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CLOSE-TO-CONVEX MULTIVALENT FUNCTIONS WITH RESPECT TO WEAKLY STARLIKE FUNCTIONS

BY

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ABSTRACT. It is the object of this article to define close-to-convex multivalent functions in terms of weakly starlike multivalent functions. Six classes are defined, and shown to be equal. These generalize the class of close-to-convex functions developed by Livingston in the article, *p-valent close-to-convex functions*, Trans. Amer. Math. Soc. **115** (1965), 161–179.

1. In this paper we consider several ways to define multivalent close-to-convex functions with respect to weakly starlike functions, and we study the relationships between the classes defined. The theory is based on J. A. Hummel's paper [1] on p-valent weakly starlike functions, and the paper by A. E. Livingston [3] on p-valent close-to-convex functions.

This section is devoted to the definitions and theorems which we will use.

Definition 1. Let $S_0(1)$ be the class of functions f univalent in $U = \{z: |z| < 1\}$ such that $f(0) = 0$ and $f(U)$ is starshaped with respect to 0. $S_0(1)$ is the class of *starlike univalent* functions.

Note that we do not insist that $f'(0) = 1$. It is well known that

$f \in S_0(1)$ if and only if f is regular in U, has one zero there (counting multiplicity) and $\operatorname{Re}(zf'(z)/f(z)) > 0$ for all $z \in U$.

Definition 2. Let $S_a(p)$ be the class of functions f regular in U, with p zeros there (counting multiplicity) and such that $\operatorname{Re}(zf'(z)/f(z)) > 0$ for all z in some annulus $A_\rho = \{z: \rho < |z| < 1\}$.

These are the "standard" multivalent starlike functions. The extension to the class of *weakly starlike functions*, as developed by Hummel [1], is direct:

Definition 3. Let $S_w(p)$ be the class of functions f regular in U, with p zeros there and such that

$$\liminf_{r\to 1-}\left[\min_{|z|=r} \operatorname{Re}\frac{zf'(z)}{f(z)}\right] \geq 0. \tag{1}$$

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Definition 4. We will say that the sequence $\{f_n\}$ of functions in U converges *almost uniformly* to f if $\{f_n\}$ converges uniformly to f on each compact subset of U.

The following results on weakly starlike functions may be found in Hummel [1].

Proposition 1. *$f \in S_w(p)$ if and only if f has p zeros and f is the almost uniform limit of a sequence of functions in $S_a(p)$.*

Since functions in $S_a(p)$ are p-valent, this shows that functions in $S_w(p)$ are also p-valent. It also shows that f' has at most $p-1$ zeros.

Let $\Psi(z, z_0) = (z - z_0)(1 - \bar{z}_0 z)/z$, $\Psi(z, 0) \equiv 1$.

Proposition 2. *$f \in S_w(p)$ if and only if there is a function $h \in S_0(1)$ such that*

$$f(z) = [h(z)]^p \prod_{i=1}^{p} \Psi(z, z_i), \qquad |z_i| < 1. \tag{2}$$

Here each z_i is a zero of f.

We follow Livingston in our definition of *close-to-convex functions.*

Definition 5. Let $K(p)$ be the class of functions F, regular in U, with $F(0) = 0$, such that there is a function $f \in S_a(p)$ for which $f(0) = 0$ and

$$\operatorname{Re}[zF'(z)/f(z)] > 0 \quad \text{in some annulus } A_\rho = \{z\colon \rho < |z| < 1\}. \tag{3}$$

We note that Livingston singles out the subclass for which F and f are both regular on $C_1 = \{z\colon |z| = 1\}$ and satisfy (3) on C_1.

The class $K(1)$ is automatically reduced to close-to-convex univalent functions of Kaplan [2]:

$H \in K(1)$ if and only if there is a function $h \in S_0(1)$ such that

$$\operatorname{Re}[zH'(z)/h(z)] > 0 \quad \text{for all } z \in U. \tag{4}$$

Proposition 3. *Let $F \in K(p)$. Then F is at most p-valent in U, and F' has exactly $p-1$ zeros in U.*

The proof of this proposition may be found in Livingston [3]. It depends on a lemma by Umezawa [5].

2. **Weakly close-to-convex functions.** In this section we define two classes of functions which both extend the class $K(p)$.

Definition A. Let F be regular and nonconstant in U $(p \geq 1)$. *F belongs to the class $K_{WA}(p)$* if $F(0) = 0$ and there is a function $f \in S_w(p)$, $f(0) = 0$, such that

$$\liminf_{r\to 1-}\left[\min_{|z|=r}\operatorname{Re}\frac{zF'(z)}{f(z)}\right]\geq 0. \tag{5}$$

Definition B. *F belongs to the class* $K_{WB}(p)$ if there are $F_n \in K(p)$ and $f_n \in S_a(p)$, $f_n(0)=0$, $n=1, 2, \cdots$, such that

$$F_n \longrightarrow F \text{ almost uniformly in } U \text{ and } F \not\equiv 0,$$
$$f_n \longrightarrow f \text{ almost uniformly in } U \text{ and } f \in S_w(p),$$

and

$$\operatorname{Re}\{zF_n'(z)/f_n(z)\} > 0 \quad \text{for } \rho_n < |z| < 1,\ n = 1, 2, \cdots. \tag{6}$$

Remark. If the F_n, f_n, $n = 1, 2, \cdots$, were given the added condition that they all be regular on C_1 and if (6) were changed to be true on $z \in C_1$, then the same class of functions would result. This is because we may choose a sequence $\{r_n\}$ such that $r_n \longrightarrow 1$, $F_n(r_n z) \in K(p)$, $f_n(r_n z) \in S_a(p)$.

Both of these classes are obvious extensions of $K(p)$. We prove

Proposition 4. $K(p) \subseteq K_{WA}(p) \subseteq K_{WB}(p)$.

Proof. We only need show that $K_{WA}(p) \subseteq K_{WB}(p)$.

Let $F \in K_{WA}(p)$ and $f \in S_w(p)$, $f(0) = 0$, satisfy (5). Suppose $f(z) = [h(z)]^p \Pi \Psi(z, z_i)$ as in Proposition 2. Let

$$H_t(z) = \int_0^z \left[\frac{h(tz)}{h(z)}\right]^p \left(F'(z) + (1-t)\frac{f(z)}{z}\right) dz$$

where $0 < t < 1$. That $f_t(z) = [h(tz)]^p \Pi \Psi(z, z_i)$ is in $S_a(p)$ is shown in Hummel [1]. A little manipulation shows that $zH_t'(z)/f_t(z) = zF'(z)/f(z) + 1 - t$. Thus

$$\liminf_{r\to 1-}\left[\min_{|z|=r}\operatorname{Re}\frac{zH_t'(z)}{f_t(z)}\right] = \liminf_{r\to 1-}\left[\min_{|z|=r}\operatorname{Re}\frac{zF'(z)}{f(z)}\right] + 1 - t \geq 1 - t > 0.$$

Therefore, in some annulus, $\rho_t < |z| < 1$, $\operatorname{Re}(zH_t'(z)/f_t(z)) > 0$.

$f_t \longrightarrow f$ almost uniformly on U as $t \longrightarrow 1$ (see Hummel [1]).

We will show that $H_t \longrightarrow F$ almost uniformly on U as $t \longrightarrow 1$. This will complete the proof that $F \in K_{WB}(p)$.

Let $|z| \leq r < 1$. Then

$$|H_t(z) - F(z)| \leq \int \left| \left[\frac{h(tz)}{h(z)}\right]^p \left(F'(z) + (1-t)\frac{f(z)}{z}\right) - F'(z)\right| |dz|$$

where the path of integration is the line segment from 0 to z. But the quantity under the integral is

$$\left| F'(z)\left(\left[\frac{h(tz)}{h(z)}\right]^p - 1\right) + (1-t)\frac{f(z)}{z}\left[\frac{h(tz)}{h(z)}\right]^p \right|$$

which can be made arbitrarily small for t near 1 while $|z| \le r$. Therefore, $H_t \to F$ almost uniformly as $t \to 1$. This completes the proof.

3. **More weakly starlike functions.** In this section we define a class of functions which is a slight extension of $S_w(p)$. We then show that functions close-to-convex with respect to this larger class have some simple and useful properties.

Definition 6. Let $S_{wc}(p)$ be the class of functions f of the form

$$f(z) = [h(z)]^p \prod_{i=1}^{p} \Psi(z, z_i), \qquad |z_i| \le 1, \ 1 \le i \le p, \tag{7}$$

and $h \in S_0(1)$. Let $S_{wc}(0)$ be the set of nonzero constant functions.

The only difference between (7) and (2) is that in (7) we may have $|z_i| = 1$. This is significant in terms of being "starlike" with respect to zero. A function in $S_{wc}(p)$ may have no zero in U. Nevertheless, the author shows in another article [4] that most of the theory of the class $S_w(p)$ carries over to $S_{wc}(p)$.

We will use the uniform convergence property of $S_{wc}(p)$. Namely,

Proposition 5. *$f \in S_{wc}(p)$ if and only if $f \not\equiv 0$ and f is the almost uniform limit of a sequence of functions in $S_a(p)$.*

This is much the same as Proposition 1, and a proof need not be included.

Definition C. A function $F \not\equiv 0$ *belongs to the class* $K_{WC}(p)$ if there are $F_n \in K(p)$, $n = 1, 2, \cdots$, such that $F_n \to F$ almost uniformly in U.

Note that here we may assume there are $f_n \in S_a(p)$, $f_n(0) = 0$, $n = 1, 2, \cdots$, such that $f_n \to f$ almost uniformly in U, $f \in S_{wc}(p)$, and

$$\operatorname{Re}\{zF_n'(z)/f_n(z)\} > 0, \qquad \rho_n < |z| < 1, \ n = 1, 2, \cdots.$$

Thus $K_{WC}(p)$ is trivially an extension of $K_{WB}(p)$.

Definition D. A function F *belongs to the class* $K_{WD}(p)$ if there is a function $H \in K(1)$ and a function $g \in S_{wc}(p-1)$ such that

$$F(z) = \int_0^z g(z)H'(z)\,dz. \tag{8}$$

Proposition 6. $K_{WB}(p) \subseteq K_{WC}(p) \subseteq K_{WD}(p)$.

Proof. We only need show that $K_{WC}(p) \subseteq K_{WD}(p)$.

Let $F \in K_{WC}(p)$ and suppose F_n and f_n, $n = 1, 2, \cdots$, are as defined in Definition C. We may then assume without loss of generality that the F_n and the f_n are regular on C_1 and $\operatorname{Re}\{zF_n'(z)/f_n(z)\} > 0$ on C_1.

Let $f_n(z) = [h_n(z)]^p \Pi\Psi(z, z_i)$. Suppose that the zeros of F_n' are $\alpha_1^{(n)}, \cdots, \alpha_{p-1}^{(n)}$.

Then

$$\operatorname{Re}\left\{\frac{zF_n'(z)\left[\Pi\Psi(z, \alpha_i^{(n)})\right]^{-1}}{[b_n(z)]^p}\right\} > 0 \quad \text{on } C_1.$$

But this is the real part of a regular function, so it is positive throughout U. Also, $H_n'(z) = F_n'(z)/([b_n(z)]^{p-1}\Pi\Psi(z, \alpha_i^{(n)}))$ is regular in U, and $\operatorname{Re}\{zH_n'(z)/b_n(z)\} > 0$ in $|z| \le 1$.

Now $b_n \to b$ almost uniformly in U, and $b \in S_0(1)$.

$$H_n'(z) \to \frac{F'(z)}{[b(z)]^{p-1}\Pi\Psi(z, \alpha_i)} \quad \text{almost uniformly in } U,$$

where $\alpha_i^{(n)} \to \alpha_i$, $1 \le i \le p-1$. Let $H'(z) = F'(z)/([b(z)]^{p-1}\Pi\Psi(z, \alpha_i))$. Thus H' is not identically zero. Furthermore, $zH_n'(z)/b_n(z) \to zH'(z)/b(z)$ almost uniformly on U, so $\operatorname{Re}\{zH'(z)/b(z)\} \ge 0$ for all $z \in U$. For $H(z) = \int_0^z H'(z)dz$, $H \in K(1)$. That is, if $zH'(z)/b(z)$ is not constant, then $\operatorname{Re}\{zH'(z)/b(z)\} > 0$ for all $z \in U$. If $zH'(z)/b(z) = i\alpha$ for some $\alpha \in \mathbf{R}$, then $i\alpha b(z) \in S_0(1)$ and $zH'(z)/i\alpha b(z) = 1$, which has positive real part.

Let $g(z) = [b(z)]^{p-1}\Pi\Psi(z, \alpha_i)$. Thus $F'(z) = g(z)H'(z)$, where $H \in K(1)$ and $g \in S_{wc}(p-1)$, $F \in K_{WD}(p)$. This completes the proof.

Definition E. Let F be regular in U, with $F(0) = 0$. *F belongs to the class* $K_{WE}(p)$ if there is a function $f \in S_{wc}(p)$ such that $f(0) = 0$ and

$$\operatorname{Re}\{zF'(z)/f(z)\} > 0 \quad \text{for all } z \in U. \tag{9}$$

Proposition 7. $K_{WD}(p) \subseteq K_{WE}(p)$.

Proof. Let $F \in K_{WD}(p)$. Then there is an $H \in K(1)$ and a $g \in S_{wc}(p-1)$ such that $F'(z) = g(z)H'(z)$. Also there is a function $b \in S_0(1)$ such that $\operatorname{Re}\{zH'(z)/b(z)\} > 0$ in U. Thus

$$\operatorname{Re}\{zF'(z)/g(z)b(z)\} = \operatorname{Re}\{zg(z)H'(z)/g(z)b(z)\} > 0 \quad \text{in } U.$$

Now $g(z) = [g_1(z)]^{p-1}\Pi\Psi(z, z_i)$, where $g_1 \in S_0(1)$, so $b(z)[g_1(z)]^{p-1} = [f_1(z)]^p$ for some $f_1 \in S_0(1)$. Thus $g(z)b(z) = [f_1(z)]^p\Pi\Psi(z, z_i) \equiv f(z)$. Hence $\operatorname{Re}\{zF'(z)/f(z)\} > 0$ in U, and $f \in S_{wc}(p)$, $f(0) = 0$, $F \in K_{WE}(p)$. This completes the proof.

4. **A subclass of** $K_{WA}(p)$. In this section we unify our results. First it is desirable to define a final class of close-to-convex functions.

Definition O. Let F be regular and nonconstant in U. *F belongs to the class* $K_{WO}(p)$ if $F(0) = 0$ and there is a function $b \in S_0(1)$ such that

$$\liminf_{r \to 1-} \left[\min_{|z|=r} \operatorname{Re}\left\{\frac{zF'(z)}{[b(z)]^p}\right\}\right] \ge 0. \tag{10}$$

It is completely trivial that $K_{WO}(p) \subseteq K_{WA}(p)$. It is less trivial, but true, that $K_{WE}(p) \subseteq K_{WO}(p)$. When we show this we will have

Theorem 1. *Let p be a positive integer. Then $K_{WA}(p) = K_{WB}(p) = K_{WC}(p) = K_{WD}(p) = K_{WE}(p) = K_{WO}(p)$. If we let $K_W(p)$ stand for this class, then any function $F \in K_W(p)$ is at most p-valent and F' has at most $p-1$ zeros in U. Furthermore, $K(p) \subsetneqq K_W(p)$ when $p > 1$.*

That a function $F \in K_W(p)$ is at most p-valent follows from the characterization $K_{WB}(p)$. By the integral (8), F' has exactly the same number of zeros as g. Since g may have fewer than $p-1$ zeros when $p > 1$, F need not be in $K(p)$.

In order to show that $K_{WE}(p) \subseteq K_{WO}(p)$ we prove a couple of lemmas.

Lemma 1. *Let*

$$g(z, t) = \frac{1 + e^{-it}z}{1 - e^{-it}z} \cdot \prod_{i=1}^{n} \Psi(z, z_i)$$

where $|z_i| \le 1$, $1 \le i \le n$, $z \in \mathbb{C}$ and $t \in [0, 2\pi]$. Then $\liminf_{r\to 1^-}[\min \operatorname{Re} g(z, t)] \ge 0$ where the minimum is taken over all $z \in C_r$ and all $t \in [0, 2\pi]$.

Proof. For t fixed, $g(z, t)$ is continuous on C_1, except possibly at $z = e^{it}$, and pure imaginary on C_1. If $g(z, t)$ is continuous on C_1, then

$$\lim_{r\to 1-}\left[\min_{|z|=r} \operatorname{Re} g(z, t)\right] = 0.$$

If $g(z, t)$ has a pole at $z = e^{it}$, then it has a pole of order one, so that $g(z, t)$ is conformal in some disc D centered on e^{it}. Since $g(z, t)$ is pure imaginary on C_1, either $\operatorname{Re} g(z, t)$ is positive for all $z \in D \cap U$ or $\operatorname{Re} g(z, t)$ is negative for all $z \in D \cap U$. Since $g(z, t)$ has a pole at $z = e^{it}$, $z_i \ne e^{it}$, $1 \le i \le n$. It is easily seen that for $z = re^{it}$, $z \in D$ and $\operatorname{Re} g(z, t) > 0$ provided r, which is less than 1, is sufficiently near 1. Therefore, $\operatorname{Re} g(z, t) > 0$ for all $z \in D \cap U$.

In any case, for t fixed,

$$\liminf_{r\to 1-}\left[\min_{|z|=r} \operatorname{Re} g(z, t)\right] \ge 0.$$

For each positive real number ϵ, let

$$r(t) = r(t, \epsilon) = \inf\{r\colon \operatorname{Re} g(re^{i\theta}, t) > -\epsilon \text{ for all } \theta\}.$$

Let $\epsilon_1 > 0$ be given with $r(t) + \epsilon_1 < 1$. If we choose r such that $r(t) < r < r(t) + \epsilon_1$, then $\operatorname{Re} g(re^{i\theta}, t) > -\epsilon + \alpha$ for all θ, where α is some positive number. Since g is continuous in t, $\operatorname{Re} g(re^{i\theta}, t') > -\epsilon$ for all θ in $[0, 2\pi]$ and all t' in some neighborhood of t. Therefore $r(t') < r(t) + \epsilon_1$. This means, by definition, $r(t)$ is upper-semicontinuous on $[0, 2\pi]$. But any such function takes its maximum. That is, there is a $t_0 \in [0, 2\pi]$ such that $r(t) \le r(t_0) < 1$ for all $t \in [0, 2\pi]$.

Therefore $\inf \operatorname{Re} g(z, t) \geq -\epsilon$ where the infimum is taken over all z with $r(t_0) < |z| < 1$, and $t \in [0, 2\pi]$. This completes the proof of Lemma 1.

Lemma 2. *Suppose that f is regular in U and that $\operatorname{Re} f(z) > 0$ for all $z \in U$. Then*

$$\liminf_{r \to 1-} \left[\min_{|z|=r} \operatorname{Re} \left\{ f(z) \prod_{i=1}^{n} \Psi(z, z_i) \right\} \right] \geq 0$$

whenever $|z_i| \leq 1$ for $1 \leq i \leq n$.

Proof. Assume that $f(0) = 1$. With this normalization we can apply the Herglotz representation theorem for functions with positive real part:

There is an increasing function α: $[0, 2\pi] \to [0, 1]$ such that $\alpha(0) = 0$, $\alpha(2\pi) = 1$, and

$$f(z) = \int_0^{2\pi} \frac{1 + e^{-it}z}{1 - e^{-it}z} \, d\alpha(t).$$

See, for instance, Wall [6, p. 275]. Therefore,

$$\operatorname{Re} f(z) \prod \Psi(z, z_i) = \int_0^{2\pi} \operatorname{Re} \left\{ \frac{1 + e^{-it}z}{1 - e^{-it}z} \prod \Psi(z, z_i) \right\} d\alpha(t).$$

Let $\epsilon > 0$ be given. By Lemma 1, there is an $r_\epsilon < 1$ such that

$$\operatorname{Re} \left\{ \frac{1 + e^{-it}z}{1 - e^{-it}z} \prod \Psi(z, z_i) \right\} > -\epsilon$$

whenever $|z| > r_\epsilon$. Thus, for $|z| > r_\epsilon$,

$$\operatorname{Re} f(z) \prod \Psi(z, z_i) > \int_0^{2\pi} -\epsilon \, d\alpha(t) = -\epsilon.$$

This proves Lemma 2 for the case $f(0) = 1$.

Suppose $f(0) = a + ib$. Then $a > 0$, and $g(z) = (f(z) - ib)/a$ is regular in U with $g(0) = 1$ and has positive real part. Since $f(z)\Pi\Psi(z, z_i) = ag(z)\Pi\Psi(z, z_i) + ib\Pi\Psi(z, z_i)$ and $\lim_{r \to 1-} \operatorname{Re} ib\,\Pi\Psi(z, z_i) = 0$, Lemma 2 is seen to be correct in any case.

Proposition 8. $K_{WE}(p) \subseteq K_{WO}(p)$.

Proof. Let $F \in K_{WE}(p)$. Then there is a function $f \in S_{wc}(p)$ such that $f(0) = 0$ and $\operatorname{Re}\{zF'(z)/f(z)\} > 0$ for all $z \in U$. Let $f(z) = [h(z)]^p \Pi_{i=1}^{p-1} \Psi(z, z_i)$. Now

$$\liminf_{r \to 1-} \left[\min_{|z|=r} \operatorname{Re} \left\{ \frac{zF'(z)}{[h(z)]^p} \right\} \right]$$

$$= \liminf_{r \to 1-} \left[\min_{|z|=r} \operatorname{Re} \left\{ \left(\frac{zF'(z)}{f(z)} \right) \prod \Psi(z, z_i) \right\} \right] \geq 0 \quad \text{by Lemma 2.}$$

Therefore $F \in K_{WO}(p)$. This completes the proof of Proposition 8 and thus also the proof of Theorem 1.

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