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PRIMITIVE IDEALS OF C^*-ALGEBRAS ASSOCIATED WITH TRANSFORMATION GROUPS

BY

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ABSTRACT. We extend results of E. G. Effros and F. Hahn concerning their conjecture that if (G, Z) is a second countable locally compact transformation group, with G amenable, then every primitive ideal of the associated C^*-algebra arises as the kernel of an irreducible representation induced from an isotropy subgroup. The conjecture is verified if all isotropy subgroups lie in the center of G and either (a) the restriction of each unitary representation of G to some open subgroup contains a one-dimensional subrepresentation, or (b) G has an open abelian subgroup and orbit closures in Z are compact and minimal.

1. **Introduction.** E. G. Effros and F. Hahn have conjectured in [8, §7.4] that if (G, Z) is a second countable locally compact transformation group, with G amenable, then every primitive ideal of the associated C^*-algebra $\mathfrak{U}(G, Z)$ arises as the kernel of an irreducible representation induced from an isotropy subgroup. They have verified their conjecture for the case of a discrete group acting freely [8, Corollary 5.16], and in this paper we extend their results.

§3 contains preliminary results concerning positive-definite measures on groups. In §4 we verify the above conjecture when all isotropy subgroups lie in the center of G and the unitary part V (see §4) of each irreducible representation of $\mathfrak{U}(G, Z)$ satisfies (∗): the restriction of V to some open subgroup of G contains a one-dimensional subrepresentation. Each unitary representation of a group which is either totally disconnected or has a compact open abelian subgroup satisfies (∗).

In §5 we prove that if orbit closures are compact and minimal, isotropy subgroups are central and G has an open abelian subgroup, then every irreducible representation of $\mathfrak{U}(G, Z)$ weakly contains (in the sense of [9, p. 426]) a representation whose unitary part satisfies (∗). The results of §4 imply that in this case also, the conjecture of Effros and Hahn is true.

§6 contains examples and a brief discussion of the relevance to our problem

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of a generalized imprimitivity theorem, due to a number of authors.

Although the use of transformation groups to define both operator algebras and function algebras is well established (see, for example, [19, pp. 192–209], [20] and [5, pp. 310–321]), the explicit construction of an algebra $\mathfrak{U}(G, Z)$ whose representations correspond to the representations of a transformation group (G, Z), and the study in this context of induced representations, first appears in [8] and [11]. The construction and many results of [8] and [11] have been extended to the case of twisted group actions on objects more general than a locally compact space Z (see [4], [7], [10], [21] and [22]), but except for Corollary 5.16 of [8] and Theorem 5.15 of [22], both valid only for G discrete, little is known even in the transformation group case concerning those representations of the constructed algebra which are not obtainable as induced representations. Our aim in this paper is to obtain results concerning such representations. We purposely avoid the complications inherent in the more general situations and also the question of whether or not our results extend, and consider only transformation groups (G, Z) with both G and Z second countable locally compact Hausdorff spaces. All representations are on separable Hilbert space $\mathcal{H}$ and all representations of algebras are nondegenerate.

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2. **Notation and preliminaries.** If X is a second countable locally compact Hausdorff space, we denote by $\mathcal{K}(X)$ the continuous functions on X of compact support, with the inductive limit topology, and by $M(X)$ the dual space of Radon measures on X with the weak*-topology. For $x \in X$, $\delta_x \in M(X)$ is the probability measure on X concentrated at x.

$\mathcal{K}(G \times Z)$ is a topological *-algebra [8, p. 33], and $\nu \in M(G \times Z)$ is called positive-definite if $\nu(f^* * f) \geq 0$ for all $f \in \mathcal{K}(G \times Z)$. The closed convex cone $D(G \times Z)$ of positive-definite measures defines a partial ordering $\prec$ on $M(G \times Z)$. We identify G with the trivial transformation group $G \times Z_0$, where Z_0 is the one-point space, and write $D(G)$ for $D(G \times Z_0)$, etc.

For $g \in G$, $k \in \mathcal{K}(Z)$ and $f \in \mathcal{K}(G \times Z)$, we define $g \cdot k \in \mathcal{K}(Z)$ and $g \cdot f$, $k \cdot f \in \mathcal{K}(G \times Z)$ by

$$(g \cdot k)(\phi) = k(g^{-1}\phi), \tag{2.1}$$

$$(g \cdot f)(t, \phi) = f(g^{-1}t, g^{-1}\phi) \quad \text{and} \tag{2.2}$$

$$(k \cdot f)(t, \phi) = k(\phi)f(t, \phi), \qquad t \in G, \phi \in Z. \tag{2.3}$$

We will assume the reader is somewhat familiar with the basic results concerning C^*-algebras associated with transformation groups, as developed in [8], primarily §§1, 3, 4 and 5.

3. **Positive-definite measures on groups.** Let G be a locally compact group, H a closed normal subgroup, dg and dh fixed choices of left Haar measure on G and H, respectively, and Φ the map of $\mathcal{K}(G)$ into $\mathcal{K}(G/H)$ defined by

$$(3.1) \qquad (\Phi f)(tH) = \int_H f(th)\,dh, \qquad t \in G,\ f \in \mathcal{K}(G).$$

Φ is linear, continuous and onto, and there is a left Haar measure $d\alpha$ on G/H such that

$$(3.2) \qquad \int_G f(g)\,dg = \int_{G/H} (\Phi f)(\alpha)\,d\alpha \qquad [16, \S 33].$$

The following two lemmas generalize Lemmas 4.40 and 4.41 of [8]. Although their proofs are easy and they are probably well known, we have not seen them in the literature and we include them for completeness.

Lemma 3.3. *Φ is a *-algebra homomorphism of $\mathcal{K}(G)$ onto $\mathcal{K}(G/H)$.*

Proof. For each g in G, there is a unique positive real number $\theta(g)$ such that

$$\int_H f(ghg^{-1})\,dh = \theta(g)\int_H f(h)\,dh, \quad f \in \mathcal{K}(G),$$

and furthermore,

$$\Delta_{G/H}(gH) = \Delta_G(g)/\theta(g),$$

where Δ_G and $\Delta_{G/H}$ are the modular functions of G and G/H, respectively. The proof of the lemma involves a routine application of (3.1), (3.2), the above two formulae and Fubini's theorem, and we omit the details.

Corollary. *Φ^*, the map dual to Φ, is a continuous linear map of $M(G/H)$ into $M(G)$, and maps $D(G/H)$ into $D(G)$. In particular, $\Phi^*(d\alpha) = dg$ and $\Phi^*(\delta_{eH}) = dh$, considered as a measure on G.*

Lemma 3.4. *Let G be an amenable group. There exists a net of measures ν_α in $D(G)$ and positive scalars a_α such that $\nu_\alpha \prec a_\alpha \delta_e$ and $\nu_\alpha \rightarrow dg$ in $M(G)$.*

Proof. By the proof of [8, Lemma 4.41], we need only verify that for each $k \in \mathcal{K}(G)$, there exists $K > 0$ such that

$$\delta_e(k^* * l^* * l * k) \le K\delta_e(l^* * l), \text{ for all } l \in \mathcal{K}(G).$$

Since for any $h,\ l \in \mathcal{K}(G)$,

$$\delta_e(h^* * l) = \int_G \bar{h}(g) l(g)\,dg = \langle l, h\rangle \quad \text{in } L^2(G),$$

it suffices to show that the operator $T(k)$, defined on the dense subspace $\mathcal{K}(G)$ of $L^2(G)$ by $T(k)l = l * k$, is bounded in the L^2-norm. $T(k)$ has an integral representation as $\int_G k(g^{-1})R_g\,dg$, where $(R_g l)(t) = l(tg)$ for $l \in \mathcal{K}(G)$ and $t \in G$. Since $\|R_g\| = \Delta(g^{-1})^{1/2}$ and k is in $\mathcal{K}(G)$, we have

$$\|T(k)\| \leq \int_G |k(g^{-1})|\Delta(g^{-1})^{1/2}\,dg < \infty$$

and we are done.

Corollary. *Let G be an amenable group and H a closed normal subgroup. There exists a net of measures ν_α in $D(G)$ and positive scalars a_α such that $\nu_\alpha \prec a_\alpha dh$ (considered as a measure on G) and $\nu_\alpha \to dg$ in $M(G)$.*

Proof. Apply Lemma 3.4 to the amenable group G/H and then apply the Corollary to Lemma 3.3 to lift the resulting measures to measures on G.

Remark. We use the above Corollary in §4 to prove a weak containment relation between two representations. For this purpose alone, [13, Theorem 5.1] is more than sufficient. See also [9, §5].

4. **Primitive ideals in $\mathfrak{U}(G, Z)$.** The correspondence $L = \langle V, M\rangle$ between representations L of $\mathfrak{U}(G, Z)$ on a Hilbert space $\mathcal{H}$ and representations $\langle V, M\rangle$ of (G, Z) on $\mathcal{H}$ (see [8, pp. 36–37]) is completely determined by

$$(4.1) \qquad \langle L(f)x, y\rangle = \int_G \langle Mf(g, \cdot)V(g)x, y\rangle\,dg, \qquad f \in \mathcal{K}(G \times Z),\ x, y \in \mathcal{H}.$$

Each $p \in D(G \times Z)$ determines a representation which we shall denote, unless explicitly stated otherwise, by $L^p = \langle V^p, M^p\rangle$ on $\mathcal{H}^p$ (see [8, §4] for the construction). As in [8, §5], we compare the kernels of different representations by considering representations determined by positive-definite measures and using the following result [8, pp. 65–66]: if J is a closed two-sided ideal in $\mathfrak{U}(G, Z)$, then

$$D_J(G \times Z) = \{p \in D(G \times Z)\colon \text{kernel } L^p \supseteq J\}$$

is a closed convex cone in $M(G \times Z)$ and also a face of $D(G \times Z)$.

Throughout this section and the next, we let $L = \langle V, M\rangle$ be a representation of $\mathfrak{U}(G, Z)$ on a Hilbert space $\mathcal{H}$ such that

(4.2)
(a) $M(Q) = I$ for some quasi-orbit Q in Z [8, pp. 5–6],
(b) all points in Q have the same isotropy group H and H is central in G, and
(c) there is a character c on H with $V(h) = c(h)I$ for all $h \in H$.

Note that if L is a factor representation, (a) is automatically satisfied for some quasi-orbit Q. If (b) holds also then (c) is automatically satisfied. For any $\phi \in Q$, the positive-definite measure $c\,dh \times \delta_\phi$, defined by

$$c\,db \times \delta_\phi(f) = \int_H c(h) f(h, \phi)\, dh, \qquad f \in \mathcal{K}(G \times Z),$$

determines an irreducible induced representation of $\mathfrak{U}(G, Z)$ which we shall denote by $L^\phi = \langle V^\phi, M^\phi \rangle$ on $\mathcal{H}^\phi$. If $\eta \in Q$, then kernel $L^\phi =$ kernel L^η [8, Corollary 5.13]. Conditions (b) and (c) are a mild generalization of free action, and they serve to provide almost automatically an irreducible induced representation, L^ϕ, whose kernel we can try to compare with kernel L. We note that if Q is actually an orbit, then L is unitarily equivalent to a multiple of L^ϕ.

Theorem 4.3. *Let G be an amenable group and let $L = \langle V, M \rangle$ be an irreducible representation of $\mathfrak{U}(G, Z)$ on $\mathcal{H}$, satisfying* (4.2). *Then* kernel $L \supseteq$ kernel L^ϕ.

Proof. The proof of Theorem 5.14 of [8] can be repeated almost exactly. Simply replace the measure δ_e used there by dh (considered as a measure on G) and observe that (with the notation of [8, Theorem 5.14]) $dh \otimes p = c\,dh \times \alpha_x$. The reasoning in the second half of the proof is justified by the Corollary to Lemma 3.4.

Theorem 4.4. *Let $L = \langle V, M \rangle$ be a representation of $\mathfrak{U}(G, Z)$ on $\mathcal{H}$, satisfying* (4.2). *If there is an open subgroup K of G such that V restricted to K has a one-dimensional invariant subspace $\mathfrak{L}$, then* kernel $L^\phi \supseteq$ kernel L.

Remark. We assume neither that L is irreducible nor that G is amenable.

Proof. Let x_0 be a unit vector in $\mathfrak{L}$ and β the probability measure on Z defined by

$$\beta(A) = \langle M(A)x_0, x_0 \rangle, \qquad A \text{ Borel } \subseteq Z. \tag{4.5}$$

By (4.2), $\exists \eta \in Q \cap \operatorname{supp} \beta$. Since kernel $L^\phi =$ kernel L^η, we need to verify that $c\,dh \times \delta_\eta \in D_J(G \times Z)$, where $J =$ kernel L. By a standard separation theorem for closed convex sets in dual pairs [3, Chapitre IV, §1, n.° 3, Proposition 3], it suffices to show that if $\operatorname{Re}(c\,dh \times \delta_\eta)(f) > 0$ for $f \in \mathcal{K}(G \times Z)$, then $\operatorname{Re} p(f) > 0$ for some $p \in D_J(G \times Z)$. Accordingly, fix f as above, with $\operatorname{supp} f \subseteq A \times B$, A and B compact subsets of G and Z, respectively.

By (4.2) we may assume without loss of generality that $K \supseteq H$ and that therefore the continuous character d on K, given by

$$V(k)x_0 = d(k)x_0, \qquad k \in K, \tag{4.6}$$

equals c on H. By the dominated convergence theorem,

$$F(k, \xi) = d(k) \int_H c(h) f(kh, \xi)\, dh, \qquad k \in K,\ \xi \in Z, \tag{4.7}$$

defines a continuous function F on $K \times Z$. Furthermore, $F(kh, \xi) = F(k, \xi)$ for $h \in H$, so F may also be regarded as a continuous function on $K/H \times Z$. Since $\operatorname{Re}(c\,dh \times \delta_\eta)(f) = \operatorname{Re} F(e, \eta) > 0$, there exist open neighborhoods N_e and N_η of

e and η, respectively, in K and Z such that $\operatorname{Re} F > 0$ on $N_e \times N_\eta$. $A \cap (N_e H)'$ is a compact set in G disjoint from H (the isotropy subgroup of $\eta \in Q$) and an obvious compactness argument implies the existence of a neighborhood U of η such that $U \subseteq N_\eta$ and $gU \cap U = \emptyset$ for all $g \in A \cap (N_e H)'$. Choose $l \in \mathcal{K}(Z)$ such that $0 \le l \le 1$, supp $l \subseteq U$ and $l(\eta) = 1$. As p in $D(G \times Z)$, defined by

$$p(u) = \langle L(u)M(l)x_0,\ M(l)x_0\rangle, \qquad u \in \mathfrak{U}(G,\ Z),$$

determines a subrepresentation L^p of L, it follows that $p \in D_J(G \times Z)$. We now show that $\operatorname{Re} p(f) > 0$.

By (4.1),

$$p(f) = \int_G \langle Mf(g,\ \cdot)V(g)M(l)x_0,\ M(l)x_0\rangle\, dg.$$

As in the proof of Theorem 5.15 of [8], the above integrand can be rewritten (see (2.1)) as $\langle Mf(g, \cdot)M((g \cdot l)l)V(g)x_0,\ x_0\rangle$. Since supp $f \subseteq A \times B$ and supp$(g \cdot l)l \subseteq gU \cap U$, the integrand is zero outside of $N_e H$. Using this, the fact that $M(gh \cdot l) = M(hg \cdot l) = M(g \cdot l)$ for $h \in H$, $g \in G$ and $l \in \mathcal{K}(Z)$, and formulas (3.1) and (3.2), we have

$$p(f) = \int_{G/H} \int_H \chi_{N_e H}(gh)\langle Mf(gh,\ \cdot)M((g\ \cdot\ l)l)V(gh)x_0,\ x_0\rangle\, dh\, d\alpha.$$

Letting π denote the canonical map of G onto G/H, and recalling that $N_e H \subseteq K$, we see from the above, (4.5) and (4.6) that

$$\begin{aligned} p(f) &= \int_{\pi(N_e)} \int_H \int_Z d(g)c(h)f(gh,\ \xi)l(g^{-1}\xi)l(\xi)\, d\beta(\xi)\, dh\, d\alpha \\ &= \int_{\pi(N_e)} \int_Z F(g,\ \xi)l(g^{-1}\xi)l(\xi)\, d\beta(\xi)\, d\alpha. \end{aligned}$$

The second equality follows from Fubini's theorem and (4.7). Thus

$$\operatorname{Re}\ p(f) = \int_{\pi(N_e)} \int_Z (\operatorname{Re}\ F)(g,\ \xi)l(g^{-1}\xi)l(\xi)\, d\beta(\xi)\, d\alpha.$$

Since $\operatorname{Re} F > 0$ on $\pi(N_e) \times N_\eta$ and supp $l \subseteq U \subseteq N_\eta$, the above integrand is a continuous function on $G/H \times Z$, ≥ 0 everywhere and > 0 at $(\pi(e),\ \eta)) \in$ supp$(d\alpha \times d\beta)$. Thus $\operatorname{Re}\ p(f) > 0$ and we are done.

Corollary. *Let $(G,\ Z)$ be a transformation group with G amenable and all isotropy subgroups central. If G is either totally disconnected or has a compact open abelian subgroup, all primitive ideals of $\mathfrak{U}(G,\ Z)$ arise as the kernels of irreducible representations induced from isotropy subgroups. If G also acts freely and minimally on Z, $\mathfrak{U}(G,\ Z)$ is a simple C^*-algebra.*

Proof. Let G be totally disconnected, V a representation of G on the Hilbert space $\mathcal{H}$, x a unit vector in $\mathcal{H}$ and N a neighborhood of the identity in G so that $\forall n \in N$, $V(n)x \in \{y \in \mathcal{H}: \|y - x\| \leq \frac{1}{2}\}$, which is a closed convex set not containing 0. N contains a compact open subgroup K, and the norm-closed convex hull of $\{V(k)x : k \in K\}$ contains a vector, clearly not 0, which is left fixed by K. The rest follows obviously.

5. **The case of compact minimal orbit closures.** We assume throughout this section that $L = \langle V, M\rangle$ is an irreducible representation of $\mathfrak{U}(G, Z)$ on a Hilbert space $\mathcal{H}$ satisfying (4.2), that the quasi-orbit Q is compact (an equivalent condition is that the orbit closure of any point in Q be compact and minimal closed G-invariant) and finally that G has an open abelian subgroup A. We prove

Theorem 5.1. *With the above hypotheses,* kernel $L^{\Phi} \supseteq$ kernel L.

Remarks. If G is amenable, Theorems 4.3 and 5.1 imply that kernel $L =$ kernel L^{Φ}. We prove Theorem 5.1 by constructing, in the following sequence of lemmas, a representation L^r of $\mathfrak{U}(G, Z)$ such that kernel $L^r \supseteq$ kernel L and L^r satisfies all the hypotheses of Theorem 4.4.

For $a \in A$ and $f \in \mathcal{K}(G \times Z)$, define

$$(5.2) \qquad (T_a f)(t, \xi) = \Delta_G(a) f(a^{-1}ta, a^{-1}\xi), \qquad t \in G,\ \xi \in Z.$$

Lemma 5.3. *Each T_a extends uniquely to define a $*$-algebra automorphism of $\mathfrak{U}(G, Z)$, which we also denote by T_a. $a \to T_a$ is a continuous group homomorphism of A into the automorphism group of $\mathfrak{U}(G, Z)$, provided with the strong topology. For any representation $R = \langle W, P\rangle$ of $\mathfrak{U}(G, Z)$, $R(T_a u) = W(a)R(u) \cdot W(a^{-1})$, $a \in A$, $u \in \mathfrak{U}(G, Z)$.*

Proof. All the relevant facts may be verified easily enough for $f \in \mathcal{K}(G \times Z)$ first, and then extended to $\mathfrak{U}(G, Z)$ by continuity. We omit the details.

Remark. Lemma 5.3 holds for any closed subgroup A of G.

We also denote by T_a the natural action of A on the dual $\mathfrak{U}'$ (with the weak $*$-topology) of $\mathfrak{U}(G, Z)$: $(T_a p)(u) = p(T_{a^{-1}}u)$, $a \in A$, $u \in \mathfrak{U}(G, Z)$, $p \in \mathfrak{U}'$. The state space P of $\mathfrak{U}(G, Z)$, consisting of positive functionals of norm less than or equal to one, is clearly a compact convex A-invariant subset of $\mathfrak{U}'$.

Lemma 5.4. *Let J be a closed two-sided ideal in $\mathfrak{U}(G, Z)$, H a closed central subgroup of G, c a character on H and F a closed G-invariant subset of Z. Then*

$$S = 0 \cup \{q \in P: q \neq 0 \text{ and (a) kernel } L^q \supseteq J, \text{ (b) supp } M^q \subseteq F \text{ and (c) } V^q(h) = c(h)I, \text{ for all } h \in H\}$$

is a compact convex A-invariant subset of P. Furthermore, S_A and P_A, the set *of A-invariant elements of S and P, respectively, are compact convex sets, and S_A is a face of P_A.*

Proof. By the construction of representations from positive-definite measures [8, §4] it follows easily that (a), (b) and (c) are equivalent, respectively, to

(a′) $q(v^* * u^* * u * v) = 0$ for all $u \in J$, $v \in \mathfrak{U}(G, Z)$,

(b′) $q((k \cdot f)^* * (k \cdot f)) = 0$ for all $f \in \mathcal{K}(G \times Z)$ and $k \in \mathcal{K}(Z)$ with supp $k \subseteq F'$ (see (2.3)) and

(c′) $q[(h \cdot f - c(h)f)^* * (h \cdot f - c(h)f)] = 0$ for all $f \in \mathcal{K}(G \times Z)$ and $h \in H$ (see (2.2)).

With these reformulations, plus the fact that by Lemma 5.3 closed two-sided ideals in $\mathfrak{U}(G, Z)$ are A-invariant, verification that S is compact, convex and A-invariant, and that S_A is a face of P_A, involves only routine calculations,which we omit.

A priori, P_A may contain only 0. We have, however,

Lemma 5.5. *Let S be defined as in Lemma 5.4, with $J =$ kernel L, $F = Q$ and H and c as in (4.2). S_A contains a nonzero element.*

Proof. Let x be a unit vector in $\mathcal{H}$ and define $p(u) = \langle L(u)x, x\rangle$, $u \in \mathfrak{U}(G, Z)$. As L is irreducible, L is unitarily equivalent to L^p, and clearly $p \in S$. A is an abelian group and thus has a normalized left-invariant mean m [12, p. 5]. Since by Lemma 5.3 $a \longrightarrow p(T_a u)$ is a bounded continuous function on A for each $u \in \mathfrak{U}(G, Z)$, we have that q, defined by $q(u) = m(p(T_a u))$, $u \in \mathfrak{U}(G, Z)$, is an A-invariant positive functional on $\mathfrak{U}(G, Z)$. It can be readily verified that $q \in S$, so $q \in S_A$. To show $q \neq 0$, we construct an element $u \in \mathfrak{U}(G, Z)$ such that $p(u) = p(T_a u) \neq 0$, for all $a \in A$. A is open and $\|x\| = 1$, so there exists $l \in \mathcal{K}(G)$ with supp $l \subseteq A$ and $\int_G l(g)\langle V(g)x, x\rangle\, dg \neq 0$. Since Q is compact, there exists $h \in \mathcal{K}(Z)$ with $h \equiv 1$ on Q, and since supp $M \subseteq Q$ it follows that $M(h) = I$. $l \otimes h \in \mathcal{K}(G \times Z) \subseteq \mathfrak{U}(G, Z)$ and

$$\begin{aligned} p(T_a(l \otimes h)) &= \langle L(T_a(l \otimes h))x, x\rangle \\ &= \langle L(l \otimes h)V(a^{-1})x,\ V(a^{-1})x\rangle, \quad \text{by Lemma 5.3,} \\ &= \int_G l(g)\langle M(h)V(g)V(a^{-1})x,\ V(a^{-1})x\rangle\, dg, \quad \text{by (4.1),} \\ &= \int_G l(g)\langle V(g)x, x\rangle\, dg, \quad \text{since supp } l \subseteq A \text{ and } A \text{ is abelian,} \\ &= p(l \otimes h) \neq 0, \quad \text{by construction.} \end{aligned}$$

Thus $q \in S_A$, $q \neq 0$.

By Lemma 5.5 and the Kreĭn-Milman theorem, S_A contains a nonzero extreme point r, which is at the same time an extreme point of P_A. Since supp $M^r \subseteq Q$ and Q was assumed to be a minimal closed G-invariant set, it follows that supp $M^r = Q$, that is, M^r is actually concentrated on the quasi-orbit Q. Thus L^r satisfies (4.2) and kernel $L^r \supseteq$ kernel L. We now prove Theorem 5.1 by verifying that L^r satisfies the hypotheses of Theorem 4.4.

Lemma 5.6. *Let $L^r = \langle V^r, M^r\rangle$ be as above. The restriction of V^r to A contains a one-dimensional invariant subspace.*

Proof. Since r is A-invariant, one can define (as in Theorem 5.3 of [20]) a unitary representation T of A on $\mathcal{H}^r$ such that $L^r(T_a u) = T(a)L^r(u)T(a^{-1})$, $a \in A$, $u \in \mathfrak{U}(G, Z)$. Furthermore, there is a unit vector x in $\mathcal{H}^r$ such that $r(u) = \langle L^r(u)x, x\rangle$, $u \in \mathfrak{U}(G, Z)$, and also $T(a)x = x$ for all $a \in A$. As in [20, Theorem 5.3], the fact that r is an extreme point of P_A implies that the commutant of the algebra generated by $\{T(a), L^r(u): a \in A,\ u \in \mathfrak{U}(G, Z)\}$ reduces to the scalars. Since $V^r(a)x = V^r(a)T(a^{-1})T(a)x = V^r(a)T(a^{-1})x$, we shall be done once we verify that for each a in A, $V^r(a)T(a^{-1})$ commutes with every operator in the above algebra. By the construction of representations of $\mathfrak{U}(G, Z)$ from positive-definite measures [8, §4], it suffices to verify the following equalities for any $f, g \in \mathcal{K}(G \times Z)$:

$$a \cdot T_{a^{-1}}(f * g) = f * (a \cdot T_{a^{-1}} g), \qquad a \in A \text{ (see (2.2) and (5.2))}$$

and

$$a \cdot T_{a^{-1}} {}_b f = T_b(a \cdot T_{a^{-1}} f), \qquad a, b \in A \text{ (see (2.2) and (5.2)).}$$

These can be checked by routine calculations (the second equality alone uses commutativity of A), and we omit the details.

Corollary. *If G is an amenable group with an open abelian subgroup and acts freely and minimally on a compact space Z, then $\mathfrak{U}(G, Z)$ is a simple C^*-algebra.*

6. Examples.

Example 1. Let θ be an irrational real number and define an action of the reals R on the two-dimensional complex vector space C^2 by $t \cdot (z, w) = (e^{it}z, e^{i\theta t}w)$, $t \in R$, $z, w \in C$. Since the abelian group C^2 is self-dual, the C^*-algebra of the semidirect product group $R \times C^2$ (Mautner's 5-parameter non-type-I solvable Lie group) is isomorphic to the C^*-algebra $\mathfrak{U}(R, C^2)$ associated with the transformation group (R, C^2). Points of the form $(z, 0)$ or $(0, w)$ have compact orbits but points of the form (z, w) with $z \neq 0$ and $w \neq 0$ have compact minimal orbit closures $Q(z, w)$ which are not orbits. $Q(z, w) = \{(\alpha, \beta) \in C^2 : |\alpha| = |z|, |\beta| = |w|\}$, and R acts freely on each such set. By Theorems 4.3 and 5.1 all irreducible representations $L = \langle V, M\rangle$ of $\mathfrak{U}(R, C^2)$ satisfying $M(Q(z, w)) = I$ have the same kernel. It follows that the conjecture of Effros and Hahn is true for this example. A listing of all the primitive ideals shows easily that the primitive ideal space is T_1 in the Jacobson topology [6, §3].

Remarks. The primitive ideals of $\mathfrak{U}(R, C^2)$ may also be determined by combining Corollary 5.16 of [8] with the observation of Fell [10, pp. 144–145] that if $\pi: C^2 \rightarrow C$ is defined by $\pi(z, w) = z$ and an action of R on C is defined by $t \cdot z = e^{it}z$, then π is a continuous R-equivariant map and the image of every quasi-orbit in C^2 is an orbit in C. Since every nonzero point in C has $K = \{2n\pi: n \text{ an integer}\}$ as an isotropy subgroup, it follows (see [1, Chapter II, Theorem 2] and Theorem 16.2 of [10]) that every irreducible representation $L = \langle V, M\rangle$ of $\mathfrak{U}(R, C^2)$ with $M(Q(z, w)) = I$, $z \neq 0$, $w \neq 0$, is "induced" from an irreducible representation of $\mathfrak{U}(K, C^2)$. Here by "induced" we mean the construction given in [21, §3] and [10, §11], which is a generalization of Mackey's construction of induced representations of groups [17, pp. 539–540]. Since K is discrete all the primitive ideals of $\mathfrak{U}(K, C^2)$ are determined by Corollary 5.16 of [8]. The primitive ideals of $\mathfrak{U}(R, C^2)$ are then completely determined by the following theorem.

Theorem 6.1. *Let (G, Z) be a transformation group, H a closed subgroup of G, L and R two representations of $\mathfrak{U}(H, Z)$ with* kernel $L \supseteq$ kernel R. *Let* ind L *and* ind R *be the representations of $\mathfrak{U}(G, Z)$ "induced" from L and R. Then* kernel(ind L) $\supseteq$ kernel(ind R).

Proof. Theorem 1 of [2] and the discussion preceding it can be repeated with only minor modifications to yield the following: if $p \in D(H \times Z)$ and a measure $\tilde{p}$ is defined on $G \times Z$ by

$$\tilde{p}(f) = p(f\Delta_G^{1/2}\Delta_H^{-1/2}|_{H\times Z}), \qquad f \in \mathcal{K}(G \times Z),$$

then $\tilde{p} \in D(G \times Z)$ and $L^{\tilde{p}}$ is unitarily equivalent to ind L^p. Since every representation can be written as the direct sum of cyclic subrepresentations, which are automatically defined by positive-definite measures, the conclusion follows by a routine use of the results in [6, §3.4] and the proof of Theorem 5.11 of [8].

The remaining examples make use of the p-adics and the adeles. We refer the reader to [15]. For each prime integer p, we let Q_p denote the completion of the rationals under the p-adic valuation, and we denote by A the ring of adeles, formed as a restricted direct product of all the Q_p's. The additive group of A is a locally compact totally disconnected self-dual abelian group.

Example 2. Let G be the group of all upper triangular 3×3 matrices with entries in A and 1's along the diagonal. An element of G can be represented as a triple (a, b, c) and the group law is given by $(a, b, c)(a', b', c') = (a + a', b + b' + ac', c + c')$. G can be formed as the semidirect product of A with $A \times A$ (the direct product), the action of A on $A \times A$ being given by $a \cdot (b', c') = (b' + ac', c')$. Since $A \times A$ is self-dual, the C^*-algebra of the group G is isomorphic to the transformation group C^*-algebra $\mathfrak{U}(A, A \times A)$. The hypotheses of

Theorems 4.3 and 4.4 are satisfied by every irreducible representation of $\mathfrak{U}(A, A \times A)$ and the conjecture of Effros and Hahn is true for this example. We note that general results concerning the primitive ideal space of nilpotent groups have been obtained in [14].

Example 3. The group G of Example 2 acts on $M(A, 3)$, the space of all 3×3 matrices with entries in A, by matrix multiplication on the left. The behaviour of the isotropy subgroups is too irregular for our theorems to apply. However, let $x = (p_1, p_2, \cdots, p_n, \cdots) \in A$, where p_n is the nth prime. Then

$$X = \left\{ \begin{Vmatrix} \alpha & \beta & 0 \\ x & \gamma & 0 \\ 0 & x^2 & 0 \end{Vmatrix} : \alpha, \beta, \gamma \in A \right\}$$

is a minimal closed G-invariant subset of $M(A, 3)$ on which G acts freely. Theorems 4.3 and 4.4 imply that $\mathfrak{U}(G, X)$ is a simple C^*-algebra. Note that X is not compact and is not an orbit.

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