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THE STRUCTURE OF PSEUDOCOMPLEMENTED DISTRIBUTIVE LATTICES. III: INJECTIVE AND ABSOLUTE SUBRETRACTS

BY

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ABSTRACT. Absolute subretracts are characterized in the classes $\mathfrak{B}_n$, $n \leq \omega$. This is applied to describe the injectives in $\mathfrak{B}_1$ (due to R. Balbes and G. Grätzer) and $\mathfrak{B}_2$.

1. **Introduction.** In Parts I and II ([4] and [6]) we have acquired a rather thorough knowledge of the structure of pseudocomplemented distributive lattices. In this paper we use this knowledge to extend the results of R. Balbes and G. Grätzer [1] on injective Stone algebras to any $\mathfrak{B}_n$. (Recall that $\mathfrak{B}_1$ is the class of Stone algebras; for the notation, see §2.) It turns out, however, that there are rather few injectives in $\mathfrak{B}_n$. It appears that weak injectives and absolute subretracts are more appropriate to investigate in general. In $\mathfrak{B}_1$ they coincide with injectives.

Accordingly, our main result is a description of weak injectives as presented in Theorem 1. An explicit construction (unique up to isomorphism) is given in (c) of Theorem 1 (corresponding to Theorem 2 of [1]) and an internal description in (d) of Theorem 1 (corresponding to Theorem 1 of [1]).

It turns out that weak injectives are injectives in $\mathfrak{B}_1$ and $\mathfrak{B}_2$; thus our result implies the results of [1], in fact, it yields a somewhat sharper form of Theorem 1 of [1]. A description of injectives in $\mathfrak{B}_2$ is given in Theorem 3.

In §2 we introduce weak injectives, absolute subretracts, and investigate their interrelationships with injectives. These observations are applied in §3, where a series of lemmas are given leading up to Theorem 1. The applications are given in §4 including a proof of a result of R. A. Day [2]. In the last section we describe a first order property Φ_n shared by the weak injectives in $\mathfrak{B}_n$ such that any first order property of weak injectives follows from Φ_n.

2. **General algebraic preliminaries.** We first recall some notations and results of Part II [4]. The equational classes of pseudocomplemented distributive lattices

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are the trivial class $\mathcal{B}_{-1}$, the classes $\mathcal{B}_n$, $n \geq 0$, and the class $\mathcal{B}_\omega$. Each $\mathcal{B}_n$, $n \geq 0$, is generated by the subdirectly irreducible pseudocomplemented distributive lattice $\bar{B}_n$, where B_n is the n-atom Boolean lattice 2^n and $\bar{B}_n$ is B_n with a new unit element. We recall that all these equational classes satisfy the Congruence Extension Property and that $\mathcal{B}_n$ satisfies the Amalgamation Property if and only if $n = -1, 0, 1, 2$, or ω.

If $\mathcal{K}$ is any class of algebras an algebra $C \in \mathcal{K}$ is said to be *injective* if, given any algebras $A, B \in \mathcal{K}$, A a subalgebra of B, and given a homomorphism $\phi: A \to C$, there is a homomorphism $\bar{\phi}: B \to C$ extending ϕ. The class $\mathcal{K}$ *has enough injectives* if each algebra in $\mathcal{K}$ is a subalgebra of an algebra injective in $\mathcal{K}$. An algebra $C \in \mathcal{K}$ is a *weak injective* if, given algebras $A, B \in \mathcal{K}$, A a subalgebra of B, and given a *surjective* (i.e. onto) homomorphism $\phi: A \to C$, there is a homomorphism $\bar{\phi}: B \to C$ extending ϕ. An algebra $C \in \mathcal{K}$ is an *absolute subretract* if C is a retract of each of its extensions in $\mathcal{K}$.

We present the various relations among these concepts. The class $\mathcal{K}$ will be assumed to be an equational class. We first have a well-known lemma.

Lemma 1. *If C is an injective in $\mathcal{K}$, then C is a weak injective. If C is a weak injective in $\mathcal{K}$, then C is an absolute subretract.*

As converses to Lemma 1, we have the following results.

Lemma 2. *Let $\mathcal{K}$ satisfy the Congruence Extension Property. Then each absolute subretract in $\mathcal{K}$ is a weak injective.*

Proof. Let C be an absolute subretract in $\mathcal{K}$, let $A, B \in \mathcal{K}$, A a subalgebra of B, and let $\phi: A \to C$ be a homomorphism onto C. By the Congruence Extension Property there is an extension C' of C in $\mathcal{K}$ and a homomorphism $\psi: B \to C'$ such that $\psi|A = \phi$. Since C is an absolute subretract there is a retraction $\rho: C' \to C$. Thus $\psi\rho$ is the required extension of ϕ, proving the lemma.

Lemma 3. *If $\mathcal{K}$ satisfies the Congruence Extension Property and the Amalgamation Property, then any absolute subretract in $\mathcal{K}$ is injective.*

Proof. Let C be an absolute subretract in $\mathcal{K}$. Let $A, B \in \mathcal{K}$ and let $\alpha: A \to B$ be an embedding. Let $\phi: A \to C$, let $C_1 = \operatorname{Im} \phi$, and let $\beta: C_1 \to C$ be the embedding. Let $\phi_1: A \to C_1$ be defined by ϕ; that is, $\phi_1\beta = \phi$. By the Congruence Extension Property there is an algebra $B_1 \in \mathcal{K}$, an embedding $\gamma: C_1 \to B_1$, and a homomorphism $\psi: B \to B_1$ such that $\alpha\psi = \phi_1\gamma$ (see Figure 1).

Since β and γ are embeddings and $\mathcal{K}$ has the Amalgamation Property there is an algebra D in $\mathcal{K}$ and embeddings $\lambda: B_1 \to D$, $\mu: C \to D$ such that $\gamma\lambda = \beta\mu$. Since C is an absolute subretract there is a homomorphism $\rho: D \to C$ such that $\mu\rho$ is the identity mapping on C. Let $\bar{\phi} = \psi\lambda\rho$; then $\alpha\bar{\phi} = \phi_1\beta = \phi$. Thus C is injective, concluding the proof of the lemma.

$$\begin{tikzcd} B \arrow[r, "\psi"] & B_1 \arrow[r, "\lambda"] & D \\ A \arrow[u, "\alpha"] \arrow[r, "\phi_1"] & C_1 \arrow[u, "\gamma"] \arrow[r, "\beta"] & C \arrow[u, "\mu"'] \end{tikzcd}$$

Figure 1

We now relate the concepts of absolute subretract and subdirect irreducibility. If A, B are algebras, A a subalgebra of B, then B is said to be an *essential extension* of A if, given any congruence Θ on B such that $\Theta_A = \omega_A$ then $\Theta = \omega_B$. (Θ_A denotes the restriction of Θ to A. If A is any algebra, ω_A denotes the congruence: $x \equiv y(\omega_A)$ if and only if $x = y$.) A *maximal subdirectly irreducible algebra* is a subdirectly irreducible algebra every proper extension of which is not subdirectly irreducible.

Lemma 4. *Any maximal subdirectly irreducible algebra A in $\mathcal{K}$ is an absolute subretract in $\mathcal{K}$.*

Proof. Let A be a subalgebra of B in $\mathcal{K}$. By Zorn's Lemma (see Lemma 3(b) of [4]) there is an essential extension B_1 of A and a surjective homomorphism $\phi: B \to B_1$ such that $x\phi = x$ for all $x \in A$. Since A is subdirectly irreducible and B_1 is an essential extension of A it follows that B_1 is subdirectly irreducible (see [2]). Thus, by the maximality of A, $B_1 = A$ and so ϕ is a retraction onto A. Thus we have shown that A is an absolute subretract.

It is well known that a retract of an injective algebra is itself injective. A weaker result holds for weak injectives. Let $(A_i | i \in I)$ be a family of algebras and let $\phi: A \to \Pi(A_i | i \in I)$ be an embedding of A as a subdirect product. If ϕ also embeds A as a retract of $\Pi(A_i | i \in I)$ we say that A is a *subdirect retract* of the family $(A_i | i \in I)$.

Lemma 5. *A subdirect retract of a family of weak injective algebras is itself a weak injective algebra.*

Proof. Let $(C_i | i \in I)$ be a family of weak injectives and let $\gamma: C \to \Pi(C_i | i \in I)$ be a representation of C as a sudirect retract; let $\rho: \Pi(C_i | i \in I) \to C$ be the retraction, that is, let $\gamma\rho = 1_C$.

Let A, B be algebras, let $\alpha: A \to B$ be an embedding, and let $\phi: A \to C$ be a surjective homomorphism. If $i \in I$ and π_i is the projection of $\Pi(C_i | i \in I)$ onto C_i, then $\gamma\pi_i$ is surjective. Thus $\phi_i: A \to C_i$ defined by $\phi_i = \phi\gamma\pi_i$ for each $i \in I$ is a surjection. Since each C_i is a weak injective, there is a homomorphism $\psi_i: B \to C_i$ such that $\alpha\psi_i = \phi_i$. Consequently there is a homomorphism $\psi: B \to$

$\Pi(C_i | i \in I)$ such that $\psi\pi_i = \psi_i$ for each $i \in I$; it follows that $\alpha\psi = \phi\gamma$. Thus if $\bar{\phi} = \psi\rho$ we find that $\alpha\bar{\phi} = \phi$, showing that C is a weak injective.

3. **Absolute subretracts in $\mathfrak{B}_n$.** Let B be a Boolean algebra and let

$$B^{[n+1]} = \{\langle x_0, x_1, \cdots, x_n\rangle \in B^{n+1} | x_0 \leq x_1 \wedge \cdots \wedge x_n\}.$$

$B^{[2]}$ was introduced in [1], and $B^{[n+1]}$ is the obvious generalization of $B^{[2]}$. It follows quite easily that $B^{[n+1]}$ is a pseudocomplemented distributive lattice and that

$$\langle x_0, x_1, \cdots, x_n\rangle^* = \langle x_1' \wedge \cdots \wedge x_n', x_1', \cdots, x_n'\rangle.$$

It is also clear that $\bar{B}_n \cong 2^{[n+1]}$, that if B' is a subalgebra of the Boolean algebra B then $(B')^{[n+1]}$ is a subalgebra of $B^{[n+1]}$, and that if $B = \Pi(B_\gamma | \gamma \in \Gamma)$ (B, B_γ, $\gamma \in \Gamma$, are Boolean algebras), then $B^{[n+1]} \cong \Pi(B_\gamma^{[n+1]} | \gamma \in \Gamma)$; thus $B^{[n+1]} \in \mathfrak{B}_n$ for any Boolean algebra B. In this section we show that the absolute subretracts in $\mathfrak{B}_n$ are precisely the pseudocomplemented distributive lattices of the form $B' \times B^{[n+1]}$, where B and B' are complete Boolean algebras.

Lemma 6. *Any complete Boolean algebra is an injective pseudocomplemented distributive lattice.*

Proof. Let B be a complete Boolean algebra, let C be a pseudocomplemented distributive lattice, and let A be a subalgebra (*-sublattice) of C. Let $\phi: A \to B$ be a *-homomorphism. Let f be the restriction of ϕ to(2) $S(A)$. Since $S(A)$ is a subalgebra of the Boolean algebra $S(C)$ and since B is an injective Boolean algebra, f lifts to a homomorphism of the Boolean algebras $\bar{f}: S(C) \to B$. The mapping $\bar{\phi}: C \to B$ given by $x\bar{\phi} = (x^{**})\bar{f}$ is a homomorphism that is the required lifting of ϕ. Thus B is an injective pseudocomplemented distributive lattice.

Lemma 7. *Let B be a complete Boolean algebra. Then $B^{[n+1]}$ is a weak injective in $\mathfrak{B}_n$.*

Proof. Since B is an injective Boolean algebra and since the equational class of Boolean algebras is generated by the two-element Boolean algebra $\mathbf{2}$, it follows that B is a subdirect retract of a family $(B_\gamma | \gamma \in \Gamma)$ of Boolean algebras all isomorphic to $\mathbf{2}$. We claim that consequently $B^{[n+1]}$ is a subdirect retract of the family $(B_\gamma^{[n+1]} | \gamma \in \Gamma)$. If $\phi: B \to B'$ is a homomorphism of Boolean algebras, then $\phi^{[n+1]}: B^{[n+1]} \to (B')^{[n+1]}$, given by $\langle x_0, \cdots, x_n\rangle \phi^{[n+1]} = \langle x_0\phi, \cdots, x_n\phi\rangle$, is a *-homomorphism. If $\pi: B \to B_\gamma$ is onto then so is $\pi^{[n+1]}: B^{[n+1]} \to (B_\gamma)^{[n+1]}$; thus $B^{[n+1]}$ is a sudirect product of $(B_\gamma^{[n+1]} | \gamma \in \Gamma)$. If ρ: $\Pi(B_\gamma | \gamma \in \Gamma) \to B$ is a retraction of Boolean algebras, then

(2) Recall that $S(A) = \{x^{**} | x \in A\}$, the *skeleton* of A, is a Boolean algebra.

$$\rho^{[n+1]}: \left(\prod(B_\gamma | \gamma \in \Gamma)\right)^{[n+1]} \cong \prod((B_\gamma)^{[n+1]} | \gamma \in \Gamma) \to B^{[n+1]}$$

is a retraction of pseudocomplemented distributive lattices; thus $B^{[n+1]}$ is a subdirect retract of the family $(B_\gamma^{[n+1]} | \gamma \in \Gamma)$. Since $B_\gamma^{[n+1]} \cong 2^{[n+1]} \cong \bar{B}_n$ we conclude, by Lemmas 2, 4, 5, that $B^{[n+1]}$ is a weak injective in $\mathfrak{B}_n$, proving the lemma.

It follows from Lemmas 6 and 7 that if B and B' are complete Boolean algebras, then $B' \times B^{[n+1]}$ is a weak injective (equivalently, by Lemmas 1 and 2, an absolute subretract) pseudocomplemented distributive lattice in $\mathfrak{B}_n$. We now characterize pseudocomplemented distributive lattices of the form $B' \times B^{[n+1]}$, B, B' Boolean. Let A be a pseudocomplemented distributive lattice. Recall that the set of *dense* elements of A, $D(A) = \{x \in A | x^* = 0\}$, is a dual ideal of A. First we present a result characterizing $\mathfrak{B}_n$.

Lemma 8. *Let L be a pseudocomplemented distributive lattice. Then $L \in \mathfrak{B}_n$ if and only if L has the following property: let $x_0, \cdots, x_n \in L$ satisfy $x_i \wedge x_j = 0$ whenever $i \neq j$; then $x_0^* \vee \cdots \vee x_n^* = 1$.*

Proof. K. B. Lee [7] gave an identity characterizing $\mathfrak{B}_n$, to wit

$$(a_1 \wedge \cdots \wedge a_n)^* \vee (a_1^* \wedge a_2 \wedge \cdots \wedge a_n)^* \vee \cdots \vee (a_1 \wedge \cdots \wedge a_{n-1} \wedge a_n^*)^* = 1.$$

Let $x_i \wedge x_j = 0$ if $i \neq j$, $i, j = 0, \cdots, n$. Then $x_i \leq x_j^*$ whenever $i \neq j$. Hence $x_0 \leq x_1^* \wedge \cdots \wedge x_n^*$, $x_1 \leq x_1^{**} \wedge x_2^* \wedge \cdots \wedge x_n^*, \cdots, x_n \leq x_1^* \wedge x_2^* \wedge \cdots \wedge x_n^{**}$. Applying Lee's identity with $a_i = x_i^*$, $i = 1, \cdots, n$, we conclude that

$$x_0^* \vee \cdots \vee x_n^* \geq (x_1^* \wedge \cdots \wedge x_n^*)^* \vee (x_1^{**} \wedge \cdots \wedge x_n^*)^* \vee \cdots \vee (x_1^* \wedge \cdots \wedge x_n^{**})^* = 1.$$

Thus the required property holds in $\mathfrak{B}_n$.

Now let L be a pseudocomplemented distributive lattice such that $x_i \wedge x_j = 0$, $i \neq j$, $i, j = 0, \cdots, n$, implies that $x_0^* \vee \cdots \vee x_n^* = 1$. Let $a_1, \cdots, a_n \in L$ and let $x_0 = a_1 \wedge \cdots \wedge a_n$, $x_1 = a_1^* \wedge a_2 \wedge \cdots \wedge a_n, \cdots, x_n = a_1 \wedge \cdots \wedge a_{n-1} \wedge a_n^*$. Then $x_i \wedge x_j = 0$ if $i \neq j$. Thus $x_0^* \vee \cdots \vee x_n^* = 1$, that is, Lee's identity holds for $a_1, \cdots, a_n$. Thus $L \in \mathfrak{B}_n$.

If L is a lattice with 0, 1 and $x \in L$, a *dual pseudocomplement* of x, denoted x^+, is the obvious dual of a pseudocomplement; $x \vee y = 1$ if and only if $y \geq x^+$.

Lemma 9. *Let L be a pseudocomplemented distributive lattice and let $n \geq 1$. The following two conditions are equivalent:*

(a) *There are Boolean algebras B and B' such that $L \cong B' \times B^{[n+1]}$.*

(b) *L has the following five properties:*

(i) *$L \in \mathfrak{B}_n$;*

(ii) *$D(L)$ has a smallest element d;*

(iii) *d has a dual pseudocomplement which is central (complemented) in L;*

(iv) *there are elements $e_1, \cdots, e_n \in L$ such that $e_i \wedge e_j = 0$ if $i \neq j$, $e_i \vee e_i^* = d$ for all i, and $e_1^* \wedge \cdots \wedge e_n^* = 0$;*

(v) *given $u_1, \cdots, u_n \in D(L)$ there is an $x \in L$ such that $(x \wedge e_i)^* \vee d = u_i$ for $i = 1, \cdots, n$.*

Proof. Let $L = B' \times B^{[n+1]}$. Then $L \in \mathfrak{B}_n$. The smallest dense element in L is $d = \langle 1, \langle 0, 1, \cdots 1\rangle\rangle$ and $d^+ = \langle 0, \langle 1, 1, \cdots, 1\rangle\rangle$ whose complement in L is $\langle 1, \langle 0, 0, \cdots, 0\rangle\rangle$. For each $i = 1, \cdots, n$, let $e_i = \langle a_{1i}, \langle 0, a_{1i}, \cdots, a_{ni}\rangle\rangle$ where $a_{ij} = 0$ if $i \neq j$ and $a_{ii} = 1$. Then (iv) of (b) holds. To establish (v) note that

$$D(L) = \{\langle 1, \langle x, 1, \cdots, 1\rangle\rangle \in B' \times B^{[n+1]}\}.$$

Let $u_i = \langle 1, \langle x_i, 1, \cdots, 1\rangle\rangle$, $i = 1, \cdots, n$. Then $x = \langle 0, \langle 0, x_1', \cdots, x_n'\rangle\rangle$.

Now let (b) hold. We show that $L \cong B' \times B^{[n+1]}$ where $B' \cong (d^{++}]$ (d^{++} exists, and equals d^{+*}, because d^+ is central) and $B \cong D(L)$.

We first show that $(d^{++}]$ is a Boolean lattice; we claim, for each $x \in L$, that $x^* \wedge d^{++}$ is the complement in $(d^{++}]$ of $x \wedge d^{++}$. Clearly $(x \wedge d^{++}) \wedge (x^* \wedge d^{++}) = x \wedge x^* \wedge d^{++} = 0$. Now $(x \wedge d^{++}) \vee (x^* \wedge d^{++}) = (x \vee x^*) \wedge d^{++}$; since $x \vee x^* \in D(L)$, $x \vee x^* \geq d \geq d^{++}$ and consequently $(x \wedge d^{++}) \vee (x^* \wedge d^{++}) = d^{++}$, establishing our claim.

That $D(L)$ is Boolean will emerge during our proof of the representation.

For each $i = 1, \cdots, n$, define $f_i: L \to D(L)$ by requiring that $xf_i = (x^* \wedge e_i)^* \vee d$. The f_i will correspond to the projections $B' \times B^{[n+1]} \to B$. We claim that each f_i is a homomorphism. First note that, for all $x, y \in L$,

$$(x \wedge y)^* = (x^{**} \wedge y)^* = (x^{**} \wedge y^{**})^*.$$

Indeed

$$(x \wedge y)^* = (x \wedge y)^{***} = (x^{**} \wedge y^{**})^*,$$

$$x \wedge y \leq x^{**} \wedge y \leq x^{**} \wedge y^{**};$$

thus

$$(x \wedge y)^* \geq (x^{**} \wedge y)^* \geq (x^{**} \wedge y^{**})^*,$$

establishing the triple equality.

We can now show that f_i preserves $\wedge$. Recall that $\cup$ denotes the join operation in $S(L)$, that is, $a \cup b = (a^* \wedge b^*)^*$. Now if $x, y \in L$ then

$$\begin{aligned}(x \wedge y)f_i &= ((x \wedge y)^* \wedge e_i)^* \vee d = ((x \wedge y)^* \wedge e_i^{**})^* \vee d \\ &= ((x \wedge y)^{**} \cup e_i^*) \vee d = ((x^{**} \wedge y^{**}) \cup e_i^*) \vee d \\ &= ((x^{**} \cup e_i^*) \wedge (y^{**} \cup e_i^*)) \vee d = ((x^{**} \cup e_i^*) \vee d) \wedge ((y^{**} \cup e_i^*) \vee d) \\ &= ((x^* \wedge e_i)^* \vee d) \wedge ((y^* \wedge e_i)^* \vee d) = xf_i \wedge yf_i.\end{aligned}$$

Clearly f_i preserves 0 and 1 (the "0" of $D(L)$ is d), since $e_i^* \leq d$. Before showing that f_i preserves $\vee$ we observe that xf_i and x^*f_i are complementary for each $x \in L$. Indeed

$$xf_i \wedge x^*f_i = 0f_i = d$$

and

$$\begin{aligned}xf_i \vee x^*f_i &= (x^* \wedge e_i)^* \vee (x^{**} \wedge e_i)^* \vee d \\ &\geq (x^* \wedge e_i)^* \vee (x^{**} \wedge e_i)^* \vee e_1^* \vee \cdots \vee e_{i-1}^* \vee e_{i+1}^* \vee \cdots \vee e_n^* = 1,\end{aligned}$$

by (iv) of (b) and (the last equality) by Lemma 8. Since, by (v) of (b), f_i is onto (recall $(x \wedge e_i)^* = (x^{**} \wedge e_i)^*$) we conclude that $D(L)$ is Boolean; denote the complement of u in $D(L)$ by u'.

We now show that f_i preserves $\vee$. Note that $xf_i = x^{**}f_i$ since $x^*f_i = (xf_i)'$. Thus $(x \vee y)f_i = (x \vee y)^{**}f_i = (x^* \wedge y^*)^*f_i = ((x^* \wedge y^*)f_i)' = (x^*f_i \wedge y^*f_i)' = ((xf_i)' \wedge (yf_i)')' = xf_i \vee yf_i$.

Now let $B' = (d^{++}]$ and let $B = D(L)$. We show that $L \cong B' \times B^{[n+1]}$. Define $\phi: L \to B' \times B^{[n+1]}$ by setting

$$x\phi = \langle x \wedge d^{++}, \langle x \vee d, xf_1, \cdots, xf_n\rangle\rangle.$$

Note that $\langle x \vee d, xf_1, \cdots, xf_n\rangle \in B^{[n+1]}$ since $x \vee d \leq x^{**} \vee d \leq xf_i$. Since the f_i are lattice homomorphisms so is ϕ. Also, $0\phi = \langle 0, \langle d, d, \cdots, d\rangle\rangle$ and $1\phi = \langle d^{++}, \langle 1, 1, \cdots, 1\rangle\rangle$; thus ϕ preserves 0, 1. To show that ϕ preserves pseudocomplements observe that

$$\begin{aligned}x^*f_1 \wedge \cdots \wedge x^*f_n &= ((x^{**} \wedge e_1)^* \wedge \cdots \wedge (x^{**} \wedge e_n)^*) \vee d \\ &= ((x^* \cup e_1^*) \wedge \cdots \wedge (x^* \cup e_n^*)) \vee d = (x^* \cup (e_1^* \wedge \cdots \wedge e_n^*)) \vee d \\ &= (x^* \cup 0) \vee d = x^* \vee d.\end{aligned}$$

Thus

$$\begin{aligned}x^*\phi &= \langle x^* \wedge d^{++}, \langle x^* \vee d, x^*f_1, \cdots, x^*f_n\rangle\rangle \\ &= \langle x^* \wedge d^{++}, \langle x^*f_1 \wedge \cdots \wedge x^*f_n, x^*f_1, \cdots, x^*f_n\rangle\rangle \\ &= \langle x^* \wedge d^{++}, \langle (xf_1)' \wedge \cdots \wedge (xf_n)', (xf_1)', \cdots, (xf_n)'\rangle\rangle\end{aligned}$$

and, since we have shown above that $x^* \wedge d^{++}$ is the complement in $(d^{++}]$ of $x \wedge d^{++}$, we conclude that $x^*\phi = (x\phi)^*$.

We now show that ϕ is an isomorphism. It suffices to show that $\phi_{D(L)}$: $D(L) \to D(B' \times B^{[n+1]})$ and $\phi_{S(L)}$: $S(L) \to S(B' \times B^{[n+1]})$ are isomorphisms. Observe that

$$D(B' \times B^{[n+1]}) = \{\langle d^{++}, \langle u, 1, \cdots, 1\rangle\rangle \mid u \in D(L)\}$$

and that $u \in D(L)$ implies $u\phi = \langle d^{++}, \langle u, 1, \cdots, 1\rangle\rangle$. Thus $\phi_{D(L)}$ is an isomorphism.

We now show that $\phi_{S(L)}$ is an isomorphism. Let $x \in S(L)$ and let $x\phi = 1$, that is, $x\phi = \langle d^{++}, \langle 1, \cdots, 1\rangle\rangle$. Thus $x \wedge d^{++} = d^{++}$, that is, $x \geq d^{++}$, and $(x^* \wedge e_i)^* \vee d = 1$ for $i = 1, \cdots, n$. Consequently $(x^* \wedge e_i)^* \geq d^+$ for each i; thus $x \cup e_i^* = (x^* \wedge e_i^{**})^* = (x^* \wedge e_i)^* \geq d^+$ and, since $e_1^* \wedge \cdots \wedge e_n^* = 0$, we conclude that $x = (x \cup e_1^*) \wedge \cdots \wedge (x \cup e_n^*) \geq d^+$. Thus $x \geq d^+ \vee d^{++} = 1$, showing that $\phi_{S(L)}$ is one-to-one.

To complete the proof of the lemma we need only show that $\phi_{S(L)}$ is onto. First observe that $d^+ \leq (d^{++} \wedge e_i)^*$ for $i = 1, \cdots, n$, since $d^+ \wedge d^{++} \wedge e_i = 0$; thus $d^+f_i = (d^{++} \wedge e_i)^* \vee d \geq d^+ \vee d = 1$, and so $d^{++}f_i = d$. Observe also that if $y \in (d^{++}]$ then $y \in S(L)$; indeed, $y^{**} \leq (d^{++})^{**} = d^{++} \leq d \leq y \vee y^*$ and, since $y = y^{**} \wedge (y \vee y^*)$, thus $y = y^{**}$. Each element of $S(B' \times B^{[n+1]})$ is of the form $\langle y, \langle u_1 \wedge \cdots \wedge u_n, u_1, \cdots, u_n\rangle\rangle$ where $y \in (d^{++}]$ and $u_1, \cdots, u_n \in D(L)$. By (v) of condition (b) there is an $x \in L$ such that $u_i = (x \wedge e_i)^* \vee d = (x^{**} \wedge e_i)^* \vee d$, that is, such that $x^*f_i = u_i$ for $i = 1, \cdots, n$. Since $d^+f_i = 1$ we conclude that $(x^* \wedge d^+)f_i = u_i$ for $i = 1, \cdots, n$. Clearly $x^* \wedge d^+ \in S(L)$; thus, since $(x^* \wedge d^+)f_i = u_i$ and $x^* \wedge d^+ \wedge d^{++} = 0$, $(x^* \wedge d^+)\phi = \langle 0, \langle u_1 \wedge \cdots \wedge u_n, u_1, \cdots, u_n\rangle\rangle$. Since $y \leq d^{++}$ and $d^{++}f_i = d$ it follows that $y\phi = \langle y, \langle d, d, \cdots, d\rangle\rangle$. Thus $(y \cup (x^* \wedge d^+))\phi = \langle y, \langle u_1 \wedge \cdots \wedge u_n, u_1, \cdots, u_n\rangle\rangle$, showing that $\phi_{S(L)}$ is onto, and thus an isomorphism.

Consequently ϕ: $L \to B' \times B^{[n+1]}$ is an isomorphism, concluding the proof of the lemma.

We are now in a position to prove the main theorem of this paper.

Theorem 1. *Let $n \geq 1$ be an integer and let L be a pseudocomplemented distributive lattice. The following four conditions are equivalent.*

(a) *L is an absolute subretract in $\mathfrak{B}_n$;*

(b) *L is a weak injective in $\mathfrak{B}_n$;*

(c) *there are complete Boolean algebras B and B' such that $L \cong B' \times B^{[n+1]}$;*

(d) *L is complete and has the following five properties:*

(i) *$L \in \mathfrak{B}_n$;*

(ii) *$D(L)$ has a smallest element d;*

(iii) *d has a dual pseudocomplement which is central in L;*

(iv) *there are elements $e_1, \cdots, e_n \in L$ such that $e_i \wedge e_j = 0$ if $i \neq j$, $e_i \vee e_i^* = d$ for all i, and $e_1^* \wedge \cdots \wedge e_n^* = 0$;*

(v) *given $u_1, \cdots, u_n \in D(L)$ there is an $x \in L$ such that $(x \wedge e_i)^* \vee d = u_i$ for $i = 1, \cdots, n$.*

Proof. Since $\mathfrak{B}_n$ satisfies the Congruence Extension Property (a) and (b) are equivalent by Lemmas 1 and 2. Note that if L is complete so are all principal ideals and principal dual ideals in L, and that if B' and B are complete so is $B' \times B^{[n+1]}$. Thus, by Lemma 8, conditions (c) and (d) are equivalent. By Lemmas 6 and 7, condition (c) implies (b). To complete the proof of the theorem we need only show that condition (a) implies (d). Let $L \in \mathfrak{B}_n$ be an absolute subretract. Since the subdirectly irreducible members of $\mathfrak{B}_n$ are subalgebras of $\overline{B}_n \cong 2^{[n+1]}$ it follows that L is a subalgebra, and thus a retract (preserving $*$) of a power of $2^{[n+1]}$. Each power of $2^{[n+1]}$ is of the form $B^{[n+1]}$, where B is a complete atomic Boolean lattice. Since $B^{[n+1]}$ is complete and satisfies (i) to (v), and since completeness and properties (i) to (v) are preserved under retraction of pseudocomplemented distributive lattices, it follows that L satisfies condition (d). We have thus concluded the proof of the theorem.

4. **Injective pseudocomplemented distributive lattices.** As was shown in Part II [4], $\mathfrak{B}_1$, the equational class of *Stone algebras*, and $\mathfrak{B}_2$ both satisfy the Congruence Extension Property and the Amalgamation Property. By Lemma 3, absolute subretracts and injectives agree in these classes; thus Theorem 1 specializes to the following two theorems.

Theorem 2. *Let $L \in \mathfrak{B}_2$. The following three conditions are equivalent:*

(a) *L is injective in $\mathfrak{B}_2$;*

(b) *there are complete Boolean algebras B and B' such that $L \cong B' \times B^{[3]}$;*

(c) *L is a complete lattice and L has the following four properties:*

(i) *$D(L)$ has a smallest element d;*

(ii) *d has a dual pseudocomplement which is central in L;*

(iii) *there is an element $e \in L$ such that $e^* \vee e^{**} = d$;*

(iv) *given $u_1, u_2 \in D(L)$ there is an $x \in L$ such that $(x \wedge e)^* \vee d = u_1$ and $(x \wedge e^*)^* \vee d = u_2$.*

Proof. We need only show that (iii) and (iv) are equivalent to (iv) and (v) of Theorem 1 for $n = 2$. Clearly (iii) and (iv) imply (iv) and (v) of Theorem 1 with $n = 2$. Set $e_1 = e$, $e_2 = e^*$. We need only establish that $e_1 \vee e_1^* = e_2 \vee e_2^* = d$; $d \leq e_1 \vee e_1^* = e \vee e^* \leq e^{**} \vee e^* = d$. On the other hand, let e_1, e_2 satisfy (iv) and (v) of Theorem 1 with $n = 2$. Then $e_1 \wedge e_2 = 0$, that is $e_2 \leq e_1^*$, implying $e_2^* \geq e_1^{**}$. Since $e_1^* \wedge e_2^* = 0$ implies $e_2^* \leq e_1^{**}$ we conclude that $e_2^* = e_1^{**}$.

Set $e = e_1^*$ and note that $(x \wedge e)^* = (x \wedge e_1^*)^* = (x \wedge e_2^{**})^* = (x \wedge e_2)^*$ and that $(x \wedge e^*)^* = (x \wedge e_1^{**})^* = (x \wedge e_1)^*$.

Theorem 3. *Let L be a Stone algebra. The following three conditions are equivalent*:

(a) *L is an injective Stone algebra*;

(b) *there are complete Boolean algebras B and B' such that $L \cong B' \times B^{[2]}$*;

(c) *L is a complete lattice and has the following three properties*:

(i) *$D(L)$ has a smallest element d*;

(ii) *d has a dual pseudocomplement which is central in L*;

(iii) *given $u \in D(L)$ there is an $x \in L$ such that $x^* \vee d = u$.*

Proof. We need only show that condition (c) is equivalent to condition (c) of Theorem 1 where $n = 1$. From (iv) of Theorem 1, $e_1 \vee e_1^* = d$ and $e_1^* = 0$, that is, $e_1 = d$, and (v) of Theorem 1 implies $x^* \vee d = (x \wedge d^{**})^* \vee d = (x \wedge d)^* \vee d = u$, since $d^{**} = 1$. Thus (c) of Theorem 1 is equivalent to (c) of the present theorem with $e_1 = d$, thereby completing the proof.

Injective Stone algebras were characterized in [1]. Condition (b) of Theorem 3 appears there, but rather than condition (c) the following characterization is given: a Stone algebra L is injective if and only if the following conditions hold:

(α) L is complete;

(β) L has a smallest dense element;

(γ) L is also a dual Stone algebra;

(δ) $a^* = b^*$ and $a^+ = b^+$ imply $a = b$.

We wish to remark that these conditions of [1] easily imply those of Theorem 3, indeed, (γ) implies that d has a dual pseudocomplement which is central, and (γ), (δ) imply (iii) of Theorem 3. For $u \in D(L)$, let $x = u^+$; we claim that $x^* \vee d = u$. Since u^+ is central $(u^+)^* = u^{++}$; thus $(x^* \vee d)^+ = (u^{++} \vee d)^+ = (u \vee d)^+ = u^+$ since $u \geq d$. Also, $(x^* \vee d)^* = 0 = u^*$. Consequently, by (δ), $x^* \vee d = u$.

To complete our discussion we present a result of R. A. Day [2], giving a proof in the spirit of this paper.

Theorem 4 (**R. A. Day** [2]). *The only nontrivial equational classes of pseudocomplemented distributive lattices that have enough injectives are $\mathfrak{B}_0$, $\mathfrak{B}_1$, and $\mathfrak{B}_2$. If $\mathfrak{B}$ is any other equational class of pseudocomplemented distributive lattices, then L is injective in $\mathfrak{B}$ if and only if L is a complete Boolean algebra.*

Proof. That $\mathfrak{B}_0$, $\mathfrak{B}_1$, and $\mathfrak{B}_2$ have enough injectives follows from the fact that their respective subdirectly irreducible generators are injective in the respective class. The injectivity of complete Boolean algebras is Lemma 6.

We now show that $\mathfrak{B}_n$, $n > 2$, has no other injectives. Let $L \in \mathfrak{B}_n$ be

injective. If L is a Boolean algebra, then L is a retract of any complete extension and so L is complete. Otherwise there is a dense element $u \in D(L)$ such that $u \neq 1$. Map $\overline{B}_1$ ($= \{0, e, 1\}$, $0 < e < 1$) to L by sending 0 to 0, 1 to 1, and e to u. Since $\overline{B}_n$ is an extension of $\overline{B}_1$, we get a *-homomorphism $\phi: \overline{B}_n \to L$ separating e (the smallest dense element in $\overline{B}_n$) from 1. Since the subdirectly irreducible members of $\mathfrak{B}_n$ are subalgebras of $\overline{B}_n$, there is a *-homomorphism $\rho: L \to \overline{B}_n$ such that $u\rho \neq 1$, hence $e\phi\rho = e$. As was noted in Part I [6], the congruence collapsing e and 1 is the smallest nontrivial congruence in $\overline{B}_n$. Consequently $\phi\rho: \overline{B}_n \to \overline{B}_n$ is an isomorphism and thus $\overline{B}_n$ is a retract of the injective algebra L and is therefore injective. This conclusion would show that $\mathfrak{B}_n$ has enough injectives and consequently satisfies the Amalgamation Property. Thus $n = 1$ or 2. Thus $\mathfrak{B}_n$, $n > 2$, can have no injectives other than the complete Boolean algebras.

For $\mathfrak{B}_\omega$, an injective Boolean algebra must be complete, as above. If $L \in \mathfrak{B}_\omega$ and L is injective and non-Boolean, then—as above—there is a *-homomorphism $\phi: \overline{B} \to L$ separating the two dense elements of $\overline{B}$ for any nontrivial Boolean algebra B. Thus ϕ is one-to-one, and to obtain the desired contradiction we need only choose B so that its cardinality is greater than that of L.

5. **First order properties of weak injectives.** The internal characterization of weak injectives in $\mathfrak{B}_n$, Theorem 1(d), is in terms of first order properties (conditions (i)–(v) of Theorem 1(d)) in addition to a second order property, namely that L be complete. In this section we show that all first order properties of weak injectives follow from a single first order property.

Let B be a Boolean lattice. We say that B *splits* if $|B| = 1$ or $B \cong B_0 \times B_1$, where B_0 is atomic and there are no atoms in B_1 (therefore, either $|B_1| = 1$ or B_1 is infinite).

Lemma 10. *Every complete Boolean lattice splits.*

Proof. Let B be a complete Boolean lattice and let a be the join of all atoms in B. Then $B \cong (a] \times (a']$ and $(a]$ is atomic while $(a']$ has no atoms. To prove that $(a]$ is atomic one has to use the Infinite Distributive Identity which is known to hold since B is complete.

It is easy to see that not all Boolean lattices split.

Lemma 11. *There is a first order sentence Φ such that, for any Boolean lattice B, Φ holds for B if and only if B splits.*

Proof. The sentence should state that there is a smallest element a such that all atoms are contained in a.

Let Φ_n be the first order sentence (in the language of pseudocomplemented

distributive lattices) that states (i)–(v) of Theorem 1(d) and requires that $(d^{++}]$ and $D(L)$ split.

This makes sense since by the proof of Lemma 9, conditions (i)–(v) imply that $(d^{++}]$ and $D(L)$ are Boolean.

Theorem 5. *The first order sentence Φ_n holds for any weak injective in $\mathfrak{B}_n$. Conversely, if Ψ is any first order sentence that holds in any weak injective in $\mathfrak{B}_n$, then Ψ follows from Φ_n.*

The first part of Theorem 5 is already known.

An equivalent form of the second part of Theorem 5 is the following:

Theorem 5′. *A pseudocomplemented distributive lattice L is elementarily equivalent to a weak injective in $\mathfrak{B}_n$ if and only if Φ_n holds in L.*

Theorem 5′ implies Theorem 5. Indeed, let Theorem 5′ hold, let Ψ be a first order sentence that holds in any weak injective in $\mathfrak{B}_n$, and assume that Φ_n does not imply Ψ. Then there exists a pseudocomplemented distributive lattice L satisfying Φ_n but not Ψ. By Theorem 5′, there exists a weak injective L_1 in $\mathfrak{B}_n$ that is elementarily equivalent to L. Hence Ψ does nor hold in L_1, contradicting the definition of Ψ.

Proof of Theorem 5′. The "only if" part is trivial. Now let L be a pseudocomplemented distributive lattice satisfying Φ_n. By Lemma 9, we can assume that $L = B' \times B^{[n+1]}$, where B' and B are Boolean lattices that split.

By A. Tarski [8], any two infinite nonatomic Boolean lattices are elementarily equivalent and (see also Theorem 38.5 of G. Grätzer [3]) every atomic Boolean lattice is elementarily equivalent to a complete and atomic Boolean lattice. Since by S. Kochen [5] finite direct products preserve elementary equivalence, there are complete Boolean lattices C and C' such that $B \equiv C$ and $B' \equiv C'$ ($\equiv$ denotes elementary equivalence).

It is easy to see that $B \to B^{[n+1]}$ commutes with prime limits. Therefore by S. Kochen [5] (or by direct computation) $B^{[n+1]} \equiv C^{[n+1]}$. Thus

$$L = B' \times B^{[n+1]} \equiv C' \times C^{[n+1]},$$

and the right side is a weak injective in $\mathfrak{B}_n$ by Theorem 1, completing the proof of Theorem 5′.

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