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NORM OF A DERIVATION ON A VON NEUMANN ALGEBRA[1]

BY

P. GAJENDRAGADKAR

ABSTRACT. A derivation on an algebra $\mathfrak{A}$ is a linear function $\mathfrak{D}: \mathfrak{A} \to \mathfrak{A}$ satisfying $\mathfrak{D}(ab) = \mathfrak{D}(a)b + a\mathfrak{D}(b)$ for all a, b in $\mathfrak{A}$. If there exists an a in $\mathfrak{A}$ such that $\mathfrak{D}(b) = ab - ba$ for b in $\mathfrak{A}$, then $\mathfrak{D}$ is called the inner derivation induced by a. If $\mathfrak{A}$ is a von Neumann algebra, then by a theorem of Sakai [7], every derivation on $\mathfrak{A}$ is inner. In this paper we compute the norm of a derivation on a von Neumann algebra. Specifically we prove that if $\mathfrak{A}$ is a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, T is in $\mathfrak{A}$, and $\mathfrak{D}_T$ is the derivation induced by T, then $\|\mathfrak{D}_T|\mathfrak{A}\| = 2\inf\{\|T - Z\|, Z \text{ in centre } \mathfrak{A}\}$.

In this paper we answer one of the questions raised by J. Stampfli in [8], viz. computing the norm of a derivation on an arbitrary C^*-algebra. It follows from [3, Lemma 3] that a derivation on a C^*-algebra acting on a Hilbert space $\mathcal{H}$ can be extended to its weak-closure in $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators acting on $\mathcal{H}$. Kaplansky's density theorem proves that this extension is achieved without increasing the norm. So there is no loss of generality in restricting our attention to von Neumann algebras. This result when T is selfadjoint has been proved by Kadison, Lance and Ringrose. But the methods have little in common.

1. **Preliminaries.** We shall use decomposition of a Hilbert space into a direct integral relative to a given abelian von Neumann algebra. We quote two theorems for future reference and refer the reader to [6] for notations and proofs.

Theorem. *To every commutative von Neumann algebra $\mathcal{C}$ on a separable Hilbert space $\mathcal{H}$ there corresponds a decomposition of $\mathcal{H}$ into direct integral with $\mathcal{C}$ as the totality of all operators of the form L_ϕ.*

Theorem. *Under the same hypothesis as in the previous theorem, for an arbitrary family $\mathfrak{A} \subseteq \mathcal{C}'$, every element A in $\mathfrak{A}$ is decomposable. Let $\mathfrak{A}(\lambda) = \{A(\lambda), A \text{ in } \mathfrak{A}\}$. Then the family $\mathfrak{A}(\lambda)$ is irreducible for almost all λ if and only if $\mathcal{C}$ is a maximal abelian subalgebra of $\mathfrak{A}'$.*

In [8], Stampfli has defined the centre of a bounded operator T as the unique

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complex number μ for which $\|T-\mu\| = \inf\{\|T-\lambda\|, \lambda$ a complex number$\}$. Following [8] we denote the centre of a bounded operator T by C_T, centre of an algebra $\mathfrak{A}$ by $\mathfrak{Z}(\mathfrak{A})$, and the complex numbers by $\mathbf{C}$. We assume that all Hilbert spaces are separable.

2. **Lemma 1.** *If* $\mathfrak{A}$ *is a von Neumann algebra acting on a Hilbert space* $\mathcal{H}$ *and if* T *is in* $\mathfrak{A}$, *then there exists* Z_0 *in* $\mathfrak{Z}(\mathfrak{A})$ *such that, for every projection* P *in* $\mathfrak{Z}(\mathfrak{A})$, $\|(T-Z_0)P\| = \inf\{\|(T-Z)P\|, Z \in \mathfrak{Z}(\mathfrak{A})\}$.

Proof. First we note that if $\alpha = \inf\{\|T-Z\|, Z \in \mathfrak{Z}(\mathfrak{A})\}$, then there exists a sequence $\{Z_n\} \subseteq \mathfrak{Z}(\mathfrak{A})$ such that $\|Z_n - T\| \to \alpha$. Also, $\|Z_n\| \le \|T-Z_n\| + \|T\|$ implies that $\{\|Z_n\|\}$ is a bounded sequence. Hence there exists a subsequence $\{Z_{n_k}\}$ such that $Z_{n_k} \to Z_0$ in the weak operator topology. It follows that $Z_0 \in \mathfrak{Z}(\mathfrak{A})$ and further,

$$\|T-Z_0\| \le \underline{\lim}\, \|T-Z_{n_k}\| = \alpha.$$

Thus $\|T-Z_0\| = \alpha$.

Now if P is a projection in $\mathfrak{Z}(\mathfrak{A})$, applying the previous case to the algebra $\mathfrak{A}P$ and operator TP, it follows that there exists $Z_P \in \mathfrak{Z}(\mathfrak{A})$ such that $\|(T-Z_P)P\| = \inf\{\|(T-Z)P\|, Z \in \mathfrak{Z}(\mathfrak{A})\}$.

If $\{P_i\}_{i=1}^n$ is a family of pairwise orthogonal projections in $\mathfrak{Z}(\mathfrak{A})$ with $\Sigma_{i=1}^n P_i = I$, we know that there exist $\{Z_i\}_{i=1}^n \subseteq \mathfrak{Z}(\mathfrak{A})$ satisfying $\|(T-Z_i)P_i\| = \inf\{\|(T-Z)P_i\|, Z \in \mathfrak{Z}(\mathfrak{A})\}$, $1 \le i \le n$. Define $Z^1 = \Sigma_{i=1}^n Z_i P_i$ and denote $Z^1 = Z_{\{P_1,\cdots,P_n\}}$. Then $\|(T-Z^1)P_i\| = \inf\{\|(T-Z)P_i\|, Z \in \mathfrak{Z}(\mathfrak{A})\}$ for $1 \le i \le n$.

Let $\mathfrak{F}$ be the class of all finite orthogonal families $\{Q_1,\cdots,Q_n\}$ with sum I. We say $\{P_1,\cdots,P_m\} \le \{Q_1,\cdots,Q_n\}$ if each P_i is a sum of a subset of Q's. Note that if $\{Q_1,\cdots,Q_n\} \in \mathfrak{F}$, then

$$\|T - Z_{\{Q_1,\cdots,Q_n\}}\| = \inf\{\|T-Z\|, Z \text{ in } \mathfrak{Z}(\mathfrak{A})\}.$$

It follows that $\{Z_{\{Q_1,\cdots,Q_n\}}, \{Q_1,\cdots,Q_n\} \in \mathfrak{F}\}$ is a bounded net. Hence there is a cofinal subset which converges to Z_0 in $\mathfrak{Z}(\mathfrak{A})$ in weak-operator-topology. We now prove that, for any projection Q in $\mathfrak{Z}(\mathfrak{A})$,

$$\|(T-Z_0)Q\| = \inf\{\|(T-Z)Q\|, Z \text{ in } \mathfrak{Z}(\mathfrak{A})\}.$$

For if Q is a projection in $\mathfrak{Z}(\mathfrak{A})$, $\{Q, I-Q\} \in \mathfrak{F}$, and if $\{Q_1,\cdots,Q_n\} \ge \{I-Q, Q\}$, we may write (with possible renumbering) $Q = Q_1 + \cdots + Q_k$, $k \le n$.

$$\begin{aligned}\|(T - Z_{\{Q_1,\cdots,Q_n\}})Q\| &= \sup_{1\le i\le k} \|(T - Z_{\{Q_1,\cdots,Q_n\}})Q_i\| \\ &\le \sup_{1\le i\le k} \|(T-Z)Q_i\| \quad \text{for all } Z \text{ in } \mathfrak{Z}(\mathfrak{A}).\end{aligned}$$

Thus $\|(T - Z_{\{Q_1,\cdots,Q_n\}})Q\| \le \|(T - Z)Q\|$ for all $Z \in \mathfrak{Z}(\mathfrak{A})$ and $\{Q_1, \cdots, Q_n\} \ge \{Q, I - Q\}$.

Since weak limits do not increase norm,

$$\|(T - Z_0)Q\| \le \|(T - Z)Q\| \quad \text{for all } Z \text{ in } \mathfrak{Z}(\mathfrak{A}).$$

Lemma 2. *Let $\mathcal{H} = \int^{\oplus} \mathcal{H}_\lambda \, d\sigma(\lambda)$ and T in $\mathfrak{B}(\mathcal{H})$ be a decomposable operator, i.e. $T = T(\lambda)$ where $\{T(\lambda)\}$ is an essentially bounded, measurable operator function. Then the scalar-valued function $C_{T(\lambda)}$ is essentially bounded and measurable.*

Proof. Let $\mathcal{H} = \int^{\oplus} \mathcal{H}_\lambda d\sigma(\lambda)$ and let T in $\mathfrak{B}(\mathcal{H})$ be written as $T = \{T(\lambda)\}$. Let $\mathfrak{Z} = \{L_\phi, \phi \in L^\infty(\Lambda, \sigma)\}$ and $\mathfrak{A} = \mathfrak{Z}'$ = the algebra of all decomposable operators on $\mathcal{H}$.

Then $T \in \mathfrak{A}$ and $\mathfrak{Z}(\mathfrak{A}) = \mathfrak{A} \cap \mathfrak{A}' = \mathfrak{Z}' \cap \mathfrak{Z} = \mathfrak{Z}$. By Lemma 1, there exists $Z_0 = L_\phi$ in $\mathfrak{Z}$ such that, for every projection P in $\mathfrak{Z}$,

$$\|(T - Z_0)P\| = \inf\{\|(T - Z)P\|, \ Z \in \mathfrak{Z}\}.$$

We assert that $\phi(\lambda) = C_{T(\lambda)}$ a.e.

If $\mu \in \mathbf{C}$ and $\epsilon > 0$, let

$$E = \{\lambda \in \Lambda, \ \|T(\lambda) - \mu I_\lambda\| \le \|T(\lambda) - \phi(\lambda) I_\lambda\| - \epsilon\}$$

and ψ_E denote the characteristic function of E. Then $Z_1 = \{\mu I_\lambda\} \in \mathfrak{Z}$ and $P = \{\psi_E(\lambda) I_\lambda\}$ is a projection in $\mathfrak{Z}$. Since $\|(T - Z_0)P\| \le \|(T - Z_1)P\|$, it follows that $\sigma(E) = 0$. If $\{\mu_n\}_1^\infty$ is a dense subset of complex numbers, write

$$F = \bigcup_{n=1}^{\infty} \{\lambda \in \Lambda, \ \|T(\lambda) - \mu_n I_\lambda\| < \|T(\lambda) - \phi(\lambda) I_\lambda\|\}.$$

It follows that $\sigma(F) = 0$.

If $\lambda \notin F$ and $\mu \in \mathbf{C}$, $\mu_{n_k} \to \mu$ for some subsequence $\{n_k\}$, and

$$\|T(\lambda) - \mu I_\lambda\| = \lim_{k \to \infty} \|T(\lambda) - \mu_{n_k} I_\lambda\| \ge \|T(\lambda) - \phi(\lambda) I_\lambda\|.$$

By uniqueness of $C_{T(\lambda)}$, it follows that, for $\lambda \notin F$, $C_{T(\lambda)} = \phi(\lambda)$. Since $\sigma(F) = 0$, this proves the claim and consequently the lemma.

Corollary 1. *The element Z obtained in Lemma 1 is unique, when $\mathfrak{A}$ is a type I algebra.*

Theorem 1. *Let $\mathfrak{A}$ be a von Neumann algebra on $\mathcal{H}$ and assume that $\mathfrak{A}'$ is abelian. Then for T in $\mathfrak{A}$, there exists Z_0 in $\mathfrak{Z}(\mathfrak{A}) = \mathfrak{A}'$ such that $\|\mathfrak{D}_T \mid \mathfrak{A}\| = 2\|T - Z_0\|$.*

Proof. Let $\mathcal{H} = \int_\lambda^{\oplus} \mathcal{H}_\lambda \, d\sigma(\lambda)$ be the decomposition corresponding to the abelian algebra $\mathcal{Z}(\mathfrak{A}) = \mathfrak{A}'$. Then $\mathcal{Z}(\mathfrak{A}) = \{L_\phi, \phi \in L^\infty(\Lambda, \sigma)\}$ and $\mathfrak{A}$ = the family of all decomposable operators on $\mathcal{H}$. Also there exists a σ-null set F such that $\lambda \notin F$ implies $\mathfrak{A}(\lambda)$ is an irreducible algebra. Define $\phi(\lambda) = C_{T(\lambda)}$ if $\lambda \notin F$ and $\phi(\lambda) = 0$ if $\lambda \in F$. Then by Lemma 2, $\phi \in L^\infty(\Lambda, \sigma)$ and hence $L_\phi \in \mathcal{Z}(\mathfrak{A})$. By Stampfli's theorem [8], for $\lambda \notin F$, $\|\mathfrak{D}_{T(\lambda)} \mid \mathfrak{A}(\lambda)\| = 2\|T(\lambda) - C_{T(\lambda)}I_\lambda\|$ and thus

$$2\|T - L_\phi\| = 2 \operatorname{ess\,sup}_\lambda \|T(\lambda) - C_{T(\lambda)}I_\lambda\| = \operatorname{ess\,sup}_\lambda \|\mathfrak{D}_{T(\lambda)} \mid \mathfrak{A}(\lambda)\|.$$

We prove that $\operatorname{ess\,sup}_\lambda \|\mathfrak{D}_{T(\lambda)} \mid \mathfrak{A}(\lambda)\| \le \|\mathfrak{D}_T \mid \mathfrak{A}\|$.

We set $\Lambda_k = \{\lambda \in \Lambda, \dim \mathcal{H}_\lambda = k\}$. Then $\Lambda = \bigcup_k \Lambda_k$ and it is sufficient to prove that

$$\sigma\{\lambda \in \Lambda_k, \|\mathfrak{D}_{T(\lambda)}\| > \|\mathfrak{D}_T \mid \mathfrak{A}\|\} = 0 \quad \text{for all } k.$$

Fix k for which $\sigma(\Lambda_k) \neq 0$ and write $\mathcal{H}_k = \int_{\Lambda_k}^{\oplus} \mathcal{H}_\lambda \, d\sigma(\lambda)$ where all $\mathcal{H}_\lambda$ can be identified with the same Hilbert space $\mathcal{K}$. Let $\{B_n, n \ge 1\}$ be a sequence which is weakly dense in the unit ball of $\mathcal{B}(\mathcal{K})$. Defining $A_n(\lambda) = B_n$ for $\lambda \in \Lambda_k$ and $A_n(\lambda) = 0$ for $\lambda \notin \Lambda_k$, we see that $A_n = A_n(\lambda)$ is an essentially bounded, measurable, operator function and hence $A_n \in \mathcal{B}(\mathcal{H})$ with $\|A_n\| \le 1$. Since by definition A_n is a decomposable operator, $A_n \in \mathfrak{A}$. Now

$$\begin{aligned}\operatorname*{ess\,sup}_{\lambda \in \Lambda_k} \|B_n T(\lambda) - T(\lambda)B_n\| &= \operatorname*{ess\,sup}_{\lambda \in \Lambda} \|A_n(\lambda)T(\lambda) - T(\lambda)A_n(\lambda)\| \\ &= \|A_n T - TA_n\| \le \|\mathfrak{D}_T \mid \mathfrak{A}\|.\end{aligned}$$

Hence if we write $E_n = \{\lambda \in \Lambda_k, \|T(\lambda)B_n - B_nT(\lambda)\| > \|\mathfrak{D}_T \mid \mathfrak{A}\|\}$ and $E = \bigcup_{n=1}^\infty E_n$, we see that $\sigma(E_n) = 0$ and hence $\sigma(E) = 0$.

Next, we will show that, for $\lambda \in \Lambda_k$, $\lambda \notin E$,

$$\|\mathfrak{D}_{T(\lambda)} \mid \mathfrak{A}(\lambda)\| \le \|\mathfrak{D}_T \mid \mathfrak{A}\|.$$

Fix $\lambda_0 \in \Lambda_k$, $\lambda_0 \notin E$ and let $A \in \mathfrak{A}$ with $\|A(\lambda_0)\| \le 1$. Then there exists a subsequence $\{B_{n_k}\}$ (depending on λ_0), such that $B_{n_k} \to A(\lambda_0)$ in weak-operator-topology. Hence

$$\|\mathfrak{D}_T(A)(\lambda_0)\| \le \underline{\lim} \|B_{n_k}T(\lambda_0) - T(\lambda_0)B_{n_k}\| \le \|\mathfrak{D}_T \mid \mathfrak{A}\|.$$

Since $\mathfrak{D}_T(A)(\lambda_0) = \mathfrak{D}_{T(\lambda_0)}A(\lambda_0)$ we have proved that

$$\|\mathfrak{D}_{T(\lambda)} \mid \mathfrak{A}(\lambda)\| \le \|\mathfrak{D}_T \mid \mathfrak{A}\| \quad \text{for all } \lambda \in \Lambda_k, \ \lambda \notin E.$$

Thus $\operatorname{ess\,sup}_\lambda \|\mathfrak{D}_{T(\lambda)} \mid \mathfrak{A}(\lambda)\| \le \|\mathfrak{D}_T \mid \mathfrak{A}\|$. Hence

$$2\|T - L_\phi\| = \operatorname{ess\,sup}_\lambda \|\mathfrak{D}_{T(\lambda)} \mid \mathfrak{A}(\lambda)\| = \|\mathfrak{D}_T \mid \mathfrak{A}\| \le 2\|T - L_\phi\|$$

and

$$2\|T - L_{\phi}\| = \|\mathfrak{D}_T \,|\, \mathfrak{A}\|.$$

Corollary 2. *If* $\mathfrak{A}$ *is a von Neumann algebra with* $\mathfrak{A}'$ *abelian and* T *is in* $\mathfrak{A}$, *then there exists* Z_0 *in* $\mathcal{Z}(\mathfrak{A})$ *such that, for every projection* P *in* $\mathcal{Z}(\mathfrak{A})$,

$$\|\mathfrak{D}_{TP} \,|\, \mathfrak{A}P\| = 2\|(T - Z_0)P\|.$$

Proof. Let P be a projection in $\mathcal{Z}(\mathfrak{A}) = \mathfrak{A}'$. Consider the algebra $\mathfrak{A}P$ and $S = TP$. Since $(\mathfrak{A}P)' = \mathfrak{A}'P$, $(\mathfrak{A}P)'$ is abelian.

Since P is a projection in $\mathcal{Z}(\mathfrak{A})$, $P = L_\eta$, where η is the characteristic function of a measurable set E. By Theorem 1, we have

$$\|\mathfrak{D}_S \,|\, \mathfrak{A}P\| = 2\|S - L_\phi\| \quad \text{where } \psi(\lambda) = C_{S(\lambda)} \text{ a.e.}$$

It follows that $C_{S(\lambda)} = \eta(\lambda)C_{T(\lambda)}$ and hence

$$\|\mathfrak{D}_S \,|\, \mathfrak{A}P\| = 2\|S - L_\psi\| = 2\operatorname{ess\,sup}_\lambda \|(T(\lambda) - C_{T(\lambda)})\eta(\lambda)\| = 2\|(T - Z_0)P\|.$$

Theorem 2. *Let* $\mathfrak{A}$ *be a von Neumann algebra on* $\mathcal{H}$, *and let* T *be in* $\mathfrak{A}$. *Then there exists* ξ *in* $\mathfrak{A}'$ *such that for every projection* P *in* $\mathcal{Z}(\mathfrak{A})$, $\|\mathfrak{D}_{TP} \,|\, \mathfrak{A}P\| = 2\|(T - \xi)P\|$.

Proof. Let $\mathfrak{A}_1$ be the type I algebra containing $\mathfrak{A}$, as constructed in [3, p. 283]. Then by Kadison's argument, $\|\mathfrak{D}_T \,|\, \mathfrak{A}\| = \|\mathfrak{D}_T \,|\, \mathfrak{A}_1\|$. Moreover, applying the same argument to $\mathfrak{A}P$ in place of $\mathfrak{A}$, we see that, for any central projection P in $\mathfrak{A}$, $\|\mathfrak{D}_{TP} \,|\, \mathfrak{A}P\| = \|\mathfrak{D}_{TP} \,|\, \mathfrak{A}_1 P\|$. By Theorem 1, there exists an element ξ in $\mathcal{Z}(\mathfrak{A}_1)$ such that $\|\mathfrak{D}_{TQ} \,|\, \mathfrak{A}_1 Q\| = 2\|(T - \xi)Q\|$ for every central projection Q in $\mathfrak{A}$. Since $\mathcal{Z}(\mathfrak{A}) \subseteq \mathcal{Z}(\mathfrak{A}_1) \subseteq \mathfrak{A}'$, this completes the proof.

Theorem 3. *Let* $\mathfrak{A}$ *be a von Neumann algebra on a separable Hilbert space and let* T *be in* $\mathfrak{A}$. *Then there exists a unique* Z_0 *in* $\mathcal{Z}(\mathfrak{A})$ *such that, for every projection* P *in* $\mathcal{Z}(\mathfrak{A})$, $\|\mathfrak{D}_{TP} \,|\, \mathfrak{A}P\| = 2\|(T - Z_0)P\|$.

Proof. By Theorem 2, there exists ξ in $\mathfrak{A}'$ such that, for every projection P in $\mathcal{Z}(\mathfrak{A})$, $\|\mathfrak{D}_{TP} \,|\, \mathfrak{A}P\| = 2\|(T - \xi)P\|$. Let $\mathcal{K}_\xi = \{\Sigma_{i=1}^n \lambda_i \mathfrak{U}_i \xi \mathfrak{U}_i^*,\ \lambda_i \geq 0,\ \Sigma \lambda_i = 1,$ $\mathfrak{U}_i$ unitary in $\mathfrak{A}'\}$, and $\mathcal{K}_\xi^=$, the norm closure of $\mathcal{K}_\xi$. If ξ' is in $\mathcal{K}_\xi$, and P is a projection in $\mathcal{Z}(\mathfrak{A})$,

$$(T - \xi')P = \sum_{i=1}^n \lambda_i (T - U_i^* \xi \mathfrak{U}_i)P = \sum_{i=1}^n \lambda_i \mathfrak{U}_i^* (TP - \xi P)\mathfrak{U}_i.$$

Hence $\|(T - \xi')P\| \leq \|(T - \xi)P\|$.

Since $\|\mathfrak{D}_{TP} \,|\, \mathfrak{A}P\| \leq 2\|(T - \xi')P\| \leq 2\|(T - \xi)P\| = \|\mathfrak{D}_{TP} \,|\, \mathfrak{A}P\|$, we have $\|\mathfrak{D}_{TP} \,|\, \mathfrak{A}P\| = 2\|(T - \xi')P\|$ for all $\xi' \in \mathcal{K}_\xi$.

By continuity, the same equality holds for all ξ' in $\mathcal{K}_\xi^=$. By [1, Theorem 1,

p. 272], $\mathcal{K}_\xi^= \cap \mathfrak{Z}(\mathfrak{A}') \neq \emptyset$. Hence if Z_0 in $\mathcal{K}_\xi^= \cap \mathfrak{Z}(\mathfrak{A})$, $2\|(T - Z_0)P\| = \|\mathfrak{D}_{TP} \mid \mathfrak{A}P\|$, for every central projection P of $\mathfrak{A}$.

Now we prove uniqueness of Z_0. If $\mathfrak{A}_1$ is the type I algebra containing $\mathfrak{A}$ as before, Q is a projection in $\mathfrak{Z}(\mathfrak{A}_1) \subseteq \mathfrak{A}'$, then we assert that $\|\mathfrak{D}_{TQ} \mid \mathfrak{A}_1 Q\| = 2\|(T - Z_0)Q\|$.

Let P be the central support of Q in $\mathfrak{Z}(\mathfrak{A})$. Then for every A in $\mathfrak{A}$, $\|AP\| = \|AQ\|$ [2].

Now,

$$\|\mathfrak{D}_{TP} \mid \mathfrak{A}P\| = 2\|(T - Z_0)P\| = 2\|(T - Z_0)Q\| \geq \|\mathfrak{D}_{TQ} \mid \mathfrak{A}_1 Q\|.$$

On the other hand, if X is in $\mathfrak{A}$,

$$\begin{aligned} \|\mathfrak{D}_T(X)P\| &= \|\mathfrak{D}_{TP}(XP)\| = \|\mathfrak{D}_T(X)Q\| = \|\mathfrak{D}_{TQ}(XQ)\| \\ &\leq \|\mathfrak{D}_{TQ} \mid \mathfrak{A}_1 Q\| \, \|XQ\| = \|\mathfrak{D}_{TQ} \mid \mathfrak{A}_1 Q\| \, \|XP\|. \end{aligned}$$

Now Corollary 1 proves the uniqueness of Z_0.

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TATA INSTITUTE OF FUNDAMENTAL RESEARCH, BOMBAY 5, INDIA

Current address: Department of Mathematics, University of Toronto, Toronto, Ontario, Canada