f(y)\, d\lambda - \pi_0(f)\lambda(Y) \text{ for each } f \in C(Z) \end{array} \right\}.$$

Approximating λ by atomic measures clearly yields

$$F = \text{linear span}\,(\overline{\text{co}}\,(X(D))).$$

Clearly, the weak* closure of F is D^*. To show that F is weak* closed it suffices to show it is norm closed [3, V, 5.9].

In order to show F is norm closed, we consider a $w \in F$. Then there is a measure λ on Y such that

$$w(f) = \int f(y)\, d\lambda - \pi_0(f)\lambda(Y)$$

for each $f \in C(Z)$. By Lemma 1.1, for each Borel set $B \subseteq Z$,

$$w(B) = \lambda(B \cap Y) - \pi_0(B)\lambda(Y).$$

For each Borel set $A \subseteq Y$ we easily get

$$\lambda(A) = w(A) + \pi_0(A)w(Y)/(1-\alpha)$$

and so w uniquely determines λ. Further

$$|\lambda|(Y) \leq |w|(Y) + \pi_0(Y)|w(Y)|/(1-\alpha)$$

and so

$$\|\lambda\| = |\lambda|(Y) \leq \|w\|(1 + \alpha/(1-\alpha)).$$

It should now be clear that F is norm closed. Hence $F = D^{\perp}$.

B. *The determination of* $A^{\perp}$. It is clear that $A = D \cap V$ and so

$$A^{\perp} = \text{weak* closure } (D^{\perp} + V^{\perp}).$$

We claim that $D^{\perp} + V^{\perp}$ is already weak* closed. From the representations for $D^{\perp}$ in A and $V^{\perp}$ in Corollary 1.3,

$$D^{\perp} + V^{\perp} = \text{linear span}\,(\overline{\text{co}}\,(X(V) \cup X(D)))$$

and so again we need only demonstrate that $D^{\perp} + V^{\perp}$ is norm closed. We let

$$W = \Big\{ w \in C^*(Z) \,\Big|\, \text{there exists a measure } \nu \text{ on } Z \text{ and a measure } \lambda \text{ on } Y \text{ such that, for each } f \in C(Z), \tag{1.3}$$
$$w(f) = \int_{Z-E} (\delta(z) - \pi_z)(f)\, d\nu + \int f(y)\, d\lambda - \pi_0(f)\lambda(Y) \Big\}.$$

Then, from the representations for $D^{\perp}$ in A and $V^{\perp}$ in Corollary 1.3, $W = D^{\perp} + V^{\perp}$. Let $w \in W$ be determined by measures ν on Z and λ on Y. Then, using Lemma 1.1, for each Borel set $B \subseteq Z$,

$$w(B) = \nu(B \cap (Z - E)) - \int_{Z-E} \pi_z(B)\, d\nu + \lambda(B \cap Y) - \pi_0(B)\lambda(Y). \tag{1.4$'$}$$

In particular, for Borel $A \subseteq Z - E - \{q_0\}$,

$$\nu(A) = w(A). \tag{1.4$''$}$$

Also, $\nu(\{q_0\}) + \lambda(\{q_0\}) = w(\{q_0\})$. We may, by transferring an atom if necessary, take $\lambda(\{q_0\}) = 0$. Hence

$$\nu(\{q_0\}) = w(\{q_0\}), \qquad \lambda(\{q_0\}) = 0. \tag{1.4$'''$}$$

Finally, for Borel $C \subseteq Y - \{q_0\}$,

$$\lambda(C) = w(C) + \int_{Z-E} \pi_z(C)\,d\nu + \pi_0(C)\lambda(Y) \tag{1.4}''''$$

with $\lambda(Y)$ found by consistency. With the choice $(1.4)'''$, $\nu|(Z - E) = w|(Z - E)$ and so we may rewrite W as

$$W = \Big\{ w \in C^*(Z) \Big| \text{ there exists a measure } \lambda \text{ on } Y \text{ without an atom at } q_0 \text{ such that, for each } f \in C(Z), \; w(f) = \int_{Z-E} (\delta(z) - \pi_z)\,(f)\,dw + \int f(y)\,d\lambda - \pi_0(f)\lambda(Y) \Big\}. \tag{1.5}$$

In this representation, we note that if $w \in C^*(Z)$ is determined by the measure λ on Y, then again

$$\|\lambda\| \le \|w\|\,(1 + \alpha/(1 - \alpha)).$$

So clearly W is norm closed. Therefore $W = A^{\perp}$.

C. *The determination of* A^*. The dual of A is $C^*(Z)/A^{\perp}$. To complete the proof we need only show that $C^*(Z)/A^{\perp}$ is isometric order isomorphic to $C^*(Z; (Z - E)\bigcup Y)$. We claim that each class of A^* contains one and only one member of $C^*(Z; (Z - E)\bigcup Y)$. Indeed, let $m \in A^*$ and suppose $n_1, n_2 \in m$ each were in $C^*(Z; (Z - E)\bigcup Y)$. Then $n_1 - n_2 = w \in W \bigcap C^*(Z; (Z - E)\bigcup Y)$. Hence, there is a measure λ on Y such that for each Borel $B \subseteq Z$ (using $(1.4)'$ and the representation (1.5) for W)

$$w(B) = w(B \bigcap (Z - E)) - \int_{Z-E} \pi_z(B)\,dw + \lambda(B \bigcap Y) - \pi_0(B)\lambda(Y). \tag{1.6}$$

If w vanishes on $Z - E$, then $w(B) = \lambda(B \bigcap Y) - \pi_0(B)\lambda(Y)$. Using $w(Y) = 0$ we get $\lambda(Y) = 0$. But then, since w vanishes on Y, $\lambda = 0$ and so $w \equiv 0$. Thus $n_1 = n_2$.

On the other hand, let $n \in m$. We define a measure $\lambda(n)$ on Y by

$$\begin{aligned} &\lambda(n)(\{q_0\}) = 0, \\ &\lambda(n)(A) = \int_{Z-E} \pi_z(A)\,dn + n(A) + \pi_0(A)\lambda(n)(Y) \end{aligned} \tag{1.7}$$

for each Borel $A \subseteq Y - \{q_0\}$, where $\lambda(n)(Y)$ is found by consistency. Let $w(n)$ be the element of W determined by n and $\lambda(n)$ by (1.3). Then $n - w(n) \in m$. Using $(1.4)''$ to $(1.4)''''$ and (1.7) one easily verifies that $w(n)(B) = n(B)$ for each Borel set $B \subseteq (Z - E)\bigcup Y$. Hence, $n - w(n) \in C^*(Z; (Z - E)\bigcup Y)$ and the claim is established.

The element $n - w(n)$ depends only on the class m and not on the particular representative n. We may therefore define a map $\phi\colon A^* \to C^*(Z; (Z - E)\bigcup Y)$ by $\phi(m) = n - w(n)$ for each $m \in A^*$ and any representative $n \in m$. Obviously, ϕ is a linear, one-to-one map of A^* onto $C^*(Z; (Z - E)\bigcup Y)$. It is positive.

Indeed, let m be positive and so n is positive. We must show that $(n - w(n))(B) \geq 0$ for each Borel $B \subseteq Z$. This is trivial for $B \subseteq (Z - E) \cup Y$ so assume $B \subseteq E - Y$. But then (1.4)$'$ yields

$$w(n)(B) = -\int_{Z-E} \pi_z(B)\, dn - \pi_0(B)\lambda(n)(Y) \tag{1.8}$$

and so $n - w(n)$ is indeed positive.

Last, we need show that ϕ is an isometry to complete the proof. Let $n \in m \in A^*$. Then

$$\|\phi(m)\| = |\phi(m)|(E - Y) = |n - w(n)|(E - Y) \leq |n|(E - Y) + |w(n)|(E - Y).$$

For any Borel $B \subseteq E - Y$, using (1.7) and (1.8), we get

$$\begin{aligned} w(n)(B) &= -\int_{Z-E} \pi_z(B)\, dn - \pi_0(B)\left(\int_{Z-E} \pi_z(Y)\, dn + n(Y)\right)\Big/(1 - \alpha) \\ &= -\int_{Z-E} \left(\pi_z(B) + \pi_z(Y)\frac{\pi_0(B)}{1-\alpha}\right) dn - n(Y)\, \frac{\pi_0(B)}{1-\alpha}. \end{aligned}$$

Then

$$\begin{aligned} |w(n)|(E - Y) &\leq \int_{Z-E} \left(\pi_z(E - Y) + \pi_z(Y)\frac{\pi_0(E-Y)}{1-\alpha}\right) d|n| + |n|(Y)\,\frac{\pi_0(E-Y)}{1-\alpha} \\ &\leq \int_{Z-E} (\pi_z(E - Y) + \pi_z(Y))\, d|n| + |n|(Y) \quad \text{by (1.1)} \\ &\leq |n|(Z - E) + |n|(Y). \end{aligned}$$

Thus, we finally get $\|\phi(m)\| \leq |n|(E - Y) + |n|(Z - E) + |n|(Y) = |n|(Z) = \|n\|$. Since $\|m\| = \inf_{n \in m} \|n\|$, we have $\|m\| = \|\phi(m)\|$.

The same proof also establishes the following [cf. 7, Theorem 1.2].

Corollary 1.5. *Let V, Z, E be as in Theorem 1.4. Let X be a closed proper subset of $Z - E$. Let $x \longrightarrow \mu_x$ be a weak* continuous map of X into $P_1(C(Z))$. Let $X = X_1 \cup X_2$ where $X_2 = \{x \in X \mid \mu_x = \delta(x)\}$. We assume*

1. *For each $x \in X \cap (Z - E)$, $\mu_x = \pi_x$.*
2. *For each $x \in X_1$, $\mu_x(X_1 \cup (Z - E)) = 0$.*
3. *$X_1 \cup (Z - E) \neq Z$.*

Then $A = \{f \in V \mid f(x) = \mu_x(f)$ for each $x \in X\}$ is a nontrivial simplex space with the relative norm and order. Further, A^* is isometrically order isomorphic to $C^*(Z; X_1 \cup (Z - E))$.

2. **The characterization.** A subset F of a convex compact set K is called a *face* if it is convex and satisfies the following condition: if $\alpha x + (1 - \alpha)y \in F$ with $x, y \in K$ and $0 < \alpha < 1$, then $x, y \in F$. The following extension theorem is a well-known consequence of the Edwards separation theorem [4].

Lemma 2.1. *Let F be a closed face of a simplex K. Suppose f_1, f_2 are continuous* *affine functions on K. Let* $\bar{g}$ *be a continuous affine function on F satisfying* $\alpha \geq \bar{g} \geq f_1, f_2$ *for some* $\alpha \in \mathbf{R}$. *Then there exists a continuous affine extension* g *of* $\bar{g}$ *to K which satisfies* $\alpha \geq g \geq f_1, f_2$.

Lemma 2.2. *Let V be a simplex space. Let* $q \in Z(V) - EP_1(V)$. *Suppose there exists* $p \in \operatorname{supp} \pi_q \cap EP_1(V)^+$ *and a net* $\{x_\beta\} \subseteq EP_1(V)^+ - \{p\}$ *which converges weak* to q. Suppose, further, there is an element* $f \in V$ *such that*

1. $x_\beta(f) = 0 = q(f)$ *for all* β.
2. $p(f) > 0$.

Then there exists a collection of closed maximal ideals I_β *such that* $\bigcap I_\beta$ *is not an ideal. If the net is a sequence, then the collection of ideals is countable.*

Proof. Let $F_\beta = \{\alpha x_\beta \mid 0 \leq \alpha \leq 1\}$. Then F_β is a maximal face containing zero of $P_1(V)$. Let I_β be the annihilator of F_β within V. Then I_β is a closed maximal ideal [5, Corollary 3.2]. Since $x_\beta(f) = 0$, $f \in I_\beta$ and so $f \in \bigcap I_\beta$.

Suppose $\bigcap I_\beta$ is an ideal. Then there would be a $v \in V^+$, $v \in \bigcap I_\beta$, and $v \geq f$. Thus, $v \in I_\beta$ for each β and so $x_\beta(v) = 0$. Therefore $q(v) = \lim x_\beta(v) = 0$. Since $v \in V^+$, it is zero on the smallest closed face containing q and so $p(v) = 0$. However, this contradicts the assumptions that $p(f) > 0$ and $v \geq f$.

We are now prepared for our characterization.

Theorem 2.3. *Let V be a separable simplex space. Then the following are equivalent*:

1. *V is an M-space.*
2. *The intersection of an arbitrary collection of closed (maximal) ideals is always an ideal.*
3. *The intersection of a countable collection of closed (maximal) ideals is always an ideal.*

Proof. (1) ⟹ (2) ⟹ (3) are obvious.

Not (1) ⟹ not (3). Assume V is not an M-space. Let $Z = Z(V)$ and $E = EP_1(V)^+$. Then there is an element $q \in Z - E$ such that $\operatorname{supp} \pi_q$ has at least two points ([10, Theorem 2], [6, Corollary 2.6]). So choose distinct points p_1, $p_2 \in E \cap \operatorname{supp} \pi_q$. Since q is in the closure of E, there is a sequence $\{x_n\} \subseteq E - \{p_1, p_2\}$ such that $\lim x_n = q$.

Let A be defined by $A = \{v \in V \mid x_n(v) = q(v) = \pi_q(v),\ n = 1, 2, \cdots\}$. Then A is a nontrivial simplex space with $EP_1(A)^+ = E - \{x_n \mid n = 1, 2, \cdots\}$ by Theorem 1.4. Hence, p_1 and p_2 are in $EP_1(A)^+$. Let $F = \{\alpha p_1 + \beta p_2 \mid 0 \leq \alpha, 0 \leq \beta, \alpha + \beta \leq 1\}$. Then F is a closed face of the simplex $P_1(A)$. On F, define continuous affine functions $\bar{g}_1$ and $\bar{g}_2$ by

$$\bar{g}_1(\alpha p_1 + \beta p_2) = \alpha, \qquad g_2(\alpha p_1 + \beta p_2) = \beta.$$

Then $\bar{g}_i \geq 0$, $i = 1, 2$, on F and so there exist elements $g_1, g_2 \in A$ such that $g_i \geq 0$ and $g_i|F = \bar{g}_i$ by Lemma 2.1. Since $p_1(g_1) = 1$ and $p_2(g_2) = 1$, we must have $q(g_1) > 0$ and $q(g_2) > 0$. Let

$$f = g_1 - (q(g_1)/q(g_2))g_2.$$

Then obviously $f \in A \subseteq V$, $q(f) = 0$ and $p_1(f) > 0$. Since $f \in A$, we also have $x_n(f) = 0$, $n = 1, 2, \cdots$. Hence, Lemma 2.2 implies that (3) is not true.

We note that Theorem 1.4 and Lemma 2.2 allow us to conclude more than just Theorem 2.3. Let V be a standard simplex space and suppose $q \in Z(V) - EP_1(V)^+$ does not lie in the rays of $P_1(V)$. Suppose we could find a net $\{x_\beta\} \subseteq EP_1(V)^+$ converging to q such that $Y = \{x_\beta\}^-$ satisfies the hypotheses of Theorem 1.4. We could then conclude the existence of a collection of closed ideals whose intersection is not an ideal.

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