ABSTRACT. In this paper strictly irreducible *-representations of Banach *-algebras and the positive functionals associated with these representations are studied.

Introduction. Let \( A \) be a Banach *-algebra, and let \( \alpha \rightarrow \pi(\alpha) \) be a representation of \( A \) on a Hilbert space \( \mathcal{H} \). A subspace \( K \subset \mathcal{H} \) is \( \pi \)-invariant if \( \pi(\alpha)K \subset K \) for every \( \alpha \in A \). The representation \( \pi \) is irreducible if \( \pi \) is nonzero and the only closed \( \pi \)-invariant subspaces of \( \mathcal{H} \) are \( \mathcal{H} \) and \( \{0\} \). \( \pi \) is strictly irreducible if \( \pi \) is nonzero and the only \( \pi \)-invariant subspaces of \( \mathcal{H} \) are \( \mathcal{H} \) and \( \{0\} \). In the case where \( A \) is a \( B^* \)-algebra, R. V. Kadison proved the remarkable result that every irreducible *-representation of \( A \) is strictly irreducible [6, Theorem 1].

Aside from this theorem of Kadison, there are only a few minor isolated results concerning strictly irreducible *-representations of Banach *-algebras. In this paper we study strictly irreducible *-representations and certain positive functionals associated with these representations which we call strictly pure states (a positive functional \( \alpha \) on \( A \) is a strictly pure state if \( \alpha \) is a pure state and the *-representation of \( A \) determined by \( \alpha \) is strictly irreducible). We give necessary and sufficient conditions that a pure state of \( A \) be strictly pure in \( \S 2 \). In \( \S \S 3 \) and 4 some of the special properties of strictly pure states and strictly irreducible representations are presented. In \( \S 5 \) some examples of Banach *-algebras with the property that every irreducible *-representation is strictly irreducible are provided.

1. Notation and preliminaries. Throughout this paper \( A \) denotes a Banach *-algebra. A linear functional \( \alpha \) on \( A \) is positive if \( \alpha(\alpha^*\alpha) \geq 0 \) for all \( \alpha \in A \). When \( \alpha \) is a positive functional on \( A \), let

\[
M(\alpha) = \sup \left\{ \frac{|\alpha(a)|^2}{\alpha(a^*a)} \middle| a \in A, \ \alpha(a^*a) \neq 0 \right\}.
\]

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The set of all positive functionals $\alpha$ on $A$ with the properties $\alpha(a^*) = \overline{\alpha(a)}$ for all $a \in A$ and $M(\alpha) < +\infty$, we denote by $\mathcal{P}$. $\mathcal{P}_1$ is the set of all $\alpha \in \mathcal{P}$ with $M(\alpha) \leq 1$. Let $A_h$ be the real linear space of hermitian elements of $A$. $\mathcal{P}_1$ is a convex subset of $A_h^*$, the dual space of $A_h$, and $\mathcal{P}_1$ is compact in the weak $*$-topology on $A_h^*$ (see [4, Theorem (21.33), p. 328]). The extreme points of $\mathcal{P}_1$ are called pure states. For $\alpha \in \mathcal{P}$, the left kernel of $\alpha$, denoted $K_\alpha$, is the set of all $a \in A$ such that $\alpha(a^*a) = 0$. $K_\alpha$ is a closed left ideal of $A$. The quotient space $A - K_\alpha$ is a pre-Hilbert space in the inner-product $(a + K_\alpha, b + K_\alpha) = \alpha(b^*a)$. Let $H_\alpha$ denote the Hilbert space which is the completion of this pre-Hilbert space. A $*$-representation $\alpha \rightarrow \pi_\alpha(a)$ of $A$ on $H_\alpha$ is constructed by first defining $\pi_\alpha(a)(b + K_\alpha) = ab + K_\alpha$ for $b + K_\alpha \in A - K_\alpha$. Then $\pi_\alpha(a)$ is a bounded operator on $A - K_\alpha$ which extends uniquely to a bounded operator on $H_\alpha$ (also denoted by $\pi_\alpha(a)$). For details of this construction see the proof of Theorem (21.24) in [4]. It is a well-known theorem that $\alpha \in \mathcal{P}$ is a pure state of $A$ if and only if $M(\alpha) = 1$ and the $*$-representation $\pi_\alpha$ is irreducible on $H_\alpha$ [4, Theorem (21.34), p. 328]. We define $\alpha \in \mathcal{P}$ to be a strictly pure state of $A$ if $\alpha$ is a pure state of $A$ and $\alpha \rightarrow \pi_\alpha(a)$ is strictly irreducible on $H_\alpha$.

When $X$ is a normed linear space with norm $\| \cdot \|$ and $Y$ is a closed subspace of $X$, then the quotient norm $\| \cdot \|_q$ on the quotient space $X - Y$ is defined as usual by

$$\|x + Y\|_q = \inf \{\|x - y\| : y \in Y\}.$$ 

$H$ is always a Hilbert space and $B(H)$ is the algebra of all bounded operators on $H$.

2. Necessary and sufficient conditions for a pure state to be strictly pure.

When $\alpha \in \mathcal{P}$, the quotient space $A - K_\alpha$ is an inner product space with inner product defined by $(a + K_\alpha, b + K_\alpha) = \alpha(b^*a)$. Let $\|a + K_\alpha\|_2 = (\alpha(a^*a))^{1/2}$. We prove that a pure state $\alpha$ of $A$ is strictly pure if and only if $A - K_\alpha$ is complete in the norm $\|a + K_\alpha\|_2$.

**Theorem 2.1.** Assume that $A$ is a Banach $*$-algebra and that $\alpha$ is a pure state of $A$. Then $\alpha$ is a strictly pure state of $A$ if and only if $A - K_\alpha$ is complete in the norm $\|a + K_\alpha\|_2 = \alpha(a^*a)^{1/2}$. Also when $\alpha$ is a strictly pure state of $A$, then $K_\alpha$ is a modular maximal left ideal of $A$.

**Proof.** Assume first that $\alpha$ is a strictly pure state of $A$. By the construction of $H_\alpha$, $A - K_\alpha$ is an invariant subspace for $\pi_\alpha(a)$ whenever $a \in A$. Then $H_\alpha = A - K_\alpha$, so that $A - K_\alpha$ is complete in the norm $\|a + K_\alpha\|_2$.

Conversely assume that $A - K_\alpha$ is complete in this norm. We prove first that the two norms $\| \cdot \|_2$ and $\| \cdot \|_q$ are equivalent on $A - K_\alpha$. By the Closed Graph Theorem it suffices to prove that $\| \cdot \|_q$ dominates $\| \cdot \|_2$. This is exactly the same
as proving that the identity map \( a + K_\alpha \rightarrow a + K_\alpha \) is a continuous map from 
\((A - K_\alpha, \| \cdot \|_q)\) onto \((A - K_\alpha, \| \cdot \|_q)\). Again using the Closed Graph Theorem, it 
suffices to show that this map is closed. Therefore assume that \( \{a_n\} \subseteq A, a \in A, \)
\[ \|a_n + K_\alpha\|_q \rightarrow 0, \quad \text{and} \quad \|a_n - a\|_q + \|K_\alpha\|_q \rightarrow 0. \]
Then there exists a sequence \( \{k_n\} \subseteq K_\alpha \) such that \( \|a_n + k_n\| \rightarrow 0. \) Therefore \( \|a^*a_n + a^*k_n\| \rightarrow 0, \) and this implies
\[ \alpha(a^*a_n) = \alpha(a^*a_n + a^*k_n) \rightarrow 0. \] But also \( |\alpha(a^*(a_n - a))| = |(a_n - a + K_\alpha a + K_\alpha)| \)
\[ \leq \|a_n - a\|_q + \|K_\alpha\|_q \rightarrow 0. \] Therefore \( \alpha(a^*a) = 0, \) so that \( a + K_\alpha = 0. \)

Now define a functional \( \overline{\alpha} \) on the Hilbert space \( H_\alpha = A - K_\alpha \) by \( \overline{\alpha}(a + K_\alpha) = \alpha(a). \) Since \( K_\alpha \) is contained in the null space of \( \overline{\alpha}, \) \( \overline{\alpha} \) is well defined. Also,
\[ \|\overline{\alpha}\|^2 = \sup \left\{ \frac{|\alpha(a)|^2}{\alpha(a^*a)} \bigg| a \in A, \alpha(a^*a) \neq 0 \right\} = M(\alpha) = 1. \]
Since \( A - K_\alpha \) is a Hilbert space, there exists \( v \in A \) such that \( \overline{\alpha}(a + K_\alpha) = (a + K_\alpha, v + K_\alpha) = \alpha(v^*a) \) for all \( a \in A. \) Therefore \( \alpha(a) = \alpha(v^*a) \) for all \( a \in A. \)
Given any \( a \in A, \)
\[ \alpha((a(1 - v))^*(a(1 - v))) = \alpha(a^*a(1 - v)) - \alpha(v^*a*a(1 - v)) = 0. \]
Therefore \( A(1 - v) \subseteq K_\alpha \) so that \( K_\alpha \) is a modular left ideal. Let \( K \) be a maximal left ideal of \( A \) such that \( K_\alpha \subseteq K. \) Set \( M = \{b + K_\alpha \} b \in K \}. \) \( M \) is a proper \( \pi_\alpha \)-invariant subspace of \( H_\alpha = A - K_\alpha. \) Furthermore \( M \) is \( | \cdot |_2 \)-closed. Therefore by the 
result of the previous paragraph, \( M \) is \( | \cdot |_2 \)-closed. It follows that \( K_\alpha = K. \) Then since \( K_\alpha \) is a maximal modular left ideal of \( A, \) \( \pi_\alpha(A) \) acts strictly irreducibly on \( H_\alpha = A - K_\alpha. \)

Every Banach \(*\)-algebra \( A \) has an algebra pseudonorm called the Gelfand-
Naimark pseudonorm. We denote this pseudonorm by \( |a| \), \( a \in A. \) This pseudonorm
has the properties:

1. \( |a^*a| = |a|^2 \) for all \( a \in A. \)
2. \( |\alpha(a)| \leq M(\alpha)|a| \) whenever \( \alpha \in \mathcal{P}, a \in A. \)
3. The \(*\)-radical of \( A \) is the set of all \( a \in A \) such that \( |a| = 0. \) See [8, p. 
226] for the details of these results. We prove next that a pure state \( \alpha \) of \( A \) is
strictly pure if and only if \( |a + K_\alpha|_q = \inf \{ |a + k| \mid k \in K_\alpha \} \) is a complete norm on
\( A - K_\alpha. \)

**Theorem 2.2.** Let \( | \cdot | \) denote the Gelfand-Naimark pseudonorm on \( A. \) Then
a pure state \( \alpha \) of \( A \) is strictly pure if and only if \( |a + K_\alpha|_q \) is a complete norm
on \( A - K_\alpha. \)

**Proof.** For convenience we assume in the proof that \( A \) is reduced (i.e. the
\(*\)-radical of \( A \) is 0). This assumption can be made with no loss of generality.
In this case \( | \cdot | \) is a norm on \( A \) with the \( B^* \)-property by (1) and (3) above. Let
\( B \) denote the \( B^* \)-algebra which is the completion of \( A \) in the norm \( | \cdot |. \) Let \( \alpha \)
be a pure state of $A$. By (2) above $\alpha$ is $| \cdot |$-continuous. Therefore $\alpha$ has a 
unique extension $\tilde{\alpha}$ to $B$. It is easy to verify that $\tilde{\alpha}$ is a pure state of $B$.

Now assume that $\alpha$ is a strictly pure state of $A$. Let $\text{cl}(K_{\alpha})$ denote the 
$| \cdot |$-closure of $K_{\alpha}$ in $B$. If $\text{cl}(K_{\alpha}) \neq K_{\alpha}$, then by [8, Theorem (4.9.8), p. 251]
there exists a pure state $\tilde{\beta}$ of $B$ with $\text{cl}(K_{\alpha}) \subset K_{\tilde{\beta}}$ and $\tilde{\alpha} \neq \tilde{\beta}$. Let $\beta$ be the
restriction of $\tilde{\beta}$ to $A$. $K_{\alpha} \subset K_{\beta}$ and therefore $K_{\alpha} = K_{\beta}$. By a result we prove in
the next section, Theorem 3.2, it follows that $\alpha = \beta$. But then $\tilde{\alpha} = \tilde{\beta}$, a
contradiction. Therefore $\text{cl}(K_{\alpha}) = K_{\alpha}$. By Kadison's theorem $\tilde{\alpha}$ is a strictly pure
state of $B$. Then as noted in Theorem 2.1 there exists $M > 0$ such that

$$M \tilde{\alpha}(b^*b)^{1/2} \geq |b + K_{\alpha}|_q$$

for all $b \in B$.

Also using (2) above we have, for $a \in A$, $k \in K_{\alpha}$,

$$|a + K_{\alpha}|_2 = \alpha(((a + k)^*(a + k))^{1/2} \leq (a + k)^*(a + k)^{1/2} = |a + k|.$$

Therefore $|a + K_{\alpha}|_2 \leq |a + K_{\alpha}|_q$. Then for all $a \in A$,

$$M|a + K_{\alpha}|_2 = M \tilde{\alpha}(a^*a)^{1/2} \geq |a + K_{\alpha}|_q = |a + K_{\alpha}|_q \geq |a + K_{\alpha}|_2.$$

The norm $|a + K_{\alpha}|_2$ is complete on $A - K_{\alpha}$ by Theorem 2.1. Therefore $|a + K_{\alpha}|_q$
is a complete norm on $A - K_{\alpha}$.

Conversely assume that $|a + K_{\alpha}|_q$ is a complete norm on $A - K_{\alpha}$. Given
$b \in K_{\alpha}$, choose $\{b_n\} \subset A$ such that $|b_n - b| \to 0$. Then $(b_n - b) + K_{\alpha} \to 0$
as $n, m \to +\infty$. Therefore there exists $a \in A$ such that $(b_n - a) + K_{\alpha} \to 0$.

Choose $\{k_n\} \subset K_{\alpha}$ such that $|b_n - a + k_n| \to 0$. Then $|b - a + k_n| \to 0$, so that
$b - a \in \text{cl}(K_{\alpha})$. It follows that $a^*b - a^*a \in \text{cl}(K_{\alpha})$, and therefore that $\tilde{\alpha}(a^*b - a^*a)$
$= 0$. But $\tilde{\alpha}(a^*b) = 0$, since $b \in K_{\alpha}$. Then $\alpha(a^*a) = 0$, so that $a \in K_{\alpha}$. Therefore
$b \in \text{cl}(K_{\alpha})$. We have now shown that $K_{\alpha} = \text{cl}(K_{\alpha})$. We have $|a + K_{\alpha}|_q \geq |a + K_{\alpha}|_2$
for all $a \in A$ just as before. By Kadison's theorem $\tilde{\alpha}$ is a strictly pure state of
$B$. Then by Theorem 2.1 there exists $m > 0$ such that $|b + K_{\alpha}|_2 > m|b + K_{\alpha}|_q$
for all $b \in B$. Therefore for all $a \in A$,

$$|a + K_{\alpha}|_q \geq |a + K_{\alpha}|_2 = |a + K_{\alpha}|_2 \geq m|a + K_{\alpha}|_q = m|a + K_{\alpha}|_q.$$

It follows that $|a + K_{\alpha}|_2$ is a complete norm on $A - K_{\alpha}$, and therefore $\alpha$ is
strictly pure by Theorem 2.1.

3. Results concerning strictly pure states and strictly irreducible represen-
tations. The relationship between a pure state and its left kernel has never been
fully explored in a general Banach $*$-algebra. In fact to our knowledge none of
the following questions have been answered when $A$ is a Banach algebra with
hermitian involution.

**Question 1.** If $\alpha$ is a pure state of $A$, is $K_{\alpha}$ a maximal left ideal of $A$?

**Question 2.** If $\alpha$ and $\beta$ are pure states of $A$ and $K_{\alpha} = K_{\beta}$, does $\alpha = \beta$?
Question 3. If \( \alpha \in \mathcal{P} \), \( M(\alpha) = 1 \), and \( K_\alpha \) is a maximal left ideal of \( A \), is \( \alpha \) a pure state of \( A \)?

We add to this list another closely related question.

Question 4. If \( \alpha \to \pi(\alpha) \) and \( \alpha \to \gamma(\alpha) \) are two algebraically equivalent irreducible \( * \)-representations of \( A \) on respective Hilbert spaces, are \( \pi \) and \( \gamma \) necessarily unitarily equivalent?

The answer to all these questions is affirmative when \( A \) is a \( B^* \)-algebra. In this section we deal with special cases of these questions. To begin with, Theorem 2.1 states that when \( \alpha \) is a strictly pure state of \( A \), then \( K_\alpha \) is a modular maximal left ideal of \( A \). This answers Question 1 in the case when \( \alpha \) is strictly pure.

Next we prove a result which easily settles Question 2 if \( \alpha \) or \( \beta \) is strictly pure. Kadison proves in [6] that when \( \alpha \) is a pure state of a \( B^* \)-algebra, then \( \mathcal{H}(\alpha) = K_\alpha + K_\alpha^* \) where \( \mathcal{H}(\alpha) \) is the null space of \( \alpha \). We have the following generalization.

Proposition 3.1. If \( \alpha \) is a strictly pure state of \( A \), then \( \mathcal{H}(\alpha) = K_\alpha + K_\alpha^* \)

**Proof.** Since \( M(\alpha) = 1 \), then \( |\alpha(a)|^2 \leq \alpha(a^*a) \) for all \( a \in A \). Therefore \( K_\alpha \subset \mathcal{H}(\alpha) \), and it follows that \( K_\alpha + K_\alpha^* \subset \mathcal{H}(\alpha) \). Now we prove the reverse inclusion.

By Theorem 2.1, \( K_\alpha \) is a modular left ideal of \( A \). Therefore there exists \( u \in A \) such that \( A(1 - u) \subset K_\alpha \). When \( a \in \mathcal{H}(\alpha) \), then \( a^* \in \mathcal{H}(\alpha) \), and \( (u + K_\alpha, a + K_\alpha) = \alpha(a^*u) = \alpha(a^*u - a^*) = 0 \). Thus \( u + K_\alpha \) is orthogonal to \( a + K_\alpha \) in \( A - K_\alpha = \mathcal{H}(\alpha) \). \( \pi_\alpha(A) \) is a \( * \)-subalgebra of \( \mathcal{B}(\mathcal{H}_\alpha) \) which acts strictly irreducibly on \( \mathcal{H}_\alpha \). Let \( B \) be the closure of \( \pi_\alpha(A) \) in the operator norm. By the transitivity theorem [3, Théorème (2.8.3)] there exists \( \alpha \in B \), \( T = T^* \), such that \( T(u + K_\alpha) = 0 \) and \( T(a + K_\alpha) = a + K_\alpha^* \). Then there exists \( \{v_n\} \subset A \) such that \( v_n = v_n^* \) for all \( n \) and \( \|v_n a - T\| \to 0 \) where \( |\cdot| \) denotes the operator norm. Therefore \( \|v_n (u + K_\alpha)\|_2 \to 0 \) and \( \|(v_n^* a - a) + K_\alpha)\|_2 \to 0 \). Also \( v_n = v_n^* (1 - u) + v_n u \) and \( v_n (1 - u) \in K_\alpha \) for all \( n \). Then \( \|v_n + K_\alpha\|_2 \to 0 \), and finally \( \|a^*v_n + K_\alpha\|_2 \to 0 \). From the proof of Theorem 2.1 it follows that \( \|a^*v_n + K_\alpha\|_q \to 0 \) and \( \|(v_n a - a) + K_\alpha\|_q \to 0 \).

Assume for the moment the \( * \) is continuous on \( A \). There exists \( \{k_n\}, \{j_n\} \subset K_\alpha \) such that \( \|a^*v_n - k_n\| \to 0 \) and \( \|(a - v_n a) - j_n\| \to 0 \). Then \( \|a - (j_n + k_n^*)\| \leq \|v_n a - k_n\|^q + \|(a - v_n a) - j_n\|^q \to 0 \). Therefore in this case \( \mathcal{H}(\alpha) = K_\alpha + K_\alpha^* \).

In the general case, let \( P_\alpha \) be the kernel of the representation \( \pi_\alpha \). \( A/P_\alpha \) is a semisimple Banach \( * \)-algebra. Note that \( P_\alpha \subset K_\alpha \cap K_\alpha^* \). Define \( \alpha_0 \) on \( a + P_\alpha \in A/P_\alpha \) by \( \alpha_0(a + P_\alpha) = \alpha(a) \). Then \( \alpha_0 \) is a strictly pure state of \( A/P_\alpha \). By Johnson’s theorem [5, Theorem 2] the involution on \( A/P_\alpha \) is continuous. Therefore \( \mathcal{H}(\alpha_0) = K_{\alpha_0} + K_{\alpha_0}^* \) by our previous argument. Then when \( a \in \mathcal{H}(\alpha_0) \), there exists \( \{k_n\}, \{j_n\} \subset K_\alpha \) such that \( \|(a - (k_n + j_n^*)) + P_\alpha\|_q \to 0 \). Then there exists
\{p_n\} \subset P_\alpha such that \|a - (k_n + p_n + j^*_n)\| \to 0. This proves the proposition.

We are now in a position to answer Question 2 affirmatively when \(\alpha\) is assumed to be a strictly pure state of \(A\).

**Theorem 3.2.** Let \(\alpha\) be a strictly pure state of \(A\). Assume that \(\beta \in \mathcal{P}\), \(M(\beta) = 1\), and \(K_\alpha = K_\beta\). Then \(\alpha = \beta\).

**Proof.** \(K_\beta + K_\beta^* \subset \mathfrak{H}(\beta)\). Therefore

\[
\mathfrak{H}(\alpha) = \overline{K_\alpha + K_\alpha^*} = \overline{K_\beta + K_\beta^*} \subset \mathfrak{H}(\beta).
\]

It follows that there is a scalar \(\lambda > 0\) such that \(\alpha = \lambda \beta\). Then \(1 = M(\alpha) = \lambda M(\beta) = \lambda\).

The next theorem answers Question 4 in a special case.

**Theorem 3.3.** Assume that \(K\) and \(K\) are Hilbert spaces, and \(\pi \to \pi(\alpha)\) and \(\gamma \to \gamma(\alpha)\) are strictly irreducible *-representations of \(A\) on \(K\) and \(K\) respectively. Then if \(\pi\) and \(\gamma\) are algebraically equivalent, then \(\pi\) and \(\gamma\) are unitarily equivalent.

**Proof.** By hypothesis there exists a linear operator \(V\) which maps \(K\) in a one-to-one manner onto \(K\) with the property that \(V^{-1}\pi(\alpha) V = \gamma(\alpha)\) for all \(\alpha \in A\). Take \(\xi \in K\) with \(\|\xi\| = 1\), and set \(\alpha(\alpha) = (\gamma(\alpha) \xi, \xi)\) for \(\alpha \in A\). By [8, Lemma (4.5.8), p. 217] the representation \(\gamma\) is unitarily equivalent to \(\pi_\alpha\) on \(K_\alpha\). Also \(M(\alpha) = \|\xi\|^2 = 1\) by [4, Theorem (21.25), p. 323]. Then \(\alpha\) is a strictly pure state of \(A\) by [4, Theorem (21.34), p. 328]. Now set \(\eta = V(\xi)/\|V(\xi)\|\). Define \(\beta(\alpha) = (\pi(\alpha) \eta, \eta)\) for all \(\alpha \in A\). By the same argument as just applied to \(\alpha\), \(\beta\) is a strictly pure state of \(A\), and \(\pi_\beta\) is unitarily equivalent to \(\pi\). Then

\[
a \in K_\alpha \iff \gamma(\alpha) \xi = 0 \iff \gamma(\alpha)(V^{-1}(\eta)) = 0
\]

\[
\iff V^{-1}(\pi(\alpha)(\eta)) = 0 \iff \pi(\alpha) \eta = 0 \iff a \in K_\beta.
\]

Thus \(K_\alpha = K_\beta\), and it follows from Theorem 3.2 that \(\alpha = \beta\). Then \(\pi_\alpha = \pi_\beta\), so that \(\pi\) and \(\gamma\) are unitarily equivalent.

To conclude this section we consider an answer to Question 3 when \(A\) has a very special property. We hypothesize that every maximal left ideal of \(A\) is the left kernel of a strictly pure state of \(A\). In this case assume that \(\alpha\) is as in Question 3, that is, \(\alpha \in \mathcal{P}\), \(M(\alpha) = 1\), and \(K_\alpha\) is a maximal left ideal of \(A\). By the special hypothesis on \(A\) there is a strictly pure state \(\beta\) of \(A\) such that \(K_\beta = K_\alpha\). Then by Theorem 3.2, \(\alpha = \beta\). We state this result as a proposition.

**Proposition 3.4.** Assume that every maximal (modular) left ideal of \(A\) is the left kernel of a strictly pure state of \(A\). When \(\alpha \in \mathcal{P}\), \(M(\alpha) = 1\), and \(K_\alpha\) is a maximal (modular) left ideal of \(A\), then \(\alpha\) is a strictly pure state of \(A\).
4. Irreducible representations which are similar to *-representations. Let 
\( a \to \pi(a) \) be a strictly irreducible representation (but not necessarily a *-representation) of \( \mathcal{A} \) on a Hilbert space \( \mathcal{H} \). If \( \xi \in \mathcal{H}, \xi \neq 0 \), then a straightforward algebraic argument proves that \( K_\xi = \{ a \in \mathcal{A} | \pi(a)\xi = 0 \} \) is a modular maximal left ideal of \( \mathcal{A} \). We show in the next theorem that when \( K_\xi \) is the left kernel of a strictly pure state \( \alpha \) of \( \mathcal{A} \), then \( \pi \) is similar to a *-representation of \( \mathcal{A} \) on \( \mathcal{H} \).

**Theorem 4.1.** Let \( a \to \pi(a)\xi \) and \( K_\xi \) be as above. Assume that \( \alpha \) is a strictly pure state of \( \mathcal{A} \) with \( K_\alpha = K_\xi \). Then there exists a *-representation \( a \to \rho(a) \) of \( \mathcal{A} \) on \( \mathcal{H} \) and a positive operator \( V \in \mathcal{B}(\mathcal{H}) \) such that, for all \( a \in \mathcal{A} \),

\[
\pi(a) = V^{-1} \rho(a) V.
\]

**Proof.** Since \( \pi \) is strictly irreducible, \( a \to \pi(a)\xi \) is a linear map from \( \mathcal{A} \) onto \( \mathcal{H} \). We define a sesquilinear form \([\cdot, \cdot]\) on \( \mathcal{H} \times \mathcal{H} \) by

\[
[\pi(a)\xi, \pi(b)\xi] = \alpha(b^*a),
\]

for all \( a, b \in \mathcal{A} \). Whenever \( c \in K_\alpha \) and \( d \in \mathcal{A} \), then \( \alpha(d^*c) = 0 \). This implies that \([\cdot, \cdot]\) is well defined.

Next we prove that \([\cdot, \cdot]\) is a bounded form. By a theorem of B. E. Johnson [5, Theorem 1, p. 537], \( \pi \) is a continuous map of \( \mathcal{A} \) into \( \mathcal{B}(\mathcal{H}) \). If \( k \in K_\xi, a \in \mathcal{A}, \)

\[
||\pi(a)\xi|| = ||\pi(a + k)\xi|| \leq ||\pi|| ||\xi|| ||a + k||.
\]

Therefore for any \( a \in \mathcal{A} \),

(1) \[
||\pi(a)\xi|| \leq ||\pi|| ||\xi|| ||a + K_\xi||_q.
\]

Then by the Closed Graph Theorem there exists \( N > 0 \) such that, for all \( a \in \mathcal{A} \),

(2) \[
||a + K_\xi||_q \leq N ||\pi(a)\xi||.
\]

As shown in the proof of Theorem 2.1, the norms \( ||\cdot||_2 \) and \( ||\cdot||_q \) are equivalent on \( \mathcal{A} - K_\alpha \). In particular there exists \( J > 0 \) such that \( ||a + K_\alpha||_2 \leq J ||a + K_\alpha||_q \) for all \( a \in \mathcal{A} \). Therefore for all \( a, b \in \mathcal{A} \),

(3) \[
|\alpha(b^*a)| = ||(a + K_\alpha, b + K_\alpha)|| \leq J^2 ||a + K_\alpha||_q ||b + K_\alpha||_q.
\]

Then combining (2) and (3) we have

\[
|[\pi(a)\xi, \pi(b)\xi]| = |\alpha(b^*a)| \leq J^2 ||a + K_\alpha||_q ||b + K_\alpha||_q \leq J^2 N^2 ||\pi(a)\xi|| ||\pi(b)\xi||.
\]

This proves that \([\cdot, \cdot]\) is bounded on \( \mathcal{H} \times \mathcal{H} \).

The form \([\cdot, \cdot]\) is a symmetric, positive definite, bounded sesquilinear form on \( \mathcal{H} \times \mathcal{H} \). Therefore there exists an operator \( U \in \mathcal{B}(\mathcal{H}) \) such that \( U = U^* \), \( U \geq 0 \), and \([\phi, \psi] = (U\phi, \psi)\), when \( \phi, \psi \in \mathcal{H} \).

By (3), for all \( a \in \mathcal{A} \),

\[
||a + K_\alpha||_2^2 = \alpha(a^*a) \leq J^2 (||a + K_\alpha||_q)^2.
\]
By the proof of Theorem 2.1 there exists \( P > 0 \) such that, for all \( a \in A \),
\[
\| a + K_a \|_q \leq P \| a + K_a \|_2.
\]

Given \( a \in A \), set \( \psi = \pi(a)\xi \). Then by (1),
\[
\| \psi \| = \| \pi(a)\xi \| \leq \| \pi \| \| \xi \| \| a + K_a \|_q.
\]

Set \( M = \| \pi \| \| \xi \| P \). Then
\[
\| \psi \|^2 \leq M^2(\| a + K_a \|_2)^2 = M^2\alpha(a^*a) = M^2[\pi, \psi].
\]
Therefore
\[
\| \psi \|^2 \leq M^2[\psi, \psi] = M^2(\psi, \psi) \leq M^2\| \psi \| \| \psi \|.
\]

Finally \( \| \psi \| \leq M^2\| \psi \| \), and this proves that \( U^{-1} \in \mathcal{B}(\mathcal{H}) \).

Now set \( V = V^{1/2} \). Then \( [\phi, \psi] = (V\phi, V\psi) \) for all \( \phi, \psi \in H \). Let \( \rho(a) = V\pi(a)V^{-1} \), \( a \in A \). Given \( \psi_1, \psi_2 \in H \), there exists \( \phi_1, \phi_2 \in H \) and \( a_1, a_2 \in A \) such that
\[
\psi_i = V\phi_i \quad \text{and} \quad \phi_i = \pi(a_i)\xi, \quad i = 1, 2.
\]

Then
\[
(\rho(a)\psi_1, \psi_2) = (V\pi(a)V^{-1}V\phi_1, V\phi_2)
= [\pi(a)\phi_1, \phi_2] = [\pi(a)\pi(a_1)\xi, \pi(a_2)\xi]
= \alpha(a^*(a_1)) = \alpha((a^*a_2)^*a_1)
= [\pi(a_1)\xi, \pi(a)\pi(a_2)\xi] = [\phi_1, \pi(a^*)\phi_2]
= (V\phi_1, V\pi(a^*)\phi_2) = (V\phi_1, V\pi(a^*)V^{-1}V\phi_2) = (\psi_1, \rho(a^*)\psi_2).
\]

Therefore \( \rho(a^*) = \rho(a)^* \) for all \( a \in A \) which completes the proof of the theorem.

**Corollary 4.2.** Assume that every modular maximal left ideal of \( A \) is the left kernel of a strictly pure state of \( A \). Let \( a \rightarrow \pi(a) \) be a strictly irreducible representation of \( A \) on a Hilbert space \( H \). Then there exists a \(*\)-representation \( a \rightarrow \rho(a) \) of \( A \) on \( H \) and a positive operator \( V \in \mathcal{B}(H) \) such that, for all \( a \in A \),
\[
\pi(a) = V^{-1}\rho(a)V.
\]

5. Some examples. When \( A \) is \( B^*\)-algebra, then \( A \) has the following two properties:

(I) Every pure state of \( A \) is strictly pure.

(II) Every modular maximal left ideal of \( A \) is the left kernel of a strictly pure state of \( A \).

Also when \( G \) is a compact topological group and \( 1 \leq p < +\infty \), then
A = L^p(G) (or C(G), the continuous functions on G) has properties (I) and (II). Here the multiplication is, as usual, convolution. All the irreducible *-representations of A in this case are finite dimensional. In this section we present two examples of algebras which have properties (I) and (II), but which are not in general B*-algebras, and which need not in general have any finite dimensional *-representations.

Example 5.1. Let A be a Banach algebra which is also a dense *-ideal in a B*-algebra B. Any full Hilbert algebra is a particular example of such a Banach algebra; see [1]. Assume that \( a \rightarrow \pi(a) \) is an irreducible *-representation of A on a Hilbert space \( \mathcal{H} \). Then as shown in [1, Proposition 4.1] \( \pi \) extends uniquely to a *-representation \( b \rightarrow \widehat{\pi}(b) \) of B on \( \mathcal{H} \). Therefore by Kadison's theorem \( \widehat{\pi}(B) \) acts strictly irreducibly on \( \mathcal{H} \). Since A is a dense ideal of B, \( \pi(A) = \widehat{\pi}(A) \) is a non-zero ideal in \( \widehat{\pi}(B) \). Given \( \xi \in \mathcal{H} \), \( \pi(A)\xi \) is a \( \pi(B)\)-invariant subspace of \( \mathcal{H} \). Therefore \( \pi(A)\xi = \mathcal{H} \), so that \( a \rightarrow \pi(a) \) is strictly irreducible on \( \mathcal{H} \). It follows that A has property (I).

Now assume that \( M \) is a modular maximal left ideal of A. Then there exists \( u \in A \) such that \( A(1 - u) \subseteq M \). Let \( N = \{ b \in B | bu \in M \} \). \( N \) is a left ideal of B and \( M = N \cap A \). Furthermore if \( b \in B \), \( b(1 - u)u = bu(1 - u) \in M \) since \( bu \in A \). Therefore \( N \) is a proper modular left ideal of B. By [8, Theorem (4.9.8), p. 251] there exists \( \tilde{\alpha} \) a pure state of B with \( N \subseteq K_{\tilde{\alpha}} \). Then \( M = K_{\tilde{\alpha}} \cap A \). It follows that \( \alpha \), the restriction of \( \tilde{\alpha} \) to A, is a strictly pure state of A with \( K_\alpha = M \). We have shown that A has property (II).

Example 5.2. Assume that \( \Omega \) is a compact Hausdorff space and B is a B*-algebra with identity \( e \). Let \( C(\Omega, B) \) be the algebra of all continuous B-valued functions on \( \Omega \). \( C(\Omega, B) \) is a B*-algebra with identity. Assume that A is a Banach algebra which is a *-subalgebra of \( C(\Omega, B) \) containing the identity. We also assume that A has the properties:

1. Given \( \omega \in \Omega \) and \( b \in B \), there exists \( f \in A \) such that \( f(\omega) = b \).
2. \( f \in A \) is left invertible in A if and only if \( f(\omega) \) is invertible in B for all \( \omega \in \Omega \).

We mention a specific example of such an algebra A: Let \( \Omega \) be the interval \([0, 2\pi]\) with 0 and \( 2\pi \) identified and with the usual topology. Let B be any B*-algebra with identity. We define A to be the algebra of all functions of the form

\[
f(t) = \sum_{n = -\infty}^{+\infty} a_n e^{int}
\]

where \( t \in \Omega \) and \( \{a_n\} \) is any sequence in B such that \( \sum_{n = -\infty}^{+\infty} \|a_n\| < +\infty \). When \( f(t) = \sum_{n = -\infty}^{+\infty} a_n e^{int} \), let \( \|f\| = \sum_{n = -\infty}^{+\infty} \|a_n\|. \) The algebra A is discussed by

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Bochner and Phillips in [2]. That \( A \) has property (2) above is the assertion of [2, Theorem 1, p. 409]. The rest of the required properties of \( A \) are easily verified.

Now assume that \( A \) is any Banach *-subalgebra of \( C(\Omega, B) \) which contains the identity and satisfies (1) and (2). When \( \omega \in \Omega \) and \( N \) is a maximal left ideal of \( B \), we define

\[
K(\omega, N) = \{ f \in A \mid f(\omega) \in N \}.
\]

It is not difficult to see that \( K(\omega, N) \) is a maximal left ideal of \( A \). We prove the converse of this. Assume that \( M \) is a maximal left ideal of \( A \). For any \( \omega \in \Omega \), \( M(\omega) = \{ f(\omega) \mid f \in M \} \) is a left ideal of \( B \). Suppose that \( M(\omega) = B \) for all \( \omega \in \Omega \).

Then for each \( \omega \in \Omega \), we can choose a function \( g_\omega \in M \) such that \( g_\omega(\omega) = e \).

Therefore there exists an open set \( U_\omega \) in \( \Omega \) such that \( \omega \in U_\omega \) and \( g_\omega(y) \) is invertible in \( B \) for all \( y \in U_\omega \). Then \( (g_\omega^* g_\omega)(y) \) is invertible in \( B \) for all \( y \in U_\omega \).

Choose a finite cover \( U_{\omega_1}, \ldots, U_{\omega_\xi} \) for \( \Omega \). Set \( f = \sum_{k=1}^{\xi} g_\omega^* g_\omega \in M \). When \( b_k \in B, b_k \geq 0, 1 \leq k \leq \xi \), and \( b_j \) is invertible for some \( j \), then \( h_1 + \cdots + h_\xi \) is invertible (this is easy to verify when the \( b_k \) are positive operators on a Hilbert space, since the lower bound of the numerical range of the sum \( h_1 + \cdots + h_\xi \) is greater or equal to the lower bound of the numerical range of \( b_j \)). But then for all \( y \in \Omega \), \( f(y) \) is invertible in \( B \). By (2), \( f \) is then invertible in \( A \), which contradicts the fact that \( f \in M \). It follows that for some \( \omega \in \Omega \), \( M(\omega) \) is a proper left ideal of \( B \).

Then there exists a maximal left ideal of \( B \) such that \( M(\omega) \subset N \).

Therefore \( \omega \in \Omega \), \( M(\omega) \subset N \) by the assumption that \( M \) is maximal.

Given \( M \) a maximal left ideal of \( A \), then as we have shown above \( M = K(\omega, N) \) for some \( \omega \in \Omega \) and some maximal left ideal \( N \) of \( B \). Choose \( \beta \) a pure state of \( B \) such that \( K_\beta = N \). Define \( \alpha \) on \( A \) by \( \alpha(f) = \beta(f(\omega)) \), \( f \in A \). Then \( K_\alpha = K(\omega, N) = M \).

It is easy to verify that the norm \( |f + K_\alpha|_2 = \alpha(f^* f)^{1/2} \) is a complete norm on \( A - K_\alpha \). Therefore \( \alpha \) is a strictly pure state of \( A \). This proves that \( A \) has property (II).

Now assume that \( \alpha \) is a pure state of \( A \). Then by [3, Lemma 2.10.1, p. 50] \( \alpha \) has an extension to a pure state \( \beta \) of \( C(\Omega, B) \). By [7, Corollary, p. 337] there exists a point \( \omega \in \Omega \) and a maximal left ideal \( N \) of \( B \) such that \( K_\beta = \{ f \in C(\Omega, B) \mid f(\omega) \in N \} \).

Therefore \( K_\alpha = K_\beta \cap A = K(\omega, N) \). It follows that \( \alpha \) is a strictly pure state of \( A \) by Proposition 3.4. Therefore \( A \) has property (I).

REFERENCES


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