

A NOTE ON THE GEOMETRIC MEANS OF
 ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

BY

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ABSTRACT. Let $f(z_1, \dots, z_n)$ be an entire function of $n (\geq 2)$ complex variables. Recently Agarwal [Trans. Amer. Math. Soc. 151 (1970), 651–657] has obtained certain results involving geometric mean values of f . In this paper we have constructed examples to contradict some of the results of Agarwal and have thereafter given improvements and modifications of his results.

1. Introduction. Let

$$f(z_1, z_2) = \sum_{m, n \geq 0} a_{mn} z_1^m z_2^n$$

be an entire function of two complex variables (we consider the two variables case for the sake of simplicity). Let

$$(1.1) \quad M(r_1, r_2) = \max_{|z_1| \leq r_1, |z_2| \leq r_2} |f(z_1, z_2)|;$$

$$G(r_1, r_2) = \exp \left\{ \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})| d\theta_1 d\theta_2 \right\};$$

$$(1.2) \quad g_{k, \lambda}(r_1, r_2) = \exp \left\{ \frac{(k+1)(\lambda+1)}{r_1^{k+1} r_2^{\lambda+1}} \int_0^1 \int_0^1 x_1^k x_2^\lambda \log G(x_1, x_2) dx_1 dx_2 \right\},$$

where $0 < k, \lambda < \infty$, be the geometric means of $f(z_1, z_2)$. The term $g_{k, \lambda}(r_1, r_2)$ and its various properties were probably first considered as early as in 1962 by the author [2] in terms of an entire function of a single variable. Recently, Agarwal [1] has generalised some of the results in [3] in terms of $G(r_1, r_2)$ and $g_{k, \lambda}(r_1, r_2)$ when $k = \lambda$, and in addition has also proved the following:

$$(1.3) \quad \lim_{r_1, r_2 \rightarrow \infty} \left\{ \begin{array}{l} \sup \\ \inf \end{array} \frac{\log \log g_{k, \lambda}(r_1, r_2)}{\log(r_1, r_2)} \right\} = \left\{ \begin{array}{l} \rho \\ \mu \end{array} \right. \quad (k = \lambda)$$

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where⁽¹⁾

$$(1.4) \quad \lim_{r_1, r_2 \rightarrow \infty} \left\{ \begin{array}{l} \sup \\ \inf \end{array} \frac{\log \log M(r_1, r_2)}{\log(r_1, r_2)} \right\} = \left\{ \begin{array}{l} \rho, \\ \mu. \end{array} \right.$$

Apart from giving certain growth results involving $G(r_1, r_2)$ and $g_{k,\lambda}(r_1, r_2)$, our chief aim is to present an example which violates (1.3)—an improvement of which (i.e. a correct version of (1.3)) is given in §2 that follows now.

2. A counterexample for (1.3) and its improvement. Let $f(z_1, z_2) = e^{z_1 z_2}$. Then $M(r_1, r_2) = e^{r_1 r_2}$. Therefore $\rho = \mu = 1$. Now

$$\log |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})| = r_1 r_2 \cos \theta_1 \cos \theta_2 - r_1 r_2 \sin \theta_1 \sin \theta_2$$

$$\Rightarrow G(r_1, r_2) = 1, \quad \text{for all } r_1, r_2 > 0.$$

Hence $\log g_{k,k}(r_1, r_2) = 0$ for all $r_1, r_2 > 0$. Thus if (1.3) is true then $\rho = \mu = -\infty$ which is absurd. We may lead to a similar discussion if $f(z_1, z_2) = \exp(z_1 + z_2)$ and the details are left to the reader. I may point out that the main fault in establishing (1.3) is the following inequality (see line 4 from above, p. 653 of [1]) which Agarwal has proved:

$$\log g_{k,k}(\alpha r_1, \alpha r_2) \geq \{(\alpha - 1)/(\alpha + 1)\}^2 \{1 - 1/\alpha^{k+1}\}^2 \log M(r_1/\alpha, r_2/\alpha), \quad \alpha > 1,$$

and which is also incorrect in view of the above example.

To offer an improvement of (1.3), let us define first

$$G^+(r_1, r_2) = \exp \left\{ \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log^+ |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})| d\theta_1 d\theta_2 \right\},$$

where

$$\log^+ |f| = \max(\log |f|, 0);$$

also let

$$\lim_{r_1, r_2 \rightarrow \infty} \left\{ \begin{array}{l} \sup \\ \inf \end{array} \frac{\log \log G(r_1, r_2)}{\log(r_1, r_2)} \right\} = \left\{ \begin{array}{l} \rho_G, \\ \mu_G. \end{array} \right.$$

We have then the following result:

Theorem 2.1. *If $f(z_1, z_2)$ is an entire function, then for $R_1 > r_1, R_2 > r_2$,*

$$(2.1) \quad \log G^+(r_1, r_2) \leq \log^+ M(r_1, r_2) \leq \frac{R_1 + r_1}{R_2 + r_2} \frac{R_2 + r_2}{R_2 - r_2} \log G^+(R_1, R_2),$$

⁽¹⁾ Agarwal's claim that ρ and μ are nonintegral is irrelevant as far as the proof of (1.3) goes.

and

$$(2.2) \quad \lim_{r_1, r_2 \rightarrow \infty} \left\{ \begin{array}{l} \sup \\ \inf \end{array} \frac{\log \log g_{k, \lambda}(r_1, r_2)}{\log(r_1, r_2)} \right\} = \left\{ \begin{array}{l} \rho_G, \\ \mu_G, \end{array} \right.$$

for any k, λ such that $0 < k, \lambda < \infty$.

Proof. (2.1) immediately follows from Poisson's inequality in two variables.

For (2.2), we observe

$$(2.3) \quad \log g_{k, \lambda}(r_1, r_2) \leq \left\{ \frac{(k+1)(\lambda+1)}{r_1^{k+1} r_2^{\lambda+1}} \int_0^{r_1} \int_0^{r_2} x_1^k x_2^\lambda dx_1 dx_2 \right\} \log G(r_1, r_2) \\ = \log G(r_2, r_2).$$

Moreover

$$\log g_{k, \lambda}(R_1, R_2) \geq \frac{(k+1)(\lambda+1)}{R_1^{k+1} R_2^{\lambda+1}} \int_{r_1}^{R_1} \int_{r_2}^{R_2} x_1^k x_2^\lambda \log G(x_1, x_2) dx_1 dx_2 \\ \geq \frac{(R_1^{k+1} - r_1^{k+1})(R_2^{\lambda+1} - r_2^{\lambda+1})}{R_1^{k+1} R_2^{\lambda+1}} \log G(r_1, r_2).$$

Hence, putting $R_1 = \alpha r_1$, $R_2 = \beta r_2$; $\alpha, \beta > 1$,

$$(2.4) \quad \log g_{k, \lambda}(\alpha r_1, \beta r_2) \geq \frac{(\alpha^{k+1} - 1)(\beta^{\lambda+1} - 1)}{\alpha^{k+1} \beta^{\lambda+1}} \log G(r_1, r_2).$$

The inequalities (2.3) and (2.4) result in (2.2).

In this section we offer improvements of Theorem 2, (3.3) and Theorem 3(ii) and (iii) of Agarwal [1].

$$(3.1) \quad \lim_{r_1, r_2 \rightarrow \infty} \left\{ \begin{array}{l} \sup \\ \inf \end{array} \frac{\log g_{k, \lambda}(r_1, r_2)}{(r_1 r_2)^{\rho_G} \phi(r_1, r_2)} \right\} = \left\{ \begin{array}{l} p \\ q \end{array} \right. \quad (0 < q \leq p < \infty),$$

$$(3.2) \quad \lim_{r_1, r_2 \rightarrow \infty} \left\{ \begin{array}{l} \sup \\ \inf \end{array} \frac{\log G(r_1, r_2)}{(r_1 r_2)^{\rho_G} \phi(r_1, r_2)} \right\} = \left\{ \begin{array}{l} c \\ d \end{array} \right. \quad (0 < d \leq c < \infty),$$

where $\phi(r_1, r_2)$ is as mentioned by Agarwal. Then we have

Theorem 3.1. If $f(z_1, z_2)$ is an entire function having finite nonzero value ρ_G , i.e. $0 < \rho_G < \infty$, then

$$(3.3) \quad d(k+1)(\lambda+1)/\{(k+\rho_G+1)(\lambda+\rho_G+1)\} \leq q \leq p \\ \leq c(k+1)(\lambda+1)/\{(k+\rho_G+1)(\lambda+\rho_G+1)\}.$$

Proof. The proof is sketched as follows: Let $0 < \alpha, \beta < 1, 0 < r_1^0 < r_1, 0 < r_2^0 < r_2$. Then

$$\begin{aligned} & \log g_{k, \lambda}(r_1 + \alpha r_1, r_2 + \beta r_2) \\ & < \frac{A}{r_1^{k+1} r_2^{\lambda+1}} \\ & + \frac{1}{(\alpha + 1)^{k+1} (\beta + 1)^{\lambda+1}} \left\{ \left(\frac{r_1^0}{r_1} \right)^{k+1} \left[\left\{ 1 - \left(\frac{\beta r_2^0}{r_2} \right)^{\lambda+1} \right\} \log G(r_1^0, r_2) \right. \right. \\ & \qquad \qquad \qquad \left. \left. + \{(1 + \beta)^{\lambda+1} - 1\} \log G(r_1^0, r_2 + \beta r_2) \right] \right. \\ & \qquad \qquad \qquad \left. + \left(\frac{r_2^0}{r_2} \right)^{\lambda+1} \left[\left\{ 1 - \left(\frac{\alpha r_1^0}{r_1} \right)^{k+1} \right\} \log G(r_1, r_2^0) \right. \right. \\ & \qquad \qquad \qquad \left. \left. + \{(1 + \alpha)^{k+1} - 1\} \log G(r_1 + \alpha r_1, r_2^0) \right] \right. \\ & + \frac{(c + \epsilon)(k + 1)(\lambda + 1)}{r_1^{k+1} r_2^{\lambda+1}} \int_{r_1^0}^{r_1} \int_{r_2^0}^{r_2} x_1^{\rho_G + k} x_2^{\rho_G + \lambda} \phi(x_1, x_2) dx_1 dx_2 \\ & + \frac{(k + 1)((1 + \beta)^{\lambda+1} - 1)}{r_1^{k+1}} \int_{r_1^0}^{r_1} x_1^k \log G(x_1, r_2 + \beta r_2) dx_1 \\ & + \frac{(\lambda + 1)((1 + \alpha)^{k+1} - 1)}{r_2^{\lambda+1}} \int_{r_2^0}^{r_2} x_2^\lambda \log G(r_1 + \alpha r_1, x_2) dx_2 \\ & \left. + \{(1 + \alpha)^{k+1} - 1\} \{(1 + \beta)^{\lambda+1} - 1\} \log G(r_1 + \alpha r_1, r_2 + \beta r_2) \right\}. \end{aligned}$$

Next, observe that the seventh, eighth, and ninth lines of the foregoing inequality are respectively equal at most to the following estimates:

- (i) $\frac{(c + \epsilon)(k + 1)(\lambda + 1)}{(1 + \alpha)^{k+1} (1 + \beta)^{\lambda+1} (\rho_G + k + 1)(\rho_G + \lambda + 1)} \phi(r_1, r_2) (r_1 r_2)^{\rho_G},$
- (ii) $\frac{(k + 1)(1 + \beta)^{\rho_G} ((1 + \beta)^{\lambda+1} - 1)}{(1 + \alpha)^{k+1} (1 + \beta)^{\lambda+1} (\rho_G + k + 1)} \phi(r_1, r_2 + \beta r_2) (r_1 r_2)^{\rho_G},$
- (iii) $\frac{(\lambda + 1)(1 + \alpha)^{\rho_G} ((1 + \alpha)^{k+1} - 1)}{(1 + \alpha)^{k+1} (1 + \beta)^{\lambda+1} (\rho_G + \lambda + 1)} \phi(r_1 + \alpha r_1, r_2) (r_1 r_2)^{\rho_G}.$

Making use of these estimates in the corresponding terms of the above inequality, then dividing the complete expression by

$$(r_1 + \alpha r_1)^{\rho_G} (r_2 + \beta r_2)^{\rho_G} \phi(r_1 + \alpha r_1, r_2 + \beta r_2)$$

and finally proceeding to the limit as $r_1, r_2 \rightarrow \infty$, one gets the following: namely,

$$p \leq \frac{c}{(1 + \alpha)^{k+1} (1 + \beta)^{\lambda+1}} \left\{ \frac{(k + 1)(1 + \beta)^{\rho_G} ((1 + \beta)^{\lambda+1} - 1)}{\rho_G + k + 1} - \frac{(\lambda + 1)(1 + \alpha)^{\rho_G} ((1 + \alpha)^{k+1} - 1)}{\rho_G + \lambda + 1} + \frac{(k + 1)(\lambda + 1)}{(\rho_G + k + 1)(\rho_G + \lambda + 1)} \right\}.$$

But, α, β are arbitrary and so making $\alpha, \beta \rightarrow 0$, we find that the right-hand inequality in (3.3) is established.

Next, we have from (1.2) for all sufficiently large values of r_1 and r_2 ,

$$\begin{aligned} & \log g_{k, \lambda}(r_1 + \alpha r_1, r_2 + \beta r_2) \\ & > \frac{(d - \epsilon)(k + 1)(\lambda + 1)}{r_1^{k+1} r_2^{\lambda+1} (1 + \alpha)^{k+1} (1 + \beta)^{\lambda+1}} \int_{r_1^0}^{r_1} \int_{r_2^0}^{r_2} x_1^{\rho_G+k} x_2^{\rho_G+\lambda} \phi(x_1, x_2) dx_1 dx_2 \\ & + \frac{1}{(1 + \alpha)^{k+1} (1 + \beta)^{\lambda+1}} \left\{ \frac{(k + 1)((1 + \beta)^{\lambda+1} - 1)}{r_1^{k+1}} \int_{r_1^0}^{r_1} x_1^k \log G(x_1, r_2) dx_1 \right. \\ & \quad + \frac{(\lambda + 1)((1 + \alpha)^{k+1} - 1)}{r_2^{\lambda+1}} \int_{r_2^0}^{r_2} x_2^\lambda \log G(r_1, x_2) dx_2 \\ & \quad \left. + \{(1 + \alpha)^{k+1} - 1\} \{(1 + \beta)^{\lambda+1} - 1\} \log G(r_1, r_2) \right\}. \end{aligned}$$

Observe that

$$\log G(x_1, r_2) > (d - \epsilon)(x_1 r_2)^{\rho_G} \phi(x_1, r_2), \quad \text{for } x_1 > r_1^0,$$

$$\log G(r_1, x_2) > (d - \epsilon)(r_1 x_2)^{\rho_G} \phi(r_1, x_2), \quad \text{for } x_2 > r_2^0,$$

$$\log G(r_1, r_2) > (d - \epsilon)(r_1 r_2)^{\rho_G} \phi(r_1, r_2), \quad \text{for } r_1 > r_1^0, r_2 > r_2^0.$$

Hence

$$\begin{aligned} (1 + \alpha)^{\rho_G + k + 1} (1 + \beta)^{\rho_G + \lambda + 1} q &\geq \frac{(k + 1)(\lambda + 1)d}{(\rho_G + k + 1)(\rho_G + \lambda + 1)} + \frac{(k + 1)((1 + \beta)^{\lambda + 1} - 1)d}{k + \rho_G + 1} \\ &+ \frac{(\lambda + 1)((1 + \alpha)^{k + 1} - 1)d}{\lambda + \rho_G + 1} \\ &+ \{(1 + \alpha)^{k + 1} - 1\}\{(1 + \beta)^{\lambda + 1} - 1\}d, \end{aligned}$$

and making now $\alpha, \beta \rightarrow 0$, the left-hand inequality in (3.3) is obtained.

Invoking Theorem 2 and the technique of its proof as envisaged in [3] together with the method adopted in the proof of the above theorem, one may now easily prove the following:

Theorem 3.2. *If $f(z_1, z_2)$ is an entire function, such that $c = d$, then $p = q = (k + 1)(\lambda + 1)c / \{k + \rho_G + 1\}\{\lambda + \rho_G + 1\}$, and*

$$\lim_{r_1, r_2 \rightarrow \infty} \frac{\log g_{k, \lambda}(r_1, r_2)}{\log G(r_1, r_2)} = \frac{(k + 1)(\lambda + 1)}{(k + \rho_G + 1)(\lambda + \rho_G + 1)}.$$

Remark. The author is of the view that the results (3.4) and (3.5) of Agarwal may not be generalised in terms of $\log g_{k, \lambda}(r_1, r_2)$ when $k \neq \lambda$ and are arbitrary. Attempts towards these generalisations involve enormous calculations without yielding any solid solution.

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