MAXIMAL REGULAR RIGHT IDEAL SPACE
OF A PRIMITIVE RING

BY

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ABSTRACT. If $R$ is a ring, let $X(R)$ be the set of maximal regular right ideals of $R$ and $\mathfrak{P}(R)$ be the lattice of right ideals. For each $A \in \mathfrak{P}(R)$, define $\text{supp}(A) = \{ I \in X(R) \mid I \not\subseteq A \}$. We give a topology to $X(R)$ by taking $\{ \text{supp}(A) \mid A \in \mathfrak{P}(R) \}$ as a subbase. Let $R$ be a right primitive ring. Then $X(R)$ is the union of two proper closed sets if and only if $R$ is isomorphic to a dense ring with nonzero socle of linear transformations of a vector space of dimension two or more over a finite field. $X(R)$ is a Hausdorff space if and only if either $R$ is a division ring or $R$ modulo its socle is a radical ring and $R$ is isomorphic to a dense ring of linear transformations of a vector space of dimension two or more over a finite field.

Introduction. For a ring $R$, define $X(R)$ to be the set of maximal regular right ideals of $R$. Then $X(R)$ is a nonempty set if and only if $R$ is not a radical ring. If $A$ is a right ideal of a ring $R$, define the support of $A$ to be the set of maximal regular right ideals of $R$ which do not contain $A$. We topologize $X(R)$ by defining that a subset is open if and only if it is an arbitrary union of finite intersections of the supports of right ideals in $R$; that is, the supports of the right ideals form a subbasis for this topology. We will call $X(R)$ together with this topology the maximal regular right ideal space of the ring $R$. Recall that a topological space is irreducible (refer to [3, p. 13]) if it is not the union of two proper closed subsets, and it is reducible if it is not irreducible. Our main results in this paper are as follows: Let $R$ be a (right) primitive ring. Then $X(R)$ is reducible if and only if $R$ is isomorphic to a dense ring with nonzero socle of linear transformations of a vector space of dimension two or more over a finite field. $X(R)$ is a Hausdorff space if and only if either $R$ is a division ring or $R$ is isomorphic to a dense ring of linear transformations of a vector space over a finite field such that $R$ modulo its socle is a radical ring. If $R$ has 1, then $X(R)$ is a Hausdorff space if and only if either $R$ is a division ring or a finite ring.

1. Preliminaries.

1.1 Definition. If $A$ is a right ideal of a ring $R$, the support of $A$ is the...
set of maximal regular right ideals of $R$ which do not contain $A$. It will be denoted by $\text{supp}(A)$.

1.2 Definition. For a ring $R$, let $X(R)$ be the set of maximal regular right ideals in $R$. We give a topology to $X(R)$ which is generated by the subbasis consisting of all supports of the right ideals in $R$. We will call $X(R)$ together with this topology the \textit{maximal regular right ideal space} of $R$. It will simply be denoted by $X(R)$.

1.3 Definition. If $x$ is an element of $X(R)$ for some ring $R$, then $x$ is also a right ideal of the ring $R$. Therefore, it is convenient to make a distinction by writing $j(x)$ for the right ideal $x$ and if $Y$ is a subset of $X(R)$, $j(Y) = \bigcap \{j(x)\mid x \in Y\}$. If $E$ is a subset of $R$, we define $b(E) = \{x \in X(R)\mid j(x) \supseteq E\}$. $b(E)$ is called the \textit{bull} of $E$ and we write $\text{supp}(E) = X(R) \setminus b(E)$.

1.4 Definition. A topological space is called \textit{irreducible} [3, p. 155] if it is not the union of two proper closed subsets. A space which is not irreducible is \textit{reducible}.

1.5 Definition. If $X$ is a set which is a finite union of subsets $Y_1, Y_2, \ldots, Y_n$, we say that $X$ is an \textit{irredundant} union of $Y_i$ provided that $X \neq Y_1 \cup Y_2 \cup \cdots \cup Y_{i-1} \cup Y_{i+1} \cup \cdots \cup Y_n$ for every $i$ such that $1 \leq i \leq n$.

1.6 Proposition. If $R$ is a ring and $J(R)$ is the Jacobson radical of $R$ then $X(R)$ is homeomorphic to $X(R/J(R))$.

\textbf{Proof.} Straightforward.

1.7 Proposition. If $R$ is a ring with a unit element, then $X(R)$ is a compact space.

\textbf{Proof.} In view of the Alexander subbase theorem, it suffices to show that if

$$X(R) = \bigcup \{\text{supp}(A_\alpha)\mid A_\alpha \in \Lambda\},$$

then there exists a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ in $\Lambda$ such that $X(R) = \bigcup_{i=1}^n \text{supp}(A_{\alpha_i})$. Since $X(R) = \bigcup \{\text{supp}(A_\alpha)\mid \alpha \in \Lambda\}$, $b(\Sigma_{\alpha \in \Lambda} A_\alpha) = \bigcap \{b(A_\alpha)\mid \alpha \in \Lambda\} = \emptyset$, and hence $\Sigma_{\alpha \in \Lambda} A_\alpha = R$. Thus $1 = a_{\alpha_1} + a_{\alpha_2} + \cdots + a_{\alpha_n}$ for some $a_{\alpha_i} \in A_{\alpha_i}$, $i = 1, 2, \ldots, n$, and $\bigcap_{i=1}^n b(A_{\alpha_i}) = \emptyset$. Therefore, $X(R) = \bigcup_{i=1}^n \text{supp}(A_{\alpha_i})$.

1.8 Proposition. If $Y$ is an irreducible subset of $X(R)$, then the closure $\overline{Y}$ of $Y$ is equal to $b(j(Y))$.

\textbf{Proof.} Clearly, $\overline{Y} \subseteq b(j(Y))$. Let $x \in b(j(Y))$. If $x \notin \overline{Y}$, then there exists a finite number of right ideals $A_1, A_2, \ldots, A_n$ in $R$ such that $x \in \bigcap_{i=1}^n \text{supp}(A_i)$ and $Y \cap (\bigcap_{i=1}^n \text{supp}(A_i)) = \emptyset$. Hence $Y \subseteq \bigcup_{i=1}^n b(A_i)$. Since $Y$ is irreducible, $Y \subseteq b(A_i)$ for some $A_i$. Hence $A_i \subseteq j(Y) \subseteq j(x)$. This is impossible since $x \notin \text{supp}(A_i)$. 

1.9 Corollary. $X(R)$ is a $T_1$-space.

Proof. Let $x \in X(R)$. Then $\{x\} = b(j(x)) = \{x\}$ by 1.8.

1.10 Example. It is not true, in general, that if $Y$ is a subset of $X(R)$ then $\overline{Y} = b(j(Y))$. For example, let $R$ be the ring of $2 \times 2$ matrices over the field of real numbers and let $x = \{(a \ b) | a, b \text{ are real numbers}\}$ and $y = \{(0 \ c) | c, d \text{ are real numbers}\}$. Then $\{x, y\} = \{x, y\}$, but $b(j(x, y)) = X(R)$.

1.11 Proposition. $X(R)$ is reducible if and only if there exists a finite number of right ideals $A_1, A_2, \ldots, A_n$, $n \geq 2$, in $R$ such that $X(R)$ is an irredundant union of $b(A_1), b(A_2), \ldots, b(A_n)$.

Proof. Straightforward.

1.12 Proposition. If $R$ is a primitive ring and $A$ is a right ideal of $R$, then either $\text{supp}(A) = \emptyset$ or $j(\text{supp}(A)) = \{0\}$.

Proof. Suppose $\text{supp}(A) \neq \emptyset$ and $j(\text{supp}(A)) \neq \{0\}$. Let $B = j(\text{supp}(A))$. Then $B$ is a nonzero proper right ideal of $R$, $\text{supp}(A) \subseteq b(B)$ and $\text{supp}(B) \cap \text{supp}(A) = \emptyset$. Since $B$ is a nonzero right ideal of a primitive ring, $\text{supp}(B) \neq \emptyset$ and $X(R) = b(B) \cup b(A)$.

Let $M$ be a faithful simple (right) $R$-module. Since for each $0 \neq m \in M$, $m^+ = \{r \in R | mr = 0\}$ is a maximal regular right ideal of $R$, either $mA = 0$ or $mB = 0$.

Let $M_1 = \{m \in M | mA = 0\}$ and $M_2 = \{m \in M | mB = 0\}$. Then $M_1$ and $M_2$ are subgroups of $M$ and $M = M_1 \cup M_2$. Hence either $M = M_1$ or $M = M_2$. Therefore either $MA = 0$ or $MB = 0$. This is impossible since $M$ is faithful.

1.13 Remark. If $R$ is a ring with 1 and $X(R)$ is irreducible, then $R$ is isomorphic to the ring of global sections of the simple $R$-sheaf over $X(R)$ (refer to [3, p. 45] for the definition of a simple sheaf). Hence $R$ can be identified with the ring of all continuous functions from $X(R)$ to $R$. To see this, let $\hat{r}(x) = (x, r)$ for every $r \in R$ and $x \in X(R)$. Then $\hat{r}$ is a global section of the simple sheaf over $X(R)$. If $f$ is an arbitrary global section, then $f(X(R)) = X(R) \times \{r\}$ for some $r \in R$ since $X(R)$ is irreducible (hence it is connected). Thus $f = \hat{r}$. Clearly, $r \mapsto \hat{r}$ is an isomorphism of $R$ onto the ring of all global sections.

2. Primitive rings with reducible maximal regular right ideal spaces. In this section, we will give a structure theorem of primitive rings whose maximal regular right ideal spaces are reducible. The basic facts about a primitive ring, which we will use freely in this section, could be found in [2].

2.1 Lemma. Let $V$ be a vector space over a division ring $D$. Assume that $V = V_1 \cup \cdots \cup V_n$, where the $V_i$ are subspaces of $V$, $n \geq 2$, and the union is irredundant. Then $D$ is a finite field, and the dimension of $V/(V_1 \cap \cdots \cap V_n)$ is finite.
Proof. The fact that $D$ is a finite field follows from Lemma 2 of [1, p. 32]. Suppose that the dimension of $V/(V_1 \cap \cdots \cap V_n)$ is infinite. Since $V/\bigcap_{i=1}^n V_i$ is the irredundant union of proper subspaces $V_i/\bigcap_{j \neq i} V_j$, we may assume, without loss of generality, that $\bigcap_{i=1}^n V_i = \{0\}$. Since $V$ is infinite dimensional, there is a subspace $V_{t_1}$ for some $t_1$, $1 \leq t_1 \leq n$, such that $V_{t_1}$ is infinite dimensional. Let $i$ be a positive integer less than or equal to $n$ such that $i \neq t_1$. Let $v_i \in V_i \setminus \bigcup_{k \neq i} V_k$. Let $N(t_1) = \{v_i + w | w \in V_{t_1}\}$. Since $N(t_1) \cap V_{t_1} = \emptyset$, $N(t_1) \subseteq \bigcup_{k \neq t_1} V_k$. Since $N(t_1)$ is an infinite set, there is a subspace $V_{t_2}$ for some $t_2$ such that $1 \leq t_2 \leq n$ and $t_2 \neq t_1$ and $V_{t_2}$ contains infinitely many $v_i + w$'s, say $v_i + w_1$, $v_i + w_2$, \ldots, where $w_j \in N(t_1)$. It follows that $w_1 - w_j = (v_i + w_1) - (v_i + w_j) \in V_{t_2}$ for infinitely many $w_j$'s in $V_{t_1}$. Hence $V_{t_1} \cap V_{t_2}$ is an infinite set and hence it is infinite dimensional since $V_{t_1} \cap V_{t_2}$ is a vector space over the finite field $D$. Now assume that $V_{t_1} \cap V_{t_2} \cap \cdots \cap V_{t_k}$ is infinite dimensional for some distinct positive integers $t_1$, $t_2$, \ldots, $t_k$ each of which is less than or equal to $n$. Let $i$ be a positive integer less than or equal to $n$ such that $i \notin \{t_1, t_2, \ldots, t_k\}$. Let $v_i \in V_i \setminus \bigcup_{j \neq i} V_i$. Define $N(t_1, t_2, \ldots, t_k) = \{v_i + w | w \in \bigcap_{j=1}^k V_{t_j}\}$. Then $N(t_1, t_2, \ldots, t_k) \cap V_{t_j} = \emptyset$ for every $t_j$, $1 \leq j \leq k$. Hence there is a subspace $V_{t_{k+1}}$ for some $1 \leq t_{k+1} \leq n$ such that $V_{t_{k+1}}$ contains infinitely many elements of $N(t_1, t_2, \ldots, t_k)$; hence $\bigcap_{i=1}^{k+1} V_{t_i}$ is infinite dimensional. Thus, by inductive argument, $\bigcap_{i=1}^n V_i$ is an infinite dimensional space, which is absurd.

2.2 Remark. If $V$ is a vector space over a finite field, say $D$, such that $\dim V \geq 2$, then $V$ is a finite union of proper subspaces. Let $v_1$, $v_2$ be linearly independent elements in $V$ and let $N$ be a subspace such that $V = Dv_1 \oplus Dv_2 \oplus N$. For every pair $(\alpha, \beta) \in D \times D$, define $U(\alpha, \beta) = D(\alpha v_1 + \beta v_2) \oplus N$. Then $U(\alpha, \beta) \in D \times D$.

2.3 Lemma. Let $V$ be a vector space of dimension at least 2 over a finite field $D$. Let $R$ be a dense ring of linear transformations of $V$, such that the socle $S$ of $R$ is not zero. Let $v$ and $w$ be linearly independent vectors of $V$. Then there is a subspace $W$ of $V$ such that

(a) $V = W \oplus Dv \oplus Dw$;

(b) for each $\alpha, \beta$ in $D$, $U(\alpha, \beta) = W \oplus D(\alpha v + \beta w)$ and $U(\alpha, \beta)^\perp = \{r \in R | U(\alpha, \beta)r = 0\}$; and

(c) $X(R) = U_{\alpha, \beta \in D}(U(\alpha, \beta)^\perp) \cup b(S)$.

Proof. Since $S$ is also a dense ring of linear transformations of $V$, and $v$, $w$ are linearly independent, there is an $s$ in $S$ such that $vs = v$ and $ws = 0$. Since $Vs$ is a finite dimensional subspace of $V$, there exist $v_2$, $v_3$, \ldots, $v_n$ in $Vs$ such that $\{v, v_2, v_3, \ldots, v_n\}$ is a basis for $Vs$. Let $t \in R$ such that $vt = v$.
and \( v_i t = v \) for all \( i \) such that \( 2 \leq i \leq n \). Let \( a = st \). Then \( V a = D v \), \( v a = v \) and \( w a = 0 \). In a similar manner, we can choose \( b \) in \( R \) such that \( V b = D w \), \( w b = w \) and \( v b = 0 \). Let \( W = \ker a \cap \ker b \). Then \( W = \ker(a + b) \) since \( D v \cap D w = \{0\} \). Since \( v a = v \), \( w b = w \), and \( w a = 0 = v b \), for any \( x \) in \( V \), \( (x(a + b) - x) \cdot (a + b) = 0 \). Hence \( V = W \oplus D v \oplus D w \) and \( W^\perp \supseteq \{a, b\} \). For \( \alpha, \beta \) in \( D \), let \( U(\alpha, \beta) = W \oplus D(\alpha v + \beta w) \). We claim that the right ideal \( U(\alpha, \beta)^\perp = \{r \in R | U(\alpha, \beta) r = 0\} \) is not zero. It is clear that if either \( \alpha = 0 \) or \( \beta = 0 \), then either \( a \) or \( b \) is an element of \( U(\alpha, \beta) \). So assume \( \alpha \neq 0 \) and \( \beta \neq 0 \). Then \( \beta w b \neq 0 \) and there exists \( r \) in \( R \) such that \( \beta w b r = \alpha \omega a \). Now \( (\alpha v + \beta w) b r = \beta w b r = \alpha \omega a = (\alpha v + \beta w) a \). Hence \( (\alpha v + \beta w) (b r - a) = 0 \) and \( b r - a \in U(\alpha, \beta)^\perp \).

If \( x \) is an element of \( X(R) \), then \( \langle x \rangle \supseteq S \) or \( R/\langle x \rangle \cong V \) as \( R \)-modules. Hence if \( x \notin b(S) \), then \( \langle x \rangle = \{r \in R | r v = 0\} \) for some \( \neq v \in V \). In this case, \( v \in U(\alpha, \beta) \) for some \( \alpha, \beta \) in \( D \) and \( \langle x \rangle \supseteq U(\alpha, \beta)^\perp \).

2.4 Theorem. Let \( R \) be a (right) primitive ring. Then \( X(R) \) is reducible if and only if \( R \) is isomorphic to a dense ring with nonzero socle of linear transformations of a vector space of dimension two or more over a finite field.

Proof. Assume that \( X(R) \) is reducible. Then by 1.11, there exists a finite number of right ideals \( A_1, A_2, \ldots, A_n, n \geq 2 \), in \( R \) such that \( X(R) = \bigcup_{i=1}^{n} b(A_i) \) and such that this union is irredundant. Let \( V \) be a faithful simple (right) \( R \)-module and let \( D = \text{End}_R(V) \). Then \( D \) is a division ring, \( V \) is a left vector space over \( D \) and \( R \) is a dense ring of linear transformations of \( V \) over \( D \). For each \( i, 1 \leq i \leq n \), define \( V_i = \{v \in V | v A_i = 0\} \). Then \( V_i \) is a subspace of \( V \) and \( V_i \neq V \) for every \( i \) since \( A_i \neq 0 \) and \( V \) is faithful. For any \( 0 \neq v \in V \), \( v^\perp = \{r \in R | r v = 0\} \) is a member of \( X(R) \). Therefore \( v^\perp \) is a member of some \( b(A_i) \) and hence \( v \in V_i \) and \( V = \bigcup_{i=1}^{n} V_i \). Thus, by 2.1, \( D \) is a finite field and the dimension of \( V \cap \bigcap_{i=1}^{n} V_i \) is finite. Therefore, there is a finite dimensional subspace \( M_i \) such that \( M_i \oplus V_i = V \) for each \( i \) and \( \dim V \geq 2 \) since \( V_i \neq V \). Thus every element in the right ideal \( A_i \) is of finite rank and the socle of \( R \) is not zero. Conversely, assume that \( R \) is isomorphic to a dense ring with nonzero socle \( S \) of linear transformations of a vector space \( V \) of dimension two or more over a finite field \( D \). Then by 2.3(c), \( X(R) = \bigcup_{\alpha, \beta \in D} b(U(\alpha, \beta)^\perp) \cup b(S) \) and hence \( X(R) \) is reducible.

2.5 Theorem. Let \( R \) be a primitive ring and \( S \) be the socle of \( R \). If \( X(R) \) is a Hausdorff space, then either \( R \) is a division ring or \( R/S \) is a radical ring and \( R \) is a dense ring of linear transformations of a vector space over a finite field.
Proof. If \( X(R) \) is a Hausdorff space and \( R \) is not a division ring, then certainly \( X(R) \) is reducible. Hence by 2.4, \( R \) has nonzero socle and it is isomorphic to a dense ring of linear transformations of a vector space \( V \) of dimension two or more over a finite field \( D \). If \( R/S \) is not a radical ring, then \( b(S) \neq \emptyset \). Let \( x, y \) be two points in \( X(R) \) such that \( x \notin b(S) \) and \( y \in b(S) \). We shall show that \( x \) and \( y \) cannot be separated in \( X(R) \). Since \( x \notin b(S) \), \( f(x) = v^\perp \) for some \( 0 \neq v \in V \).

Suppose there exist right ideals \( S_1, S_2, \ldots, S_p, T_1, T_2, \ldots, T_q \) such that \( x \in \bigcap_{i=1}^p \text{supp}(S_i) \), \( y \in \bigcap_{j=1}^q \text{supp}(T_j) \) such that \( \bigcap_{i=1}^p \text{supp}(S_i) \cap \bigcap_{j=1}^q \text{supp}(T_j) = \emptyset \). First we note that the separation still holds if we replace \( S_i \) by \( S_i S \) for every \( i, 1 \leq i \leq p \), since \( x \notin \text{supp}(S_i) \cap \text{supp}(S) = \text{supp}(S S) \). Thus, without loss of generality, we may assume that \( S_i \subseteq S \) for every \( i \). Since \( f(x) = v^\perp \) and \( x \in \bigcap_{i=1}^p \text{supp}(S_i) \), \( v S_i \neq 0 \) for every \( i \). Let \( s_i \in S_i \) such that \( v s_i \neq 0 \) for every \( i \). Since each \( s_i \) is of finite rank, \( V/\text{Ker} s_i \) is finite dimensional. Thus, \( V/\bigcap_{i=1}^p \text{Ker} s_i \) being a subdirect sum of the vector spaces \( V/\text{Ker} s_i \) is finite dimensional. Note that \( T_j \notin j(y) \) for every \( j, 1 \leq j \leq q \), and \( S \subseteq f(y) \). Therefore \( T_j \subseteq S \neq \emptyset \) for every \( j \). Let \( t_j \in T_j \setminus S \) and let \( w_j = \sum_{i=1}^p \text{Ker} s_i |w_{t_j} \in \Sigma_{i=1}^q \text{Dut}_i \) for every \( j, 1 \leq j \leq q \). We claim that \( \bigcap_{i=1}^p \text{Ker} s_i /W_j \) is infinite dimensional for every \( j \). For if \( \bigcap_{i=1}^p \text{Ker} s_i /W_j \) is finite dimensional for some \( j \) and if \( \bigcap_{i=1}^p \text{Ker} s_i = W_j \) for some finite dimensional subspace \( U_j \), then \( \bigcap_{i=1}^p \text{Ker} s_i /W_j = W_j t_j + U_j \) is a finite dimensional subspace of \( V \). Since \( V/\bigcap_{i=1}^p \text{Ker} s_i \) is finite dimensional, there is a finite dimensional subspace \( W \) such that \( V = (\bigcap_{i=1}^p \text{Ker} s_i) \oplus W \). Then \( Vt_j = (\bigcap_{i=1}^p \text{Ker} s_i) t_j + Wt_j \) is also finite dimensional and \( t_j \notin S \). This is impossible since, by choice, \( t_j \notin S \). Thus \( \bigcap_{i=1}^p \text{Ker} s_i /W_j \) is infinite dimensional for each \( j \) and \( \bigcap_{i=1}^p \text{Ker} s_i /W_j \neq \bigcup_{i=1}^q W_j \). Hence there exists \( w \in \bigcap_{i=1}^p \text{Ker} s_i \) such that \( w \notin \bigcup_{i=1}^q W_j \). This means that \( w s_i = 0 \) for every \( i, 1 \leq i \leq p \), and \( w t_j \notin \Sigma_{i=1}^q \text{Dut}_i \) for every \( j, 1 \leq i \leq q \). Now, we consider the vector \( v + w \). Then \( (v + w)s_i = vs_i \neq 0 \) for every \( i \) such that \( 1 \leq i \leq p \) and \( (v + w)t_j \neq 0 \) for every \( j, 1 \leq j \leq q \). For if \( (v + w)t_j = 0 \) for some \( j \), then \( w t_j = -v t_j \in \Sigma_{i=1}^q \text{Dut}_i \). Thus \( (v + w)^\perp \notin (\bigcap_{i=1}^p \text{supp}(S_i)) \cap (\bigcap_{j=1}^q \text{supp}(T_j)) = \emptyset \). This is a contradiction.

2.6 Theorem. Let \( R \) be a primitive ring and \( S \) be the socle of \( R \). If \( R/S \) is a radical ring and \( R \) is isomorphic to a dense ring of linear transformations of a vector space \( V \) over a finite field \( D \), then \( X(R) \) is a Hausdorff space.

Proof. Let \( x, y \) be two distinct members of \( X(R) \). Since \( R/S \) is a radical ring, there exist \( v \) and \( w \) in \( V \) such that \( f(x) = v^\perp \), \( f(y) = w^\perp \) and \( Dv \cap Dw = \{0\} \). Moreover, \( v^\perp \cap S \notin w^\perp \) and \( w^\perp \cap S \notin v^\perp \). Let \( a \in (w^\perp \cap S) \setminus v^\perp \) and \( b \in (v^\perp \cap S) \setminus w^\perp \). Then \( aw = wb = 0 \), \( va \neq 0 \), and \( wb \neq 0 \). Since \( a \in S \), \( Va = Dva \oplus U \) for some finite dimensional subspace \( U \) of \( V \). Hence there is \( r \) in \( R \) such that
Ur = 0 but var = v. Therefore, Var = Dv, war = 0, and V = Ker ar ⊕ Dv. Likewise there is r₀ in R such that Vbr₀ = Dw, vbr₀ = 0, wbr₀ = w, and V = Ker br₀ ⊕ Dw. Since Ker ar ≠ Ker br₀ and the codimensions of Ker ar and Ker br₀ are 1, V = Ker ar + Ker br₀. Thus dim(Ker ar/Ker ar ∩ Ker br₀) = dim(V/Ker br₀) = 1. Hence codim(Ker ar ∩ Ker br₀) = dim(V/Ker br₀) + dim(Ker ar/Ker ar ∩ Ker br₀) = 2 and therefore, V = (Ker ar ∩ Ker br₀) ⊕ Dw ⊕ Dv. For every α, β in D, define U(α, β) = (Ker ar ∩ Ker br₀) ⊕ D(αv + βw). Then V = ∪ₐ,β∈D U(α, β). We shall show that U(α, β) ≠ 0 for each pair (α, β) ∈ D × D. Clearly ar ∈ U(0, β)⊥ and br₀ ∈ U(α, 0)⊥. Assume α ≠ 0 and β ≠ 0. Then there is c ∈ R such that αvc = βw. Consequently, (αv + βw)arc = αvarc = αvc = βw = (αv + βw)br₀ since var = v and wbr₀ = w. Hence D(αv + βw) · (arc - br₀) = 0. Clearly, arc - br₀ ≠ 0 since var = v and wbr₀ = w. Thus 0 ≠ arc - br₀ ∈ U(α, β)⊥. Let

Ω₁ = ∩ₐ,β∈D×D supp(U(α, β)⊥) and Ω₂ = ∩ₐ,β∈D×D (α, β) ≠ 0 supp(U(α, β)⊥).

Recall j(x) = v⊥ and j(y) = w⊥. If x ∉ Ω₁ then U(α, β)⊥ ∩ v⊥ for some β ≠ 0 in D. For every f ∈ U(α, β)⊥, uf = 0. Consequently, wf = 0 also since β = 0 and hence Vf = 0. This means that U(α, β)⊥ = 0, a contradiction. Thus x ∈ Ω₁. A similar argument shows that y ∈ Ω₂. We now claim that Ω₁ ∩ Ω₂ = ∅. For if z ∈ Ω₁ ∩ Ω₂ then j(z) = v₀⊥ for some v₀ ∈ V and v₀ = v' + αv + βw for some v' ∈ Ker ar ∩ Ker br₀ and α, β ∈ D. It follows that v₀⊥ ≥ U(α, β)⊥ and z ∉ Ω₁ ∩ Ω₂, a contradiction. Therefore, X(R) is Hausdorff.

2.7 Corollary. If R is a primitive ring with 1, then X(R) is a Hausdorff space if and only if either R is a division ring or R is a finite ring.

Proof. If R is a finite ring or a division ring then certainly X(R) is a finite T₁-space. Hence it is a Hausdorff space. Conversely, if X(R) is a Hausdorff space, then by 2.5, R is either a division ring or a dense ring of linear transformations of finite rank of a vector space over a finite field. In the latter case, since 1 ∈ R, R must be the complete ring of linear transformations of a finite dimensional vector space over a finite field. Thus, R is a finite ring.

2.8 Example. Let Z be the ring of integers and let V be the set of finite sequences over Z/(2). Then V becomes an ℤ₀-dimensional vector space over Z/(2). Let R be the ring of linear transformations on V and S be the ideal of linear transformations of finite rank. Then R/S is a simple ring with 1 (refer to [2, Theorem 1, p. 93]). Hence by 2.5, X(R) is not a Hausdorff space. However, X(R) is a reducible space by 2.4 and X(S) is a Hausdorff space by 2.6.
2.9 Example. Let $R$ be the ring of infinite row finite matrices of the following form:

$$\begin{pmatrix}
A_n & * \\
0 & 0 \\
0 & 
\end{pmatrix}$$

where $A_n$ is an $n \times n$ matrix over $\mathbb{Z}/(2)$ for some $n$. Then $R$ is a dense ring of the vector space of sequences over $\mathbb{Z}/(2)$. If $S$ is the socle of $R$ then $R/S$ is a radical ring.

3. Finite dimensional maximal regular right ideal space of a primitive ring.

If $X$ is a topological space, then the combinatorial dimension of $X$, $\dim X$, is the supremum of the positive integer $n$ such that there is a strictly ascending chain of nonempty closed irreducible subsets of $X$, $\emptyset \neq F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n$ (refer to [4, p. 156]).

3.1 Theorem. If $R$ is a dense ring of linear transformations of a vector space $M$ over an infinite division ring $D$, then $\dim M = n + 1$ if and only if $\dim X(R) = n$.

Proof. Assume $\dim M = n + 1$ and let $\{m_1, m_2, \ldots, m_{n+1}\}$ be a basis for the vector space $M$. Then $\bigcap_{i=1}^{n+1} m_i = \{0\}$. Hence $X(R) = b(\bigcap_{i=1}^{n+1} m_i) \supsetneq b(\bigcap_{i=2}^{n+1} m_i) \supsetneq \cdots \supsetneq b(m_1 \cap m_2) \supsetneq b(m_1) = \{m_1\}$. Since $R$ is a simple artinian ring, if $x \in X(R)$ then $v(x) = v^k$ for some vector $v \neq 0$ in $M$. Hence if $x \in b(\bigcap_{i=1}^{n+1} m_i)$, then $v \in \sum_{i=1}^{n+1} Dm_i$. Therefore, if $b(\bigcap_{i=1}^{n+1} m_i)$ were reducible, then, as in the case of proof of 2.4, the vector space $\sum_{i=1}^{n+1} Dm_i$ would be a finite union of proper subspaces and $D$ would be a finite field by 2.1. Thus $\dim X(R) \geq n$. Now let $F_{n+1} \supsetneq F_n \supsetneq \cdots \supsetneq F_1 \supsetneq F_0 = \emptyset$ be a strictly descending closed irreducible subsets of $X(R)$. Let $A_i = j(F_i)$, $0 \leq i \leq n + 1$. Then $A_n+1 \supsetneq A_n \supsetneq \cdots \supsetneq A_1 \supsetneq A_0$ is a strictly ascending chain of right ideals of $R$ since $b(j(F_i)) = F_i$ for each $i$ by 1.8. Since $R$ is a simple artinian ring, every right ideal of $R$ is a direct summand of $R$. Hence $A_0 = K_1 \oplus A_1$, $A_1 = K_2 \oplus A_2$, $\cdots$, $A_n = K_{n+1} \oplus A_{n+1}$ for some nonzero right ideals $K_1$, $K_2$, $\cdots$, $K_{n+1}$, and $R = K_0 \oplus K_1 \oplus \cdots \oplus K_{n+1} \oplus A_{n+1}$ for some right ideal $K_0$ of $R$. This means that $R$ is a direct sum of at least $n + 2$ minimal right ideals. This is impossible since $R$ is a direct sum of $n + 1$ minimal right ideals and the number of summands is unique. Conversely, let us assume now that $\dim X(R) = n$. Let $B$ be a basis for the vector space $M$. If $B$ is a finite set then, by the first part of the theorem, the number of elements in $B$ must be $n + 1$. So suppose that $\dim M = \infty$. Let $Y = \{m^1 \mid m \in M, m \neq 0\}$. Then $Y$ is a nonempty subspace of $X(R)$ and $\dim Y \leq \dim X(R) = n$ by [4, 9.3, p. 156].
Let $b_1, b_2, \ldots, b_k, \ldots$ be distinct elements in $B$. Then the chain of subsets $b(b_1^k) \cap Y \subseteq b(b_1^k \cap b_2^k) \cap Y \subseteq \ldots \subseteq b(\bigcap_{i=1}^k b_i^k) \cap Y \subseteq \ldots$ is strictly ascending. Since $D$ is an infinite division ring, each $b(\bigcap_{i=1}^k b_i^k) \cap Y$ is irreducible in $Y$ as in the case of proof of 2.4. Hence dim $Y$ is not finite and this is a contradiction.

3.2 Theorem. Let $R$ be a primitive ring. Then $R$ is right artinian if and only if $X(R)$ satisfies the descending chain condition on the subbasic open sets.

Proof. If $R$ is right artinian, then $R$ is a simple artinian ring. Hence if $A$ is a right ideal of $R$, then $j(b(A)) = A$. Hence if $\supp(A_1) \supseteq \supp(A_2) \supseteq \ldots \supseteq \supp(A_n) \supseteq \ldots$ is a chain of subbasic open sets for some right ideals $A_1, A_2, \ldots, A_n, \ldots$ then $b(A_1) \subseteq b(A_2) \subseteq \ldots \subseteq b(A_n) \subseteq \ldots$ and $j(b(A_1)) = A_1 \supseteq j(b(A_2)) = A_2 \supseteq \ldots \supseteq j(b(A_n)) = A_n \supseteq \ldots$. Thus the chain must terminate. Conversely, assume that the descending chain condition holds on the subbasic open sets. Let $V$ be a faithful simple right $R$-module. To prove that $R$ is artinian, it suffices to show that $V$ is a finite dimensional vector space. So suppose the dimension $V$ is infinite. Then there exist infinite independent vectors $v_1, v_2, \ldots$ such that $\supp(v_1^k) \supseteq \supp(v_1^k \cap v_2^l) \ldots$. This is a contradiction.

Acknowledgement. We are deeply indebted to the referee for many invaluable suggestions for the preparation of this paper.

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