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MAXIMAL REGULAR RIGHT IDEAL SPACE OF A PRIMITIVE RING

BY

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ABSTRACT. If R is a ring, let $X(R)$ be the set of maximal regular right ideals of R and $\mathfrak{L}(R)$ be the lattice of right ideals. For each $A \in \mathfrak{L}(R)$, define $\operatorname{supp}(A) = \{I \in X(R) \mid A \not\subseteq I\}$. We give a topology to $X(R)$ by taking $\{\operatorname{supp}(A) \mid A \in \mathfrak{L}(R)\}$ as a subbase. Let R be a right primitive ring. Then $X(R)$ is the union of two proper closed sets if and only if R is isomorphic to a dense ring with nonzero socle of linear transformations of a vector space of dimension two or more over a finite field. $X(R)$ is a Hausdorff space if and only if either R is a division ring or R modulo its socle is a radical ring and R is isomorphic to a dense ring of linear transformations of a vector space of dimension two or more over a finite field.

Introduction. For a ring R, define $X(R)$ to be the set of maximal regular right ideals of R. Then $X(R)$ is a nonempty set if and only if R is not a radical ring. If A is a right ideal of a ring R, define the support of A to be the set of maximal regular right ideals of R which do not contain A. We topologize $X(R)$ by defining that a subset is open if and only if it is an arbitrary union of finite intersections of the supports of right ideals in R; that is, the supports of the right ideals form a subbasis for this topology. We will call $X(R)$ together with this topology the maximal regular right ideal space of the ring R. Recall that a topological space is irreducible (refer to [3, p. 13]) if it is not the union of two proper closed subsets, and it is reducible if it is not irreducible. Our main results in this paper are as follows: Let R be a (right) primitive ring. Then $X(R)$ is reducible if and only if R is isomorphic to a dense ring with nonzero socle of linear transformations of a vector space of dimension two or more over a finite field. $X(R)$ is a Hausdorff space if and only if either R is a division ring or R is isomorphic to a dense ring of linear transformations of a vector space over a finite field such that R modulo its socle is a radical ring. If R has 1, then $X(R)$ is a Hausdorff space if and only if either R is a division ring or a finite ring.

1. Preliminaries.

1.1 **Definition.** If A is a right ideal of a ring R, the *support* of A is the

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set of maximal regular right ideals of R which do not contain A. It will be denoted by $\mathrm{supp}(A)$.

1.2 **Definition.** For a ring R, let $X(R)$ be the set of maximal regular right ideals in R. We give a topology to $X(R)$ which is generated by the subbasis consisting of all supports of the right ideals in R. We will call $X(R)$ together with this topology the *maximal regular right ideal space* of R. It will simply be denoted by $X(R)$.

1.3 **Definition.** If x is an element of $X(R)$ for some ring R, then x is also a right ideal of the ring R. Therefore, it is convenient to make a distinction by writing $j(x)$ for the right ideal x and if Y is subset of $X(R)$, $j(Y) = \bigcap\{j(x) \mid x \in Y\}$. If E is a subset of R, we define $h(E) = \{x \in X(R) \mid j(x) \supseteq E\}$. $h(E)$ is called the *hull* of E and we write $\mathrm{supp}(E) = X(R) \setminus h(E)$.

1.4 **Definition.** A topological space is called *irreducible* [3, p. 155] if it is not the union of two proper closed subsets. A space which is not irreducible is *reducible.*

1.5 **Definition.** If X is a set which is a finite union of subsets $Y_1, Y_2, \cdots, Y_n$, we say that X is an *irredundant* union of Y_i provided that $X \neq Y_1 \cup Y_2 \cup \cdots \cup Y_{i-1} \cup Y_{i+1} \cup \cdots \cup Y_n$ for every i such that $1 \leq i \leq n$.

1.6 **Proposition.** *If R is a ring and $J(R)$ is the Jacobson radical of R then $X(R)$ is homeomorphic to $X(R/J(R))$.*

Proof. Straightforward.

1.7 **Proposition.** *If R is a ring with a unit element, then $X(R)$ is a compact space.*

Proof. In view of the Alexander subbase theorem, it suffices to show that if

$$X(R) = \bigcup\{\mathrm{supp}(A_\alpha) \mid \{A_\alpha\}_{\alpha \in \Lambda} \text{ is a family of right ideals indexed by a set } \Lambda\},$$

then there exists a finite subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ in Λ such that $X(R) = \bigcup_{i=1}^{n} \mathrm{supp}(A_{\alpha_i})$. Since $X(R) = \bigcup\{\mathrm{supp}(A_\alpha) \mid \alpha \in \Lambda\}$, $h(\Sigma_{\alpha \in \Lambda} A_\alpha) = \bigcap\{h(A_\alpha) \mid \alpha \in \Lambda\} = \emptyset$, and hence $\Sigma_{\alpha \in \Lambda} A_\alpha = R$. Thus $1 = a_{\alpha_1} + a_{\alpha_2} + \cdots + a_{\alpha_n}$ for some $a_{\alpha_i} \in A_{\alpha_i}$, $i = 1, 2, \cdots, n$, and $\bigcap_{i=1}^{n} h(A_{\alpha_i}) = \emptyset$. Therefore, $X(R) = \bigcup_{i=1}^{n} \mathrm{supp}(A_{\alpha_i})$.

1.8 **Proposition.** *If Y is an irreducible subset of $X(R)$, then the closure $\overline{Y}$ of Y is equal to $h(j(Y))$.*

Proof. Clearly, $\overline{Y} \subseteq h(j(Y))$. Let $x \in h(j(Y))$. If $x \notin \overline{Y}$, then there exists a finite number of right ideals $A_1, A_2, \cdots, A_n$ in R such that $x \in \bigcap_{i=1}^{n} \mathrm{supp}(A_i)$ and $Y \cap (\bigcap_{i=1}^{n} \mathrm{supp}(A_i)) = \emptyset$. Hence $Y \subseteq \bigcup_{i=1}^{n} h(A_i)$. Since Y is irreducible, $Y \subseteq h(A_i)$ for some A_i. Hence $A_i \subseteq j(Y) \subseteq j(x)$. This is impossible since $x \in \mathrm{supp}(A_i)$.

1.9 **Corollary.** *$X(R)$ is a T_1-space.*

Proof. Let $x \in X(R)$. Then $\{x\} = h(j(x)) = \overline{\{x\}}$ by 1.8.

1.10 **Example.** It is not true, in general, that if Y is a subset of $X(R)$ then $\overline{Y} = h(j(Y))$. For example, let R be the ring of 2×2 matrices over the field of real numbers and let $x = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b$ are real numbers$\}$ and $y = \{\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} | c, d$ are real numbers$\}$. Then $\overline{\{x, y\}} = \{x, y\}$, but $h(j(\{x, y\})) = X(R)$.

1.11 **Proposition.** *$X(R)$ is reducible if and only if there exists a finite number of right ideals $A_1, A_2, \cdots, A_n$, $n \geq 2$, in R such that $X(R)$ is an irredundant union of $h(A_1), h(A_2), \cdots, h(A_n)$.*

Proof. Straightforward.

1.12 **Proposition.** *If R is a primitive ring and A is a right ideal of R, then either* $\operatorname{supp}(A) = \emptyset$ *or* $j(\operatorname{supp}(A)) = \{0\}$.

Proof. Suppose $\operatorname{supp}(A) \neq \emptyset$ and $j(\operatorname{supp}(A)) \neq \{0\}$. Let $B = j(\operatorname{supp}(A))$. Then B is a nonzero proper right ideal of R, $\operatorname{supp}(A) \subseteq h(B)$ and $\operatorname{supp}(B) \cap \operatorname{supp}(A) = \emptyset$. Since B is a nonzero right ideal of a primitive ring, $\operatorname{supp}(B) \neq \emptyset$ and $X(R) = h(B) \cup h(A)$. Let M be a faithful simple (right) R-module. Since for each $0 \neq m \in M$, $m^{\perp} = \{r \in R | mr = 0\}$ is a maximal regular right ideal of R, either $mA = 0$ or $mB = 0$. Let $M_1 = \{m \in M | mA = 0\}$ and $M_2 = \{m \in M | mB = 0\}$. Then M_1 and M_2 are subgroups of M and $M = M_1 \cup M_2$. Hence either $M = M_1$ or $M = M_2$. Therefore either $MA = 0$ or $MB = 0$. This is impossible since M is faithful.

1.13 **Remark.** If R is a ring with 1 and $X(R)$ is irreducible, then R is isomorphic to the ring of global sections of the simple R-sheaf over $X(R)$ (refer to [3, p. 45] for the definition of a simple sheaf). Hence R can be identified with the ring of all continuous functions from $X(R)$ to R. To see this, let $\hat{r}(x) = (x, r)$ for every $r \in R$ and x in $X(R)$. Then $\hat{r}$ is a global section of the simple sheaf over $X(R)$. If f is an arbitrary global section, then $f(X(R)) = X(R) \times \{r\}$ for some $r \in R$ since $X(R)$ is irreducible (hence it is connected). Thus $f = \hat{r}$. Clearly, $r \mapsto \hat{r}$ is an isomorphism of R onto the ring of all global sections.

2. Primitive rings with reducible maximal regular right ideal spaces.

In this section, we will give a structure theorem of primitive rings whose maximal regular right ideal spaces are reducible. The basic facts about a primitive ring, which we will use freely in this section, could be found in [2].

2.1 **Lemma.** *Let V be a vector space over a division ring D. Assume that $V = V_1 \cup \cdots \cup V_n$, where the V_i are subspaces of V, $n \geq 2$, and the union is irredundant. Then D is a finite field, and the dimension of $V/(V_1 \cap \cdots \cap V_n)$ is finite.*

Proof. The fact that D is a finite field follows from Lemma 2 of [1, p. 32]. Suppose that the dimension of $V/(V_1 \cap \cdots \cap V_n)$ is infinite. Since $V/\bigcap_{i=1}^n V_i$ is the irredundant union of proper subspaces $V_i/\bigcap_{i=1}^n V_i$, we may assume, without loss of generality, that $\bigcap_{i=1}^n V_i = \{0\}$. Since V is infinite dimensional, there is a subspace V_{t_1} for some t_1, $1 \leq t_1 \leq n$, such that V_{t_1} is infinite dimensional. Let i be a positive integer less than or equal to n such that $i \neq t_1$. Let $v_i \in V_i \setminus \bigcup_{k \neq i} V_k$. Let $N(t_1) = \{v_i + w | w \in V_{t_1}\}$. Since $N(t_1) \cap V_{t_1} = \varnothing$, $N(t_1) \subseteq \bigcup_{k \neq t_1} V_k$. Since $N(t_1)$ is an infinite set, there is a subspace V_{t_2} for some t_2 such that $1 \leq t_2 \leq n$ and $t_2 \neq t_1$ and V_{t_2} contains infinitely many $v_i + w$'s, say $v_i + w_1, v_i + w_2, \cdots$, where $w_j \in N(t_1)$. It follows that $w_1 - w_j = (v_i + w_1) - (v_i + w_j) \in V_{t_2}$ for infinitely many w_j's in V_{t_1}. Hence $V_{t_1} \cap V_{t_2}$ is an infinite set and hence it is infinite dimensional since $V_{t_1} \cap V_{t_2}$ is a vector space over the finite field D. Now assume that $V_{t_1} \cap V_{t_2} \cap \cdots \cap V_{t_k}$ is infinite dimensional for some distinct positive integers $t_1, t_2, \cdots, t_k$ each of which is less than or equal to n. Let i be a positive integer less than or equal to n such that $i \notin \{t_1, t_2, \cdots, t_k\}$. Let $v_i \in V_i \setminus \bigcup_{t \neq i} V_t$. Define $N(t_1, t_2, \cdots, t_k) = \{v_i + w | w \in \bigcap_{j=1}^k V_{t_j}\}$. Then $N(t_1, t_2, \cdots, t_k) \cap V_{t_j} = \varnothing$ for every t_j, $1 \leq j \leq k$. Hence there is a subspace $V_{t_{k+1}}$ for some $1 \leq t_{k+1} \leq n$ such that $V_{t_{k+1}}$ contains infinitely many elements of $N(t_1, t_2, \cdots, t_k)$; hence $\bigcap_{i=1}^{k+1} V_{t_i}$ is infinite dimensional. Thus, by inductive argument, $\bigcap_{i=1}^n V_i$ is an infinite dimensional space, which is absurd.

2.2 **Remark.** If V is a vector space over a finite field, say D, such that $\dim V \geq 2$, then V is a finite union of proper subspaces. Let v_1, v_2 be linearly independent elements in V and let N be a subspace such that $V = Dv_1 \oplus Dv_2 \oplus N$. For every pair $(\alpha, \beta) \in D \times D$, define $U(\alpha, \beta) = D(\alpha v_1 + \beta v_2) \oplus N$. Then $\bigcup_{(\alpha,\beta) \in D \times D} U(\alpha, \beta) = V$.

2.3 **Lemma.** *Let V be a vector space of dimension at least 2 over a finite field D. Let R be a dense ring of linear transformations of V, such that the socle S of R is not zero. Let v and w be linearly independent vectors of V. Then there is a subspace W of V such that*

(a) $V = W \oplus Dv \oplus Dw$;

(b) *for each α, β in D, $U(\alpha, \beta)^\perp \neq 0$, where $U(\alpha, \beta) = W \oplus D(\alpha v + \beta w)$ and $U(\alpha, \beta)^\perp = \{r \in R | U(\alpha, \beta)r = 0\}$; and*

(c) $X(R) = \bigcup_{\alpha, \beta \in D} h(U(\alpha, \beta)^\perp) \cup h(S)$.

Proof. Since S is also a dense ring of linear transformations of V, and v, w are linearly independent, there is an s in S such that $vs = v$ and $ws = 0$. Since Vs is a finite dimensional subspace of V, there exist $v_2, v_3, \cdots, v_n$ in Vs such that $\{v, v_2, v_3, \cdots, v_n\}$ is a basis for Vs. Let $t \in R$ such that $vt = v$

and $v_i t = v$ for all i such that $2 \leq i \leq n$. Let $a = st$. Then $Va = Dv$, $va = v$ and $wa = 0$. In a similar manner, we can choose b in R such that $Vb = Dw$, $wb = w$ and $vb = 0$. Let $W = \operatorname{Ker} a \cap \operatorname{Ker} b$. Then $W = \operatorname{Ker}(a + b)$ since $Dv \cap Dw = \{0\}$. Since $va = v$, $wb = w$, and $wa = 0 = vb$, for any x in V, $(x(a + b) - x) \cdot (a + b) = 0$. Hence $V = W \oplus Dv \oplus Dw$ and $W^{\perp} \supseteq \{a, b\}$. For α, β in D, let $U(\alpha, \beta) = W \oplus D(\alpha v + \beta w)$. We claim that the right ideal $U(\alpha, \beta)^{\perp} = \{r \in R \mid U(\alpha, \beta)r = 0\}$ is not zero. It is clear that if either $\alpha = 0$ or $\beta = 0$, then either a or b is an element of $U^{\perp}(\alpha, \beta)$. So assume $\alpha \neq 0$ and $\beta \neq 0$. Then $\beta wb \neq 0$ and there exists r in R such that $\beta wbr = \alpha va$. Now $(\alpha v + \beta w)br = \beta wbr = \alpha va = (\alpha v + \beta w)a$. Hence $(\alpha v + \beta w)(br - a) = 0$ and $br - a \in U(\alpha, \beta)^{\perp}$. If $br = a$, then $0 = vbr = va = v$ and this is impossible. Thus $br - a \neq 0$. We assert now that $X(R) = \bigcup_{\alpha,\beta \in D} h(U(\alpha, \beta)^{\perp}) \cup h(S)$. Indeed, if $x \in X(R)$ then either $j(x) \supseteq S$ or $R/j(x) \cong V$ as R-modules. Hence if $x \notin h(S)$, then $j(x) = \{r \in R \mid vr = 0\}$ for some $0 \neq v \in V$. In this case, $v \in U(\alpha, \beta)$ for some α, β in D and $j(x) \supseteq U(\alpha, \beta)^{\perp}$.

2.4 **Theorem.** *Let R be a (right) primitive ring. Then $X(R)$ is reducible if and only if R is isomorphic to a dense ring with nonzero socle of linear transformations of a vector space of dimension two or more over a finite field.*

Proof. Assume that $X(R)$ is reducible. Then by 1.11, there exists a finite number of right ideals $A_1, A_2, \cdots, A_n$, $n \geq 2$, in R such that $X(R) = \bigcup_{i=1}^{n} h(A_i)$ and such that this union is irredundant. Let V be a faithful simple (right) R-module and let $D = \operatorname{End}_R(V)$. Then D is a division ring, V is a left vector space over D and R is a dense ring of linear transformations of V over D. For each i, $1 \leq i \leq n$, define $V_i = \{v \in V \mid vA_i = 0\}$. Then V_i is a subspace of V and $V_i \neq V$ for every i since $A_i \neq 0$ and V is faithful. For any $0 \neq v \in V$, $v^{\perp} = \{r \in R \mid vr = 0\}$ is a member of $X(R)$. Therefore $v^{\perp}$ is a member of some $h(A_i)$ and hence $v \in V_i$ and $V = \bigcup_{i=1}^{n} V_i$. Thus, by 2.1, D is a finite field and the dimension of $V/\bigcap_{i=1}^{n} V_i$ is finite. Therefore, there is a finite dimensional subspace M_i such that $M_i \oplus V_i = V$ for each i and $\dim V \geq 2$ since $V_i \neq V$. Thus every element in the right ideal A_i is of finite rank and the socle of R is not zero. Conversely, assume that R is isomorphic to a dense ring with nonzero socle S of linear transformations of a vector space V of dimension two or more over a finite field D. Then by 2.3(c), $X(R) = \bigcup_{\alpha,\beta \in D} h(U(\alpha, \beta)^{\perp}) \cup h(S)$ and hence $X(R)$ is reducible.

2.5 **Theorem.** *Let R be a primitive ring and S be the socle of R. If $X(R)$ is a Hausdorff space, then either R is a division ring or R/S is a radical ring and R is a dense ring of linear transformations of a vector space over a finite field.*

Proof. If $X(R)$ is a Hausdorff space and R is not a division ring, then certainly $X(R)$ is reducible. Hence by 2.4, R has nonzero socle and it is isomorphic to a dense ring of linear transformations of a vector space V of dimension two or more over a finite field D. If R/S is not a radical ring, then $h(S) \neq \emptyset$. Let x, y be two points in $X(R)$ such that $x \notin h(S)$ and $y \in h(S)$. We shall show that x and y cannot be separated in $X(R)$. Since $x \notin h(S)$, $j(x) = v^{\perp}$ for some $0 \neq v \in V$. Suppose there exist right ideals $S_1, S_2, \cdots, S_p, T_1, T_2, \cdots, T_q$ such that $x \in \bigcap_{i=1}^{p} \operatorname{supp}(S_i)$, $y \in \bigcap_{j=1}^{q} \operatorname{supp}(T_j)$ such that $(\bigcap_{i=1}^{p} \operatorname{supp}(S_i)) \cap (\bigcap_{j=1}^{q} \operatorname{supp}(T_j)) = \emptyset$. First we note that the separation still holds if we replace S_i by S_iS for every i, $1 \leq i \leq p$, since $x \in \operatorname{supp}(S_i) \cap \operatorname{supp}(S) = \operatorname{supp}(S_iS)$. Thus, without loss of generality, we may assume that $S_i \subseteq S$ for every i. Since $j(x) = v^{\perp}$ and $x \in \bigcap_{i=1}^{p} \operatorname{supp}(S_i)$, $vS_i \neq 0$ for every i. Let $s_i \in S_i$ such that $vs_i \neq 0$ for every i. Since each s_i is of finite rank, $V/\operatorname{Ker} s_i$ is finite dimensional. Thus, $V/\bigcap_{i=1}^{p} \operatorname{Ker} s_i$ being a subdirect sum of the vector spaces $V/\operatorname{Ker} s_i$ is finite dimensional. Note that $T_j \nsubseteq j(y)$ for every j, $1 \leq j \leq q$, and $S \subseteq j(y)$. Therefore $T_j \backslash S \neq \emptyset$ for every j. Let $t_j \in T_j \backslash S$ and let $W_j = \{w \in \bigcap_{i=1}^{p} \operatorname{Ker} s_i \mid wt_j \in \Sigma_{i=1}^{q} Dvt_i\}$ for every j, $1 \leq j \leq q$. We claim that $\bigcap_{i=1}^{p} \operatorname{Ker} s_i/W_j$ is infinite dimensional for every j. For if $\bigcap_{i=1}^{p} \operatorname{Ker} s_i/W_j$ is finite dimensional for some j and if $\bigcap_{i=1}^{p} \operatorname{Ker} s_i = W_j \oplus U_j$ for some finite dimensional subspace U_j, then $(\bigcap_{i=1}^{p} \operatorname{Ker} s_i)t_j = W_jt_j + U_jt_j$ is a finite dimensional subspace of V. Since $V/\bigcap_{i=1}^{p} \operatorname{Ker} s_i$ is finite dimensional, there is a finite dimensional subspace W such that $V = (\bigcap_{i=1}^{p} \operatorname{Ker} s_i) \oplus W$. Then $Vt_j = (\bigcap_{i=1}^{p} \operatorname{Ker} s_i)t_j + Wt_j$ is also finite dimensional and $t_j \in S$. This is impossible since, by choice, $t_j \notin S$. Thus $\bigcap_{i=1}^{p} \operatorname{Ker} s_i/W_j$ is infinite dimensional for each j and $\bigcap_{i=1}^{p} \operatorname{Ker} s_i \neq \bigcup_{j=1}^{q} W_j$ by 2.1. Hence there exists $w \in \bigcap_{i=1}^{p} \operatorname{Ker} s_i$ such that $w \notin \bigcup_{j=1}^{q} W_j$. This means that $ws_i = 0$ for every i, $1 \leq i \leq p$, and $wt_j \notin \Sigma_{i=1}^{q} Dvt_i$ for every j, $1 \leq i \leq q$. Now, we consider the vector $v + w$. Then $(v + w)s_i = vs_i \neq 0$ for every i such that $1 \leq i \leq p$ and $(v + w)t_j \neq 0$ for every j, $1 \leq j \leq q$. For if $(v + w)t_j = 0$ for some j, then $wt_j = -vt_j \in \Sigma_{i=1}^{q} Dvt_i$. Thus $(v + w)^{\perp} \in (\bigcap_{i=1}^{p} \operatorname{supp}(S_i)) \cap (\bigcap_{j=1}^{q} \operatorname{supp}(T_j)) = \emptyset$. This is a contradiction.

2.6 **Theorem.** *Let R be a primitive ring and S be the socle of R. If R/S is a radical ring and R is isomorphic to a dense ring of linear transformations of a vector space V over a finite field D, then $X(R)$ is a Hausdorff space.*

Proof. Let x, y be two distinct members of $X(R)$. Since R/S is a radical ring, there exist v and w in V such that $j(x) = v^{\perp}$, $j(y) = w^{\perp}$ and $Dv \cap Dw = \{0\}$. Moreover, $v^{\perp} \cap S \nsubseteq w^{\perp}$ and $w^{\perp} \cap S \nsubseteq v^{\perp}$. Let $a \in (w^{\perp} \cap S)\backslash v^{\perp}$ and $b \in (v^{\perp} \cap S)\backslash w^{\perp}$. Then $wa = vb = 0$, $va \neq 0$, and $wb \neq 0$. Since $a \in S$, $Va = Dva \oplus U$ for some finite dimensional subspace U of V. Hence there is r in R such that

$Ur = 0$ but $var = v$. Therefore, $Var = Dv$, $war = 0$, and $V = \text{Ker } ar \oplus Dv$. Likewise there is r_0 in R such that $Vbr_0 = Dw$, $vbr_0 = 0$, $wbr_0 = w$, and $V = \text{Ker } br_0 \oplus Dw$. Since Ker $ar \neq$ Ker br_0 and the codimensions of Ker ar and Ker br_0 are 1, $V = \text{Ker } ar + \text{Ker } br_0$. Thus $\dim(\text{Ker } ar/\text{Ker } ar \cap \text{Ker } br_0) = \dim(V/\text{Ker } br_0) = 1$. Hence $\text{codim}(\text{Ker } ar \cap \text{Ker } br_0) = \dim(V/\text{Ker } br_0) + \dim(\text{Ker } ar/\text{Ker } ar \cap \text{Ker } br_0) = 2$ and therefore, $V = (\text{Ker } ar \cap \text{Ker } br_0) \oplus Dv \oplus Dw$. For every α, β in D, define $U(\alpha, \beta) = (\text{Ker } ar \cap \text{Ker } br_0) \oplus D(\alpha v + \beta w)$. Then $V = \bigcup_{\alpha, \beta \in D} U(\alpha, \beta)$. We shall show that $U(\alpha,\beta)^{\perp} \neq 0$ for each pair $(\alpha, \beta) \in D \times D$. Clearly $ar \in U(0, \beta)^{\perp}$ and $br_0 \in U(\alpha, 0)^{\perp}$. Assume $\alpha \neq 0$ and $\beta \neq 0$. Then there is $c \in R$ such that $\alpha vc = \beta w$. Consequently, $(\alpha v + \beta w)arc = \alpha varc = \alpha vc = \beta w = (\alpha v + \beta w)br_0$ since $var = v$ and $wbr_0 = w$. Hence $D(\alpha v + \beta w) \cdot (arc - br_0) = 0$. Clearly, $arc - br_0 \neq 0$ since $warc = 0$ and $wbr_0 = w \neq 0$. Thus $0 \neq arc - br_0 \in U(\alpha, \beta)^{\perp}$. Let

$$\mathcal{O}_1 = \bigcap_{\substack{(\alpha,\beta)\in D\times D \\ \beta\neq 0}} \text{supp}(U(\alpha, \beta)^{\perp}) \quad \text{and} \quad \mathcal{O}_2 = \bigcap_{\substack{(\alpha,\beta)\in D\times D \\ \alpha\neq 0}} \text{supp}(U(\alpha, \beta)^{\perp}).$$

Recall $j(x) = v^{\perp}$ and $j(y) = w^{\perp}$. If $x \notin \mathcal{O}_1$ then $U(\alpha, \beta)^{\perp} \subseteq v^{\perp}$ for some $\beta \neq 0$ in D. For every $f \in U(\alpha, \beta)^{\perp}$, $vf = 0$. Consequently, $wf = 0$ also since $\beta = 0$ and hence $Vf = 0$. This means that $U(\alpha, \beta)^{\perp} = 0$, a contradiction. Thus $x \in \mathcal{O}_1$. A similar argument shows that $y \in \mathcal{O}_2$. We now claim that $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. For if $z \in \mathcal{O}_1 \cap \mathcal{O}_2$ then $j(z) = v_0^{\perp}$ for some $v_0 \in V$ and $v_0 = v' + \alpha v + \beta w$ for some $v' \in \text{Ker } ar \cap \text{Ker } br_0$ and $\alpha, \beta \in D$. It follows that $v_0^{\perp} \supseteq U(\alpha, \beta)^{\perp}$ and $z \notin \mathcal{O}_1 \cap \mathcal{O}_2$, a contradiction. Therefore, $X(R)$ is Hausdorff.

2.7 **Corollary.** *If R is a primitive ring with* 1, *then $X(R)$ is a Hausdorff space if and only if either R is a division ring or R is a finite ring.*

Proof. If R is a finite ring or a division ring then certainly $X(R)$ is a finite T_1-space. Hence it is a Hausdorff space. Conversely, if $X(R)$ is a Hausdorff space, then by 2.5, R is either a division ring or a dense ring of linear transformations of finite rank of a vector space over a finite field. In the latter case, since $1 \in R$, R must be the complete ring of linear transformations of a finite dimensional vector space over a finite field. Thus, R is a finite ring.

2.8 **Example.** Let $\mathcal{Z}$ be the ring of integers and let V be the set of finite sequences over $\mathcal{Z}/(2)$. Then V becomes an $\aleph_0$-dimensional vector space over $\mathcal{Z}/(2)$. Let R be the ring of linear transformations on V and S be the ideal of linear transformations of finite rank. Then R/S is a simple ring with 1 (refer to [2, Theorem 1, p. 93]). Hence by 2.5, $X(R)$ is not a Hausdorff space. However, $X(R)$ is a reducible space by 2.4 and $X(S)$ is a Hausdorff space by 2.6.

2.9 **Example.** Let R be the ring of infinite row finite matrices of the following form:

$$\begin{pmatrix} A_n & & & * \\ & 0 & & \\ & & 0 & \\ 0 & & & \ddots \end{pmatrix}$$

where A_n is an $n \times n$ matrix over $Z/(2)$ for some n. Then R is a dense ring of the vector space of sequences over $Z/(2)$. If S is the socle of R then R/S is a radical ring.

3. **Finite dimensional maximal regular right ideal space of a primitive ring.** If X is a topological space, then *the combinatorial dimension* of X, $\dim X$, is the supremum of the positive integer n such that there is a strictly ascending chain of nonempty closed irreducible subsets of X, $\emptyset \neq F_0 \subsetneqq F_1 \subsetneqq \cdots \subsetneqq F_n$ (refer to [4, p. 156]).

3.1 **Theorem.** *If R is a dense ring of linear transformations of a vector space M over an infinite division ring D, then* $\dim M = n + 1$ *if and only if* $\dim X(R) = n$.

Proof. Assume $\dim M = n+1$ and let $\{m_1, m_2, \cdots, m_{n+1}\}$ be a basis for the vector space M. Then $\bigcap_{i=1}^{n+1} m_i^\perp = \{0\}$. Hence $X(R) = h(\bigcap_{i=1}^{n+1} m_i^\perp) \supsetneqq h(\bigcap_{i=2}^{n+1} m_i^\perp) \supsetneqq \cdots \supsetneqq h(m_1^\perp \cap m_2^\perp) \supsetneqq h(m_1^\perp) = \{m_1^\perp\}$. Since R is a simple artinian ring, if $x \in X(R)$ then $v(x) = v^\perp$ for some vector $v \neq 0$ in M. Hence if $x \in h(\bigcap_{i=t}^{n+1} m_i^\perp)$, then $v \in \Sigma_{i=t}^{n+1} Dm_i$. Therefore, if $h(\bigcap_{i=t}^{n+1} m_i^\perp)$ were reducible, then, as in the case of proof of 2.4, the vector space $\Sigma_{i=t}^{n+1} Dm_i$ would be a finite union of proper subspaces and D would be a finite field by 2.1. Thus $\dim X(R) \geq n$. Now let $F_{n+1} \supsetneqq F_n \supsetneqq \cdots \supsetneqq F_1 \supsetneqq F_0 = \emptyset$ be a strictly descending closed irreducible subsets of $X(R)$. Let $A_i = j(F_i)$, $0 \leq i \leq n+1$. Then $A_{n+1} \subsetneqq A_n \subsetneqq \cdots \subsetneqq A_1 \subsetneqq A_0$ is a strictly ascending chain of right ideals of R since $h(j(F_i)) = F_i$ for each i by 1.8. Since R is a simple artinian ring, every right ideal of R is a direct summand of R. Hence $A_0 = K_1 \oplus A_1$, $A_1 = K_2 \oplus A_2, \cdots, A_n = K_{n+1} \oplus A_{n+1}$ for some nonzero right ideals $K_1, K_2, \cdots, K_{n+1}$, and $R = K_0 \oplus K_1 \oplus \cdots \oplus K_{n+1} \oplus A_{n+1}$ for some right ideal K_0 of R. This means that R is a direct sum of at least $n+2$ minimal right ideals. This is impossible since R is a direct sum of $n+1$ minimal right ideals and the number of summands is unique. Conversely, let us assume now that $\dim X(R) = n$. Let B be a basis for the vector space M. If B is a finite set then, by the first part of the theorem, the number of elements in B must be $n+1$. So suppose that $\dim M = \infty$. Let $Y = \{m^\perp \mid m \in M, m \neq 0\}$. Then Y is a nonempty subspace of $X(R)$ and $\dim Y \leq \dim X(R) = n$ by [4, 9.3, p. 156].

Let $b_1, b_2, \cdots, b_k, \cdots$ be distinct elements in B. Then the chain of subsets $h(b_1^\perp) \cap Y \subsetneqq h(b_1^\perp \cap b_2^\perp) \cap Y \subsetneqq \cdots \subsetneqq h(\bigcap_{i=1}^k b_1^\perp) \cap Y \subsetneqq \cdots$ is strictly ascending. Since D is an infinite division ring, each $h(\bigcap_{i=1}^k b_i^\perp) \cap Y$ is irreducible in Y as in the case of proof of 2.4. Hence dim Y is not finite and this is a contradiction.

3.2 **Theorem.** *Let R be a primitive ring. Then R is right artinian if and only if $X(R)$ satisfies the descending chain condition on the subbasic open sets.*

Proof. If R is right artinian, then R is a simple artinian ring. Hence if A is a right ideal of R, then $j(h(A)) = A$. Hence if $\operatorname{supp}(A_1) \supseteq \operatorname{supp}(A_2) \supseteq \cdots \supseteq \operatorname{supp}(A_n) \supseteq \cdots$ is a chain of subbasic open sets for some right ideals $A_1, A_2, \cdots, A_n, \cdots$ then $h(A_1) \subseteq h(A_2) \subseteq \cdots \subseteq h(A_n) \subseteq \cdots$ and $jh(A_1) = A_1 \supseteq jh(A_2) = A_2 \supseteq \cdots \supseteq jh(A_n) = A_n \supseteq \cdots$. Thus the chain must terminate. Conversely, assume that the descending chain condition holds on the subbasic open sets. Let V be a faithful simple right R-module. To prove that R is artinian, it suffices to show that V is a finite dimensional vector space. So suppose the dimension V is infinite. Then there exist infinite independent vectors $v_1, v_2, \cdots$ such that $\operatorname{supp}(v_1^\perp) \supset \operatorname{supp}(v_1^\perp \cap v_2^\perp) \cdots$. This is a contradiction.

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