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MÜNTZ-SZASZ TYPE APPROXIMATION AND THE ANGULAR GROWTH OF LACUNARY INTEGRAL FUNCTIONS

BY

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ABSTRACT. We consider analogues of the Müntz-Szasz theorem, as in [15] and [4], for functions regular in an angle. This yields necessary and sufficient conditions for the existence of integral functions which are bounded in an angle and have gaps of a very regular nature in their power series expansion. In the case when the gaps are not so regular, similar results hold for formal power series which converge in the angle concerned.

1. Introduction. Let

$$\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \cdots\} \tag{1.1}$$

be a given set of integers satisfying $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$. The set of all integral functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for which $a_n = 0$, $n \notin \Lambda$, is denoted by $E(\Lambda)$.

The classical Müntz-Szasz theorem, as considered in [15], [4] and [10], concerns $C[0, 1]$, the space of functions $f(t)$ continuous for $[0, 1]$ endowed with the uniform topology, and the linear manifold $V(\Lambda)$ spanned by the monomials $\{t^{\lambda_n}\}$ for $\lambda_n \in \Lambda$. We state

Theorem A. *In order that the set of functions* $\{t^{\lambda_n}\}$, $\lambda_n \in \Lambda$, *be complete in* $C[0, 1]$, *i.e. that* $\overline{V}(\Lambda) = C[0, 1]$, *it is necessary and sufficient that*

$$\lambda_0 = 0, \quad \sum_{n=1}^{\infty} \lambda_n^{-1} = \infty. \tag{1.2}$$

If, however, $\sum_{n=1}^{\infty} \lambda_n^{-1} < \infty$ *then* $\overline{V}(\Lambda)$ *consists of the restriction to the interval* $0 < x < 1$ *of functions* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *analytic in* $|z| < 1$ *and with* $a_n = 0$, $n \notin \Lambda$.

(For proofs and further details, see e.g. [4], [10] and also [3].)

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The latter half of the theorem is particularly useful in dealing with questions involving the radial growth of integral functions and, in particular, can be used to prove the following theorem of Macintyre [12].

Theorem B. *Let $f(z) \in E(\Lambda)$ for a given set Λ and suppose $f(x) = O(1)$ $(x \to \infty,\ x$ real). In order to conclude that $f(z) \equiv 0$ it is necessary and sufficient that* (1.2) *fail to hold.*

To deal with the growth of lacunary integral functions in angles of the form $\alpha \leq \arg z \leq \beta$ it seems desirable to generalize Theorem A to this situation. This is the object of the present paper. In this case the appropriate space is the space of functions $f(z)$ regular in an angle A (which, without loss of generality, we may assume to be bisected by the positive real axis) and continuous on the arms of A under the topology of uniform convergence in closed sectors $\{z: |z| \leq R,\ z \in A\}$.

It is more convenient to make the usual exponential change of variable $z = \exp(-s)$ and to omit the constant term $z^{\lambda_0} = \exp(-\lambda_0 s) = 1$. This yields the additional advantage that we need no longer consider the set Λ, defined by (1.1) as consisting of *integers*, but merely of positive real numbers such that $\lambda_{n+1} - \lambda_n > \delta > 0$ for some fixed δ and all n. Thus if $s = \sigma + it$ we denote by $R(\alpha)$ the space of functions $f(s)$ regular for $|t| < \alpha$, continuous for $|t| = \alpha$ and such that $f(s) \to 0$ uniformly in t as $\sigma \to \infty$. The topology of $R(\alpha)$ is that of uniform convergence on closed right half-strips $\{s: \sigma \geq \sigma_0,\ |t| \leq \alpha\}$. Correspondingly we denote by $V(\Lambda)$ the linear manifold of all finite linear combinations of the functions $\exp(-\lambda_n s)$, $n = 1, 2, \cdots$.

2. Results. We prove

Theorem 1. *In order that the set $\{\exp(-\lambda_n s)\}_{\lambda_n \in \Lambda}$ be complete in $R(\alpha)$, i.e. that $\overline{V}(\Lambda) = R(\alpha)$, it is necessary and sufficient that*

$$\limsup_{r \to \infty} \{\lambda(r) - (\alpha/\pi) \log r\} = \infty \tag{2.1}$$

where

$$\lambda(r) = \sum_{0 < \lambda_n \leq r} \lambda_n^{-1}. \tag{2.2}$$

Theorem 1 is briefly alluded to by Malliavin and Rubel [14, p. 204] who consider a similar problem for the space $R(\alpha)$ endowed with the topology of uniform convergence on compact subsets of the strip $\{s: -\infty < \sigma < \infty,\ |t| \leq \alpha\}$. The proof of Theorem 1 is entirely analogous to the considerations of Malliavin and Rubel (loc. cit. Theorem 9.1) but is given in detail here because of its relevance to the question, more interesting for our applications to lacunary integral functions, of what one can conclude if (2.1) is not satisfied.

The situation as regards to the appropriate generalisation of the second part of Theorem A is somewhat more complicated. This is due to the fact that the function $\lambda(r) - (\alpha/\pi)\log r$ need not be an increasing function of r, and to the possibility of the condition $\liminf_{r\to\infty}\{\lambda(r) - (\alpha/\pi)\log r\} = -\infty$ being fulfilled. These possibilities do not arise in the case $\alpha = 0$. Such questions are discussed in §§6, 7 *infra*. For sufficiently regular sets Λ we have

Theorem 2. *Suppose that* Λ, *defined by* (1.1), *is a set of integers such that the limit*

$$l = \lim_{r\to\infty}\{\lambda(r) - ((b-a)/2\pi)\log r\}$$

exists. Suppose, further, that $f(z) \in E(\Lambda)$ *and that* $|f(z)| < K$ *for* $a \le \arg z \le b$, *where* K *is some fixed constant. In order that these hypotheses imply that* $f(z)$ *is identically constant it is necessary and sufficient that* $l < \infty$.

Theorem 3. *Suppose that* Λ *is a set of positive numbers* $\{\lambda_n\}$ *such that* $0 < \lambda_1 < \lambda_2 < \cdots$, $\lambda_{n+1} - \lambda_n > \delta > 0$ *for all* n *and that*

$$-\infty < \liminf_{r\to\infty}\{\lambda(r) - (\alpha/\pi)\log r\} \le \limsup_{r\to\infty}\{\lambda(r) - (\alpha/\pi)\log r\} = l, \tag{2.3}$$

say. If the set $\{\exp(-\lambda_n s)\}_{\lambda_n\in\Lambda}$ *is incomplete in* $R(\alpha)$ (*i.e.* $l < \infty$) *and* $f(s) \in \overline{V}(\Lambda)$ *then* $f(s)$ *is the restriction to the domain* $|t| \le \alpha$ *of an integral function of the form*

$$f(s) = \sum_{n=1}^{\infty} A_n \exp(-\lambda_n s). \tag{2.4}$$

Making the substitution $z = \exp(-s)$, and considering the λ_n to be integers again, we see that, if l, defined by (2.3), is finite, i.e. if the monomials $\{z^{\lambda_n}\}$ are incomplete in the sector $|\arg z| \le \alpha$, with the appropriate topology, then only integral functions of the form

$$f(z) = \sum_{n=1}^{\infty} A_n z^{\lambda_n}$$

can be approximated in the sector $|\arg z| \le \alpha$ by polynomials containing only the exponents $\{\lambda_n\}$.

3. **Proof of Theorem** 1. The proofs are based on the following theorem of Fuchs [7], in which $A(k)$ denotes the class of functions $f(z)$ regular in $x = \operatorname{Re} z > 0$, continuous in $x \ge 0$, and such that $|f(z)| = O(\exp(k|z|))$, $z \to \infty$.

Theorem C. *If* $\Lambda = \{\lambda_\nu\}_{\nu=1}^{\infty}$ *is an increasing sequence of real numbers such that* $\lambda_{\nu+1} - \lambda_\nu > \delta > 0$ $(\nu = 1, 2, 3, \cdots)$ *then the hypotheses* $f(z) \in A(k)$, $f(\lambda_\nu) = 0$ $(\nu = 1, 2, 3, \cdots)$ *imply that* $f(z) \equiv 0$ *if and only if* $\limsup_{r\to\infty}\{\lambda(r) - (k/\pi)\log r\} = \infty$ *where* $\lambda(r)$ *is defined by* (2.2).

Proof of Theorem 2. If $\overline{V}(\Lambda) \neq R(\alpha)$ we consider $R(\alpha)$ as a subspace of the vector space of functions *continuous* in the strip $|t| \leq \alpha$, with the same topology. On applying the Hahn-Banach theorem we can determine a measure $\mu(s)$ with support in some closed right half-strip, and a function $g(s) \in R(\alpha)$ such that

$$(3.1) \qquad \begin{gathered} \int \exp(-\lambda_n s)\, d\mu(s) = 0, \qquad n = 1, 2, 3, \cdots, \\ \int g(s)\, d\mu(s) \neq 0. \end{gathered}$$

We show that if (2.1) holds a contradiction follows. The function $f(w) = \int \exp(-ws)\, d\mu(s)$ belongs to $A(\alpha)$ and $f(\lambda_n) = 0$, $n = 1, 2, 3, \cdots$. Since (2.1) holds we deduce from Theorem C above that $f(w) \equiv 0$, and, in particular, that

$$(3.2) \qquad \int \exp(-ns)\, d\mu(s) = 0, \qquad n = 1, 2, 3, \cdots.$$

If $\mu(s)$ has support in the half-strip $|t| \leq \alpha$, $\sigma > A$ say, then we may approximate to $g(s)$ uniformly in the domain by linear combinations of the functions $\exp(-ns)$, $n = 1, 2, 3, \cdots$. This follows readily from the classical theory of Runge on uniform polynomial approximation. Since (3.1) and (3.2) are then mutually contradictory we conclude that $\overline{V}(\Lambda) = R(\alpha)$. Thus (2.1) cannot hold.

Conversely, suppose that for all r

$$\lambda(r) - (\alpha/\pi) \log r < M_1.$$

For a sufficiently small positive constant δ we define $\lambda_n' = \lambda_n + \delta$ and note that

$$(3.3) \qquad \lambda'(r) = \sum_{\lambda_n' \leq r} (\lambda_n')^{-1} < M_2 + (\alpha/\pi) \log r$$

for some suitable constant M_2. Following Fuchs [7, p. 108] we consider the function

$$F(w) = (1 + w)^{-((2\alpha/\pi)w + 2)} L^w H(w),$$

where

$$H(w) = \prod_{\nu=1}^{\infty} \left\{ \frac{\lambda_\nu' - w}{\lambda_\nu' + w} \exp \frac{2w}{\lambda_\nu'} \right\},$$

and L is some suitably chosen constant. The convergence of the infinite product for $H(\omega)$ follows from the convergence of $\sum_{\nu=1}^{\infty} (\lambda_\nu')^{-2}$. Clearly, $F(\lambda_n') = 0$, $n = 1, 2, 3, \cdots$, $F(z) \not\equiv 0$. The function $F(w)$ is regular for $\operatorname{Re} w > 0$ and continuous on the boundary. Moreover, it follows from (3.3), as has been shown by Fuchs (loc. cit.) that

$$(3.4) \qquad |F(w)| < \frac{M_3}{(1 + |w|)^2} \exp \alpha |w| \qquad (\operatorname{Re} w \geq 0),$$

for some suitable constant M_3, and a suitable choice of the constant L.

To complete the proof of the theorem we show that

$$F(w) = \int \exp(-ws)\, d\mu(s) \tag{3.5}$$

for some measure $\mu(s)$ with support in a half-strip $|t| \leq \alpha, \sigma > A$ and such that

$$\int |\exp(-\delta s)\, d\mu(s)| < \infty. \tag{3.6}$$

For then we may choose w_0 so that $F(w_0) \neq 0$, and writing $g(s) = \exp((\delta - w_0)s)$ we obtain that

$$\int g(s) \exp(-\delta s)\, d\mu(s) = F(w_0) \neq 0,$$

$$\int \exp(-\lambda_n s - \delta s) d\mu(s) = F(\lambda_n') = 0, \qquad n = 1, 2, 3, \cdots.$$

If we choose $w_0 > \delta$, as we may clearly do, then $g(s) \in R(\alpha)$. Since by (3.6) there is a bounded linear functional separating $g(s)$ from the functions $\exp(-\lambda_n s)$ we conclude that the set $\{\exp(-\lambda_n s)\}$ is *not* complete in $R(\alpha)$.

The theory concerning representations of the form (3.5) has been considered by Macintyre [11, Theorem 5]. The Laplace transform

$$h(s) = \int_0^\infty \exp(st) F(t)\, dt$$

can be continued analytically into the region to the left of the contour, Γ, indicated in Figure I. Moreover, thanks to the factor $(1 + |w|)^{-2}$ in (3.4) the function $h(s)$ is actually continuous on Γ.

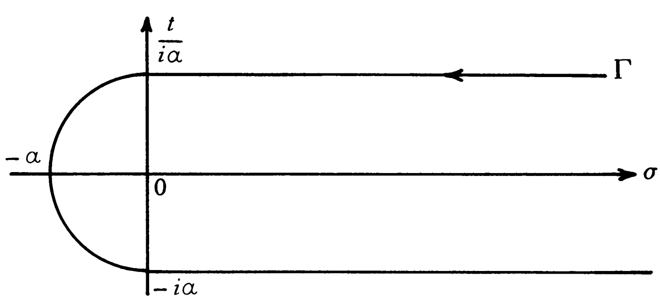

Figure I

The inverse formula

$$F(w) = \frac{1}{2\pi i} \int_\Gamma \exp(-ws)\, h(s)\, ds$$

gives the required representation (3.5). For the measure $\mu(s)$ defined as the restriction to Γ of the measure $(2\pi i)^{-1} h(s)\, ds$ has the property (3.6) since, by rotating the line of integration,

$$h(\sigma + i\alpha) = i \int_0^\infty \exp[(-i\sigma - \alpha)t]\, F(it)\, dt.$$

In view of the estimate (3.4) we obtain that $|h(\sigma + id)| < K$, for some K independent of σ. Thus $\int |\exp(-\delta s)\, d\mu(s)| < \infty$, as required, thus completing the proof of Theorem 1.

4. **Free sets of vectors.** In the case when

$$\lambda(r) - (\alpha/\pi)\log r < K_1 \tag{4.1}$$ [1]

for some suitable constant K_1, the functions $\{\exp(-\lambda_n s)\}_{\lambda_n \in \Lambda}$ are *free* in $R(\alpha)$, i.e. no one of them belongs to the closure of the linear manifold spanned by the others. We may construct functions $F_n(w)$ and $h_n(s)$ as in §3 so that

$$F_n(w) = (1+w)^{-((2\alpha/\pi)w+2)} L^{w} \prod_{\nu=1;\, \nu\neq n}^{\infty} \left\{ \frac{\lambda'_\nu - w}{\lambda'_\nu + w} \exp \frac{2w}{\lambda'_\nu} \right\},$$

$$h_n(s) = \int_0^\infty \exp(st) F_n(t)\, dt,$$

where, as before, $\lambda'_j = \lambda_j + \delta$, $j = 1, 2, 3, \cdots$, for suitably small δ. The measures $\mu_n(s) = (2\pi i)^{-1} h_n(s)\, ds$ restricted to the contour Γ of Figure I all have support in $|t| \le \alpha$, $\sigma \ge -\alpha$. It is easy to see, as before, that

$$\begin{aligned} \int \exp(-\lambda_n s - \delta s)\, d\mu_k(s) &= 0, \qquad && k \neq n, \\ &= F_n(\lambda'_n), && k = n. \end{aligned}$$

Thus the functions $\{\exp(-\lambda_n s)\}_{\lambda_n \in \Lambda}$ are free in $R(\alpha)$ as asserted.

Accordingly, to each $f(s) \in \overline{V}(\Lambda)$ we may associate an 'expansion'

$$f(s) \sim \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s).$$

In this expansion the projection maps $L_n(f) = a_n$ $(n = 1, 2, 3, \cdots)$ are continuous (and so bounded) linear functionals on $R(\alpha)$. Moreover,

$$\begin{aligned} \int f(s) \exp(-\delta s)\, d\mu_k(s) &= \sum_{n=1}^{\infty} a_n \int \exp(-\lambda_n s - \delta s)\, d\mu_k(s), \\ &= a_k F_k(\lambda'_k). \end{aligned}$$

On employing the notation, $\|f\|_\alpha = \max |f(s)|$, $\sigma \ge -\alpha$, $|t| \le \alpha$, we obtain that

$$\begin{aligned} |a_k| &\le |F_k(\lambda'_k)|^{-1} \|f\|_\alpha \int \exp(-\delta s)\, d\mu_k(s), \\ &\le K_2 (1 + \lambda'_k)^{((2\alpha/\pi)\lambda'_k + 2)} L^{-\lambda_k} \prod_{n=1;\, n\neq k}^{\infty} \left| \frac{\lambda'_k + \lambda'_n}{\lambda'_k - \lambda'_n} \exp\left(-2\frac{\lambda'_k}{\lambda'_n}\right) \right| \|f\|_\alpha. \end{aligned}$$

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[1] Throughout the rest of this paper the letter K with subscripts will be used to denote absolute constants.

Since $\lambda'_{k+1} - \lambda'_k > \delta_1 > 0$ we may use the estimate of Fuchs [6, Lemma 4] or [2, p. 159] for the infinite product to obtain

$$|a_k| \le K_3(1+\lambda'_k)^{((2\alpha/\pi)\lambda'_k+2)}\exp\left\{-2\lambda'_k\sum_{j<k}(\lambda'_j)^{-1}\right\}(AL)^{-\lambda'_k}$$

for some suitable constant A.

But

$$\sum_{j<k}(\lambda'_j)^{-1} = \lambda(\lambda_k) + O(1) \qquad (k \to \infty)$$

since $\lambda'_j = \lambda_j + \delta$, $j = 1, 2, 3, \cdots$. Thus we obtain the desired estimate

$$|a_k|^{1/\lambda_k} \le K_4\lambda_k^{2/\lambda_k}\exp\{(2\alpha/\pi)\log\lambda_k - 2\lambda(\lambda_k)\}\,\|f\|_\alpha^{1/\lambda_k}. \tag{4.2}$$

It is from this estimate that our conclusions follow, though unfortunately the one-sided condition (4.1) is of no use in this connection.

5. **Proof of Theorem 3 and Theorem 2 (sufficiency).** If (2.3) holds, we conclude from (4.2) that

$$\limsup_{k\to\infty}|a_k|^{1/\lambda_k} \le K_5 \tag{5.1}$$

so that the function $f_1(s) = \sum_{n=1}^\infty a_n \exp(-\lambda_n s)$ is analytic in the right half-plane $\sigma > \log K_5$.

The method of [1, Theorem 3], for example, enables one to show, by approximating to $f_1(s)$ and $f(s)$ in the domain $\sigma > \log K_5$, by suitable polynomials that $f_1(s) = f(s)$ in that domain. The details, being well known, are omitted; but we point out that it is here that the condition that the set $\{\exp(-\lambda_n s)\}_{\lambda_n\in\Lambda}$ be incomplete in $R(\alpha)$ (i.e. that $l < \infty$), as well as the left-hand inequality of (2.3), is used.

To complete the proof of Theorem 3 we note that the constant K_5 of (5.1) depends only on the set Λ and not on any particular function $f(s)$ in $R(\alpha)$. Thus given any $f(s) \in R(\alpha)$ we may apply the above reasoning to $f(s+B)$ for an arbitrary positive constant B. Clearly, $f(s+B) \in R(\alpha)$ and is analytic for $\sigma + B > \log K_5$. Since B is arbitrary $f(s)$ is an integral function and is evidently of the form (2.4); thus Theorem 3 is proved.

For the sufficiency part of Theorem 2 we require only that

$$\limsup_{r\to\infty}\{\lambda(r) - ((b-a)/2\pi)\log r\} < \infty \tag{5.2}$$

(for a further discussion of this condition see §7 *infra*). We assume that $b = \alpha$, $a = -\alpha$. This involves only a replacement of $f(z)$ by $f(ze^{i\theta})$ for some suitable θ. We also subtract the constant term, $a_0 z^{\lambda_0}$, and assume, without loss of generality, that $f(0) = 0$. The function $f_2(s) = f[\exp(-s+B)]$ (B real) belongs to $R(\alpha)$ and also belongs to $\overline{V}(\Lambda)$ since $f(z)$ may be approximated uniformly in any compact set by the partial sums of its power series.

The coefficients of $f_2(s)$ must satisfy the inequality (4.2). Thus

$$|a_{\lambda_k}|^{1/\lambda_k}\exp(-B) \leq K_6(\Lambda)\|f_2\|_\alpha. \tag{5.3}$$

In this expression the term $K_6(\Lambda)$ depends only on Λ and $\|f_2\|_\alpha$ is bounded above by a constant independent of B, by the hypotheses of Theorem 2. Thus in (5.3) we may fix λ_k and let $B \longrightarrow -\infty$. We conclude that $a_{\lambda_k} = 0$ for $k = 1, 2, 3, \cdots$. Thus $f(z) \equiv 0$, and the sufficiency part of Theorem 2 is established.

6. **Density conditions.** There are a number of well-known theorems relating the growth of a lacunary integral function in an angle A to its growth in the whole z-plane. Before discussing these we introduce some notation; for an integral function $f(z)$ we define

$$M(r, f) = \max|f(z)|, \qquad |z| \leq r,$$

$$M(r, A) = M(r, f, A) = \max|f(z)|, \qquad |z| \leq r, z \in A.$$

The order, ρ, and angular order $\rho(A)$ of $f(z)$ are defined by

$$\rho = \limsup_{r\to\infty}\frac{\log\log M(r)}{\log r}, \qquad \rho(A) = \limsup_{r\to\infty}\frac{\log\log M(r, A)}{\log r},$$

respectively.

The following theorem was proved independently by Edrei [5] and Malliavin [13, §10.4.2].

Theorem A. *Suppose that $f(z) \in E(\Lambda)$ for some set Λ of the form* (1.1). *Then*

$$\rho(A) = \rho \tag{6.1}$$

for every angle A of opening greater than $2\pi\Delta(1)$ where

$$\Delta(1) = \lim_{\xi\to 1-}\limsup_{x\to+\infty}\left\{\frac{\lambda(x) - \lambda(x^\xi)}{(1-\xi)\log x}\right\},$$

is the maximum logarithmic density of the set Λ.

In this theorem the conclusion (6.1) need not follow if the constant $2\pi\Delta(1)$ is replaced by any smaller constant. For an earlier result see [16]. In the case of more rapidly growing functions similar, though less precise, results have been established in [1]. In the case of sets Λ of irregular growth we must associate with Λ a 'densité extérieure' as defined by Kahane [9]. In Theorem 2 the set Λ satisfies a more stringent requirement of regularity, but, with that requirement, one can obtain a more precise theorem regarding growth of functions in angles. This regularity can be dispensed with altogether in Theorem 2 if we consider, instead of $E(\Lambda)$, the class $E(\Lambda, a, b)$ consisting of those functions $f(z)$ regular for $a \leq \arg z \leq b$ and continuous on the boundary such that, for some sequence $\{n_k\}_{k=1}^\infty$, the relationship

$$f(z) = \lim_{k\to\infty} \sum_{n=0}^{n_k} a_n z^{\lambda_n} \qquad (\lambda_n \in \Lambda)$$

is valid for $a \leq \arg z \leq b$ (see Theorem 5 below).

From inequality (4.2) one can readily deduce the following theorem.

Theorem 4. *Suppose that Λ is a set satisfying the hypotheses of Theorem 2 and that $-\infty < l < \infty$. Let $f(z) \in E(\Lambda)$, and suppose that*

$$M(r, f, A) = O\exp((\sigma+\epsilon)r^{\beta}) \qquad (r \longrightarrow \infty) \tag{6.2}$$

for each $\epsilon > 0$, where A is the angle $a \leq \arg z \leq b$. Then

$$M(r, f) = O\exp((\tau+\epsilon)r^{\beta}) \qquad (r \longrightarrow \infty) \tag{6.3}$$

for each $\epsilon > 0$, where $\tau = \sigma \exp\{\beta((b-a)/2)\}$. In particular, $\rho(A) = \rho = \beta$.

Similar, though somewhat less precise, theorems can be proved for rates of growth faster than (6.2). These are similar to the results of [1] and so we omit discussion of them here.

We remark also that in the case when $b = a$ we obtain that $\tau = \sigma$ and the theorem reduces to a result of Gaier [8, Theorem 6].

Proof of Theorem 4. We may suppose, as before, that $a = -\alpha$, $b = \alpha$. We now consider the function

$$f_R(z) = f(Rz) = f_1(\exp(-s + \log R)),$$

say, for arbitrary $R > 0$. It is clear that $f_1 \in \overline{V}(\Lambda)$ and that the inequality (4.2) applies in this situation. However, some further consideration is necessary. The constant K_4 in (4.2) arises partly from the factor L^w in the definition of $F_k(w)$. If each function $F_k(w)$ is to satisfy the inequality (3.4) as we require, it is clear, from the estimate of Fuchs [7, p. 108], that L must be chosen so that

$$K_4 \exp\{(2\alpha/\pi)\log\lambda_k - 2\lambda(\lambda_k)\} \longrightarrow K_4\exp(-2l) \leq 1.$$

It is here, of course, that the hypothesis of Theorem 2 has been used.

On applying (4.2) then, we obtain

$$|a_k| R^{\lambda_k} \leq K_7 \lambda_k^2 \|f_1\|_{\alpha}.$$

With the hypothesis of Theorem 4 this becomes

$$|a_k| \leq K_8 \lambda_k^2 R^{-\lambda_k} \exp\{(\sigma+\epsilon)(e^{\alpha}R)^{\beta}\},$$

for arbitrarily preassigned $\epsilon > 0$ and $k > k(\epsilon)$, say.

We minimize the right-hand side with respect to R by choosing $R = e^{-\alpha}(n/\beta(\sigma+\epsilon))^{1/\beta}$. The desired conclusions now follow from the well-known formulae giving the order and type of an integral function in terms of its coefficients [2, §2.2]. In particular, we note that

$$\limsup_{k\to\infty}\{\lambda_k|a_k|^{\beta\lambda_k^{-1}}\}\leq\beta(\sigma+\epsilon)\exp\{1+\alpha\beta\},$$

for arbitrary $\epsilon>0$. The estimate (6.3) now follows from [2, Theorem 2.2.10].

7. **Proof of Theorem 2 (necessity).** The proof of the remaining part of Theorem 2 is contained in the proof of the following theorem, alluded to in §6.

Theorem 5. *Suppose that Λ, given by (1.1), is a set of integers such that*

$$-\infty\leq l_1=\liminf_{r\to\infty}\{\lambda(r)-((b-a)/2\pi)\log r\}\leq\limsup_{r\to\infty}\{\lambda(r)-((b-a)/2\pi)\log r\}=\infty.$$

Then it is possible to choose a subsequence $\{n_k\}_{k=1}^{\infty}$ and coefficients $\{a_n\}_{n=1}^{\infty}$ so that the limit

$$F(w)=\lim_{k\to\infty}\sum_{n=0}^{n_k}a_nw^{\lambda_n}$$

represents a function $F(w)$, regular for $a\leq\arg w\leq b$, continuous on the arms of the angle, and such that $F(w)\to 0$ as $|w|\to\infty$ uniformly for $a\leq\arg w\leq b$. Moreover, the function so constructed will be an integral function if $l_1=\infty$.

Proof. As before, we consider only the angle $|\arg w|\leq\alpha$, and suppose the sequence $\{n_k\}_{k=1}^{\infty}$ to be such that

$$\lim_{k\to\infty}\{\lambda(n_k)-(\alpha/\pi)\log n_k\}=\infty. \tag{7.1}$$

We consider the function

$$G(z)=(1+z)^{-2}z^{(2\alpha/\pi)z}\prod_{\nu=1}^{\infty}\left\{\frac{\lambda_\nu+z}{\lambda_\nu-z}\exp\left(-2\frac{z}{\lambda_\nu}\right)\right\}. \tag{7.2}$$

Since the λ_ν are integers, $\sum_{\nu=1}^{\infty}\lambda_\nu^{-2}<\infty$, and this implies the convergence of the infinite product in (7.2). We take also that branch of $z^{(2\alpha/\pi)z}$ which is real for $z>0$. Thus, if $z=x+iy=re^{i\theta}$, the function $G(z)$ defined by (7.2) is regular for $x>0$. Moreover, as we have already seen [7, p. 108]

$$|G(z)|\leq\frac{1}{|1+z|^2}K_9^{-x}\exp\{-2x[\lambda(r)-(\alpha/\pi)\log r]-\alpha r\} \tag{7.3}$$

for $x\geq 0$ and suitable K_9.

We define

$$F_R(w)=\int_C w^zG(z)\,dz,$$

where C is the contour consisting of the semicircle $|z|=R$, $|\arg z|\leq\pi/2$ (where $R\neq\lambda_n$, $n=1,2,3,\cdots$) and the segment $-R\leq y\leq R$ of the y axis, taking, again, the principal value of w^z. Clearly,

$$F_R(w) = \sum_{\lambda_n<R} a_n w^{\lambda_n},$$

where

$$a_n = 2e^{-2}\lambda_n^{((2\alpha/\pi)\lambda_n+1)}(1+\lambda_n)^{-2}\prod_{\nu=1;\,\nu\neq n}^{\infty}\left\{\frac{\lambda_\nu+\lambda_n}{\lambda_\nu-\lambda_n}\exp\left(\frac{-2\lambda_n}{\lambda_\nu}\right)\right\}, \tag{7.4}$$

and so is of the required type.

Now, for $|\arg w| \le \alpha$, $|w^z| \le \exp(\alpha r)$. Hence from (7.1) and (7.3) we conclude that the representation

$$F(w) = \lim_{k\to\infty} F_{\alpha_k}(w) = \lim_{k\to\infty}\int_{-i\alpha_k}^{i\alpha_k} w^z G(z)\,dz,$$

where $\alpha_k = \lambda_{n_k} + \epsilon$ for small $\epsilon > 0$, is valid for $|\arg w| \le \alpha$. The estimate (7.3) shows that $\int_{-i\infty}^{i\infty} w^z G(z)\,dz$ exists for each w and tends to zero as $|w| \to \infty$, uniformly in $|\arg w| \le \alpha$.

The function $F(w)$ has all the properties required. If we apply an estimate of the type (7.3) to (7.4) we obtain, in the case when $l_1 = \infty$, that $\lim_{n\to\infty}|a_n|^{1/\lambda_n} = 0$. Therefore, the function $F_1(w) = \sum_{n=1}^{\infty} a_n w^{\lambda_n}$ is an integral function. It is not difficult to verify that $F(w) = F_1(w)$ for $|\arg w| \le \alpha$. Hence, when $l_1 = \infty$, $F(w)$ is an integral function, as required. Thus Theorem 5 is proved.

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