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EXPONENTIAL DECAY OF WEAK SOLUTIONS FOR HYPERBOLIC SYSTEMS OF FIRST ORDER WITH DISCONTINUOUS COEFFICIENTS[1]

BY

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ABSTRACT. The weak solution of the Cauchy problem for symmetric hyperbolic systems with discontinuous coefficients in several space variables satisfying the fact that the coefficients and their first derivatives are bounded in the distribution sense is identically equal to zero if it is exponential decay.

1. **Introduction.** For hyperbolic systems of first order with sufficiently smooth coefficients, the existence theorem and the uniqueness theorem of solutions for the Cauchy problem were proved by Petrovskiĭ, Leary and others (cf. Gel'fand [3]). Gel'fand [3] proposed the question of what conditions assure the existence and the uniqueness of solutions of the Cauchy problem in the case of discontinuous coefficients. Recently, Conway [1], and Hurd and Sattinger [4] investigated such problems. Conway proved the existence theorem for the case of a single linear equation in several space variables and the uniqueness theorem for a quasi-linear equation. Hurd and Sattinger proved the existence theorem for the case of first order hyperbolic systems in several space variables and the uniqueness theorem in the case of one space variable. They also proved the existence theorem and the uniqueness theorem for hyperbolic equations of second order. On the other hand, Masuda [5] proved a theorem on the exponential decay of solutions for hyperbolic equations with smooth variable coefficients. In this paper we shall prove a result on the exponential decay of weak solutions for first order hyperbolic systems with discontinuous coefficients.

2. **Preliminaries and the main result.** Let E^{n+1} be the $(n+1)$-dimensional Euclidean space with points denoted by (x, t), where x is a point in the n-dimensional Euclidean space and is denoted by its coordinates $(x_1, x_2, \cdots, x_n)$. Let $\mathfrak{D}$ be the half space $\{(x, t) \in E^{n+1} \mid t \geq 0\}$. The hyperplane $t = \tau$ is denoted by H_τ. We consider two (vector-valued) functions $u = (u_1, u_2, \cdots, u_l)$ and $v = (v_1, v_2, \cdots, v_l)$ defined in $\mathfrak{D}$ and put

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$$\langle u, v\rangle = \sum_{i=1}^{l} u_i v_i \quad \text{and} \quad |u|^2 = \langle u, u\rangle.$$

The space $L^2(\mathfrak{D})$ is the Hilbert space consisting of all measurable functions $u = (u_1, u_2, \cdots, u_l)$ for which

$$\iint_{\mathfrak{D}} |u|^2\, dx\, dt < \infty.$$

For brevity we use the notation

$$(u, u) = \|u\|^2 = \iint_{\mathfrak{D}} |u|^2\, dx\, dt.$$

In the following, $f \in L^2_{\mathrm{loc}}(\Omega)$ for a function or a vector-valued function f defined on a region Ω in some Euclidean space means that f or every component of f is square integrable on every compact subset of Ω.

Now we consider the following Cauchy problem of the first order symmetric hyperbolic system:

$$\frac{\partial}{\partial t} u + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(A^i u) + Bu + C = 0, \tag{1}$$

$$u(x, 0) = \psi(x), \tag{2}$$

where A^i $(1 \leq i \leq n)$ are $l \times l$ symmetric matrices, B is an $l \times l$ matrix, C is an $l \times 1$ matrix and each element of these matrices belongs to $L^2_{\mathrm{loc}}(\mathfrak{D})$. Moreover, we suppose that A^i, B and their first derivatives in the distribution sense are bounded in $\mathfrak{D}$.

A (vector-valued) function u is called a *weak solution* of the Cauchy problem (1), (2) for the Cauchy data $\psi(x) \in L^2_{\mathrm{loc}}(H_0)$ if $u \in L^2_{\mathrm{loc}}(\mathfrak{D})$ satisfies (2) and

$$\begin{aligned} \int_0^t \int_{H_s} \left[\left\langle u, \frac{\partial \phi}{\partial s}\right\rangle + \sum_{i=1}^{n} \left\langle A^i u, \frac{\partial \phi}{\partial x_i}\right\rangle - \langle Bu, \phi\rangle - \langle C, \phi\rangle\right] dx\, ds \\ + \int_{H_0} \langle \psi, \phi\rangle\, dx - \int_{H_t} \langle u, \phi\rangle\, dx = 0 \end{aligned} \tag{3}$$

for any t (> 0) and for any function $\phi \in C^1[0, \infty; H_0^{1,2}(E^n)]$. Here the space $H_0^{1,2}(E^n)$ is the closure of $C_0^\infty(E^n)$ by the norm

$$\|\phi\|_1 = \left(\int_{E^n} \sum_{|\alpha| \leq 1} |D_x^\alpha \phi|^2\, dx\right)^{1/2},$$

and the space $C^1[0, \infty; H_0^{1,2}(E^n)]$ consists of all functions ϕ with the following properties: (i) ϕ is measurable in E^n, (ii) for almost all $t \in [0, \infty)$, the function $\phi(x, t)$ in x belongs to $H_0^{1,2}(E^n)$, and (iii) the norm $\|\phi\|_1$ as a function of t belongs to $C^1([0, \infty))$.

The theorem which we shall prove is as follows:

Theorem. *Suppose that* $u \in L^2_{loc}(\mathfrak{D})$ *is a weak solution of the Cauchy problem for the symmetric hyperbolic system*

$$\frac{\partial}{\partial t}u + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}(A^i u) + Bu = 0, \tag{4}$$

$$u(x, 0) = \psi(x) \tag{2}$$

under the assumption stated above. Further suppose that the following conditions (i), (ii) *and* (iii) *are valid:*

(i) *The coefficients* A^i, B *and their first derivatives in the distribution sense are bounded in* $\mathfrak{D}$.

(ii) *Let* $\Delta t = t' - t$, $\Delta x_i = x_i' - x_i$, *and* $\Delta_i A^j = A^j(x_1, x_2, \cdots, x_i', \cdots, x_n, t) - A^j(x_1, x_2, \cdots, x_n, t)$. *Then there exist nonnegative locally integrable functions* $\nu(t)$, $\mu_i(t)$ $(i = 1, 2, \cdots, n)$ *bounded in* $[0, \infty]$ *such that, in* $\mathfrak{D}$,

$$|\langle B\xi, \xi\rangle| \le \nu(t)\langle \xi, \xi\rangle, \qquad \left|\left\langle \frac{\Delta_i A^i}{\Delta x_i}\xi, \xi\right\rangle\right| \le \mu_i(t)\langle \xi, \xi\rangle \qquad (i = 1, 2, \cdots, n)$$

for any vector ξ.

(iii) *For any compact set* $\tilde{H}_t$ *in* H_t, *there exists a constant* K (> 0) *such that*

$$\int_{\tilde{H}_t} \langle u, u\rangle\, dx \le Ke^{-2\alpha t}, \qquad t \in [0, \infty),$$

for any constant $\alpha > 0$.

Then u *is identically equal to zero in* $\mathfrak{D}$.

3. **The energy inequality.** We modify the coefficients of (4) by the standard mollifier method. Let $w_k^{(1)}$ $(k = 1, 2, \cdots)$ be an infinitely differentiable nonnegative function on E^n with support contained in $|x| \le 1/k$ and satisfying $\int_{E^n} w_k^{(1)}(x)\, dx = 1$. Further, let $w_k^{(2)}$ $(k = 1, 2, \cdots)$ be an infinitely differentiable function on $(-\infty, \infty)$ with compact support $|t| \le 1/k$ and satisfying $\int_{-\infty}^{\infty} w_k^{(2)}(t)\, dt = 1$.

Denote by $w_k(y, s)$ the product $w_k^{(1)}(y)\, w_k^{(2)}(s)$. For any function f locally integrable in $\mathfrak{D}$, we put

$$f_k(x, t) = f * w_k(x, t) = \iint_{\mathfrak{D}} f(y, s)\, w_k(x - y, t - s)\, dy\, ds.$$

Then f_k $(k = 1, 2, \cdots)$ is infinitely (many times) differentiable in $\mathfrak{D}$ and f_k tends to f in the L^p-norm $(p \geq 1)$ on every compact subset of $\mathfrak{D}$ as k tends to infinity.

We put $A_k(x, t) = (a^k_{ij}(x, t))$ for a matrix $A(x, t) = (a_{ij}(x, t))$ with $a_{ij} \in L^2_{\mathrm{loc}}(\mathfrak{D})$, where $a^k_{ij}(x, t) = a_{ij} * w_k(x, t)$. In such a manner, we consider matrices A^i_k $(i = 1, 2, \cdots, n)$ and B_k for matrices A^i $(i = 1, 2, 3, \cdots, n)$ and B in (4), respectively. Thus we have a system

$$\frac{\partial}{\partial t} u + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (A^i_k u) + B_k u = 0. \tag{4'}$$

First we have the following

Lemma 1. *The conditions in the Theorem imply that*

(ii)$'$ *there exists a nonnegative function $\mu(t) \in L^1_{\mathrm{loc}}([0, \infty))$ being bounded and satisfying*

$$\left|\left\langle\left(B_k + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial A^i_k}{\partial x_i}\right)\xi, \xi\right\rangle\right| \leq \mu(t)\langle \xi, \xi\rangle$$

for any ξ.

Proof. It is almost obvious that there exists $\nu_0 \in L^1_{\mathrm{loc}}([0, \infty))$ satisfying $|\langle B_k\xi, \xi\rangle| \leq \nu_0(t)\, \langle \xi, \xi\rangle$. In fact, we have

$$\begin{aligned} |\langle B_k \xi, \xi\rangle| &= \left|\sum_{i,j=1}^{l} b^k_{ij}(x, t)\, \xi_i \xi_j\right| \\ &= \left|\iint \sum_{i,j=1}^{l} b_{ij}(y, s)\, \xi_i \xi_j\, w_k(x-y, t-s)\, dy\, ds\right| \\ &\leq \left\langle \int_{E^n} w^{(1)}_k(x-y)\, dy \int_0^\infty \nu(s)\, w^{(2)}_k(t-s)\, ds\, \xi, \xi\right\rangle \\ &= \left\langle \int_0^\infty \nu(s)\, w^{(2)}_k(t-s)\, ds\, \xi, \xi\right\rangle = \nu_0(t)\langle \xi, \xi\rangle \end{aligned}$$

where $\nu_0(t) = \int_0^\infty \nu(s)\, w^{(2)}_k(t-s)\, ds \in L^1_{\mathrm{loc}}([0, \infty))$ and bounded.

Next we have only to show that there exist nonnegative functions $\mu'_i(t) \in L^1_{\mathrm{loc}}([0, \infty))$ $(i = 1, 2, \cdots, n)$ being bounded such that

$$|\langle(\partial A^i_k/\partial x_i)\,\xi, \xi\rangle| \leq \mu'_i(t)\langle \xi, \xi\rangle \qquad (i = 1, 2, 3, \cdots, n).$$

Since $\langle [\Delta_i A^i/\Delta x_i + \mu_i(t)]\, \xi, \xi\rangle \geq 0$ and since w_k is positive, the convolution of $[\Delta_i A^i/\Delta x_i + \mu_i(t)]$ and w_k is nonnegative, so $\langle [\Delta_i A^i/\Delta x_i + \mu_i] * w_k(x, t)\, \xi, \xi\rangle \geq 0$ implies that

$$\langle [\Delta_i A_k^i(x, t)/\Delta x_i + \mu_i'(t)]\, \xi, \xi \rangle \geq 0$$

where $\mu_i'(t) = \int_0^\infty \mu_i'(t-s)\, w_k^{(2)}(s)\, ds \in L^1_{loc}([0, \infty))$ is bounded. By passing through the limit, with respect to the difference quotients we have

$$\langle (\partial A_k^i/\partial x_i)\, \xi, \xi \rangle \geq -\mu_i'(t) \langle \xi, \xi \rangle, \qquad i = 1, 2, 3, \cdots, n.$$

Similarly, we have $\langle (\partial A_k^i/\partial x_i)\, \xi, \xi \rangle \leq \mu_i'(t) \langle \xi, \xi \rangle$, and hence $|\langle (\partial A_k^i/\partial x_i)\, \xi, \xi \rangle| \leq \mu_i'(t) \langle \xi, \xi \rangle$. Thus putting $\mu(t) = \nu_0(t) + \frac{1}{2} \sum_{i=1}^n \mu_i'(t)$, we then obtain (ii)$'$. Q.E.D.

Lemma 2. *If a function A on E^n and its derivatives in the distribution sense are bounded, then the smoothed functions A_k and $\partial A_k/\partial x_j$ ($j = 1, 2, \cdots, n$) are uniformly bounded (independent of k), where $A_k = A * w_k$ and w_k is a mollifier.*

It is easy to prove this lemma, so we omit the proof here (cf. [6]).

Now consider the Cauchy problem

$$\frac{\partial \phi}{\partial t} + \sum_{i=1}^n A_k^i \frac{\partial \phi}{\partial x_i} - B_k^* \phi = F, \tag{5}$$

$$\phi(x, 0) = 0 \tag{6}$$

for any given function $F \in C^\infty(\mathfrak{D})$ with compact support. It is well known in the classical theory that the solution ϕ of (5) and (6) (depending on k) is sufficiently smooth and has compact support. This support is also independent of k. Here A_k^i and B_k^* are the smooth matrices associated with the matrices A^i and the transpose B^* of B respectively.

By assuming (ii)$'$ in Lemma 1, we prove that for the solution ϕ_k of the Cauchy problem (5), (6), integrals $\int_0^t \int_{H_S} \langle \phi_k, \phi_k \rangle\, dx\, ds$ and $\int_0^t \int_{H_S} \langle \partial \phi_k/\partial x_i, \partial \phi_k/\partial x_i \rangle\, dx\, ds$ are bounded uniformly (independent of k). We start from the following lemma about the energy inequality.

Lemma 3. *Let ϕ with compact support in E^n be a solution of the Cauchy problem*

$$\frac{\partial \phi}{\partial t} + \sum_{i=1}^n A_k^i \frac{\partial \phi}{\partial x_i} - B_k^* \phi = F, \tag{5}$$

$$\phi(x, 0) = \psi(x) \tag{7}$$

for any given $\psi \in C_0^\infty(E^n)$ and $F \in C^\infty(\mathfrak{D})$ with compact support. Suppose that the coefficients of (5) satisfy the condition (ii)$'$ *in Lemma* 1. *Then the inequality*

$$\begin{aligned}\int_{H_t} \langle \phi, \phi \rangle\, dx + \int_0^t \int_{H_s} \langle \phi, \phi \rangle\, dx\, ds \\ \leq e^{2\lambda t} \left(\int_{H_0} \langle \psi, \psi \rangle\, dx + \int_0^t \int_{H_s} \langle F, F \rangle\, dx\, ds \right)\end{aligned} \tag{8}$$

holds for a sufficiently large λ *and for any* $t \in [0, \infty)$.

Proof. It is obvious that

$$\int_0^t \int_{H_s} \left\{ \left\langle \frac{\partial \phi}{\partial t}, \phi \right\rangle + \sum_{i=1}^n \left\langle A_k^i \frac{\partial \phi}{\partial x_i}, \phi \right\rangle - \langle B_k^* \phi, \phi \rangle \right\} dx\, ds = \int_0^t \int_{H_s} \langle F, \phi \rangle\, dx\, ds$$

implies

$$\int_0^t \int_{H_s} \left\{ \left\langle \frac{\partial \phi}{\partial t}, \phi \right\rangle - \sum_{i=1}^n \left\langle \frac{\partial}{\partial x_i} (A_k^i \phi), \phi \right\rangle - \langle B_k \phi, \phi \rangle \right\} dx\, ds$$

$$= \int_0^t \int_{H_s} \langle F, \phi \rangle\, dx\, ds.$$

By differentiating with respect to t, we have

$$\text{(9)} \quad \int_{H_t} \left\{ - \left\langle \frac{\partial \phi}{\partial t}, \phi \right\rangle + \sum_{i=1}^n \left\langle \frac{\partial}{\partial x_i} (A_k^i \phi), \phi \right\rangle + \langle B_k \phi, \phi \rangle \right\} dx = \int_{H_t} \langle -F, \phi \rangle\, dx.$$

It is clear that

$$\int_{H_t} \left\langle \frac{\partial}{\partial x_i} (A_k^i \phi), \phi \right\rangle dx = \int_{H_t} \left\langle \left(\frac{\partial}{\partial x_i} A_k^i \right) \phi, \phi \right\rangle dx + \int_{H_t} \left\langle A_k^i \frac{\partial \phi}{\partial x_i}, \phi \right\rangle dx$$

and

$$\int_{H_t} \left\langle \frac{\partial}{\partial x_i} (A_k^i \phi), \phi \right\rangle dx = - \int_{H_t} \left\langle A_k^i \phi, \frac{\partial \phi}{\partial x_i} \right\rangle dx = - \int_{H_t} \left\langle A_k^i \frac{\partial \phi}{\partial x_i}, \phi \right\rangle dx$$

so we have

$$\text{(10)} \qquad -\frac{1}{2} \int_{H_t} \sum_{i=1}^n \left\langle \frac{\partial A_k^i}{\partial x_i} \phi, \phi \right\rangle dx = \int_{H_t} \sum_{i=1}^n \left\langle A_k^i \frac{\partial \phi}{\partial x_i}, \phi \right\rangle dx.$$

Since $\phi \in C^\infty(\mathfrak{D})$ has compact support in E^n, we see that

$$\text{(11)} \qquad \frac{d}{dt} \int_{H_t} \langle \phi, \phi \rangle\, dx = 2 \int_{H_t} \left\langle \frac{\partial \phi}{\partial t}, \phi \right\rangle dx.$$

Set $\phi = e^{\lambda t} v$ and $f = + e^{-\lambda t} F$ for a constant λ (> 0). Then (10) and (11) are satisfied by v instead of ϕ and we have

$$\int_{H_t}\left\langle\frac{\partial v}{\partial t}, v\right\rangle dx$$

$$= \int_{H_t}\left\{+\left\langle\sum_{i=1}^{n} A_k^i \frac{\partial v}{\partial x_i}, v\right\rangle + \left\langle\sum_{i=1}^{n}\frac{\partial A_k^i}{\partial x_i} v, v\right\rangle + \langle (B_k - \lambda I) v, v\rangle + \langle f, v\rangle\right\} dx,$$

which is equivalent to (9), where I denotes the identity matrix. Applying the identity (10), we then have

$$2\int_{H_t}\left\langle\frac{\partial v}{\partial t}, v\right\rangle dx = 2\int_{H_t}\langle f, v\rangle\, dx + \int_{H_t}\left\langle\left[2(B_k - \lambda I) + \sum_{i=1}^{n}\frac{\partial A_k^i}{\partial x_i}\right] v, v\right\rangle dx.$$

Therefore, by the virtue of (11), we get

$$\frac{d}{dt}\int_{H_t}\langle v, v\rangle\, dx = 2\int_{H_t}\langle f, v\rangle\, dx + \int_{H_t}\left\langle\left[2(B_k - \lambda I) + \sum_{i=1}^{n}\frac{\partial A_k^i}{\partial x_i}\right] v, v\right\rangle dx.$$

Integrating this identity with respect to t from 0 to t, we have

$$\int_{H_t}\langle v, v\rangle\, dx - \int_{H_0}\langle v, v\rangle\, dx$$

$$(12)\qquad = 2\int_0^t\int_{H_s}\langle f, v\rangle\, dx\, ds + \int_0^t\int_{H_s}\left\langle\left(2B_k + \sum_{i=1}^{n}\frac{\partial A_k^i}{\partial x_i}\right) v, v\right\rangle dx$$

$$- 2\lambda\int_0^t\int_{H_s}\langle v, v\rangle\, dx\, ds.$$

By $v(x, 0) = \phi(x, 0) = \psi(x)$, we have

$$\int_{H_t}\langle v, v\rangle\, dx - \int_{H_0}\langle \psi, \psi\rangle\, dx$$

$$\le \int_0^t\int_{H_s}|f|^2 g^{-2}(s)\, dx\, ds + \int_0^t\int_{H_s}|v|^2 g^2(s)\, dx\, ds$$

$$+ \int_0^t\int_{H_s}\left\langle\left[2B_k + \sum_{i=1}^{n}\frac{\partial A_k^i}{\partial x_i}\right] v, v\right\rangle dx\, ds - 2\lambda\int_0^t\int_{H_s}\langle v, v\rangle\, dx\, ds$$

where we assume that $g(t)$ $(\neq 0)$ together with $g^{-1}(t)$ is an arbitrary locally square integrable function on $[0, \infty)$. Therefore we have

$$\int_{H_t}\langle v, v\rangle\, dx + \int_0^t \int_{H_s}\langle v, v\rangle\, dx\, ds$$

$$\le \int_{H_0}\langle \psi, \psi\rangle\, dx + \int_0^t \int_{H_s} g^{-2}(s)\,|f|^2\, dx\, ds$$

$$+ \int_0^t \int_{H_s}[g^2(s) + 1 - 2\lambda]|v|^2\, dx\, ds + \int_0^t \int_{H_s}\left\langle\left(2B_k + \sum_{i=1}^n \frac{\partial A_k^i}{\partial x_i}\right)v,\, v\right\rangle dx\, ds.$$

Choose $g(t)$ such that $g^2(t) = 2\lambda - 2\mu(t) - 1 \geq 1$ for a sufficiently large λ, where $\mu(t)$ is such a function as in Lemma 1. Then

$$\int_{H_t}\langle v, v\rangle\, dx + \int_0^t \int_{H_s}\langle v, v\rangle\, dx\, ds \le \int_{H_0}\langle \psi, \psi\rangle\, dx + \int_0^t \int_{H_s} g^{-2}(s)\,|f|^2\, dx\, ds$$

or

$$\int_{H_t}\langle \phi, \phi\rangle\, dx + \int_0^t \int_{H_s}\langle \phi, \phi\rangle\, dx\, ds$$
$$\le e^{2\lambda t}\left[\int_{H_0}\langle \psi, \psi\rangle\, dx + \int_0^t \int_{H_s}\langle F, F\rangle\, dx\, ds\right].$$

This proves our lemma.

An analogous inequality is satisfied by energy integrals $\|u(t)\|_1$ defined as

$$\|u(t)\|_1 = \left(\int_{H_t}(|u|^2 + |D^1u|^2)\, dx\right)^{1/2},$$

where $D^1u = \Sigma_{i=1}^n\, \partial u/\partial x_i$ and $|D^1u|^2 = \Sigma_{i=1}^n\, |\partial u/\partial x_i|^2$. It follows that

Lemma 4. *Under the assumption in the Theorem, the solution ϕ of* (5), (7) *with compact support satisfies the following inequality:*

$$\int_{H_t}(|\phi|^2 + |D^1\phi|^2)\, dx + \int_0^t \int_{H_s}(|\phi|^2 + |D^1\phi|^2)\, dx\, ds$$

$$\le Ce^{4\lambda t}\left[\int_{H_0}(|\psi|^2 + |D^1\psi|^2)\, dx + \int_0^t \int_{H_s}(|F|^2 + |D^1F|^2)\, dx\, ds\right] \tag{13}$$

or simply denoted by

$$\|\phi(t)\|_1^2 + \int_0^t \|\phi(s)\|_1^2\, ds \le Ce^{4\lambda t}\left(\|\psi\|_1^2 + \int_0^t \|F(s)\|_1^2\, ds\right)$$

where C is a constant and λ is sufficiently large.

Proof. Recall the equation

$$L\phi \equiv \sum_{i=1}^n A_k^i \frac{\partial \phi}{\partial x_i} + \frac{\partial \phi}{\partial t} - B_k^*\phi = F. \tag{5}$$

Differentiating (5) with respect to x_j, we have

$$L\frac{\partial\phi}{\partial x_j}+\sum_{i=1}^{n}\frac{\partial\phi}{\partial x_i}\cdot\frac{\partial A_k^i}{\partial x_j}-\frac{\partial B_k^*}{\partial x_j}\phi=\frac{\partial F}{\partial x_j}. \tag{14}$$

For brevity, we put $V=\sum_{i=1}^{n}\partial\phi/\partial x_j=D^1\phi$ and then

$$LV=D^1F+D^1B_k^*\cdot\phi-\sum_{i=1}^{n}\frac{\partial\phi}{\partial x_i}\cdot D^1A_k^i\equiv b,\quad \text{say.}$$

It is evident that b is continuous and has compact support, thus by the same argument as in the proof of Lemma 3, we obtain

$$\begin{aligned}\int_{H_t}\langle U,U\rangle\,dx&-\int_{H_0}\langle U,U\rangle\,dx\\ &=2\int_0^t\int_{H_s}\langle e^{-\lambda s}b,U\rangle\,dx\,ds\\ &\quad+\int_0^t\int_{H_s}\left\langle\left(2B_k+\sum_{i=1}^{n}\frac{\partial A_k^i}{\partial x_i}\right)U,U\right\rangle dx\,ds-2\lambda\int_0^t\int_{H_s}\langle U,U\rangle\,dx\,ds,\end{aligned}$$

where $U=e^{-\lambda t}V$ $(\lambda>0)$. Clearly we see

$$\begin{aligned}\int_{H_t}\langle U,U\rangle\,dx&-\int_{H_0}\langle U,U\rangle\,dx\\ &\le 2\int_0^t\int_{H_s}\langle e^{-\lambda s}b,U\rangle\,dx\,ds+\int_0^t\int_{H_s}\langle(2\mu(s)-2\lambda)U,U\rangle\,dx\,ds\\ &=2\int_0^t\int_{H_s}e^{-\lambda s}\left\langle\left(D^1F+D^1B_k^*\phi-\sum_{i=1}^{n}\frac{\partial\phi}{\partial x_i}D^1A_k^i\right),U\right\rangle dx\,ds\\ &\quad+\int_0^t\int_{H_s}\langle(2\mu(s)-2\lambda)U,U\rangle\,dx\,ds\\ &\le\int_0^t\int_{H_s}|D^1F|^2g^{-2}(s)\,dx\,ds+\int_0^t\int_{H_s}|D^1B_k^*\phi|^2g^{-2}(s)\,dx\,ds\\ &\quad+\int_0^t\int_{H_s}e^{-2\lambda s}\sum_{i=1}^{n}\left|\frac{\partial\phi}{\partial x_i}\right|^2|D^1A_k^i|^2\cdot g^{-2}(s)\,dx\,ds\\ &\quad+\int_0^t\int_{H_s}(3g^2(s)-2\lambda+2\mu(s))|U|^2\,dx\,ds,\end{aligned}$$

where $g(t)$ is locally integrable and nowhere zero in $[0,\infty)$. Let M be the bound of $D^1A_k^i$ $(i = 1, 2, \cdots, n)$ and C the bound of $D^1B_k^*$. It is known that M and C are independent of k. Choose $g^2(s) = (1/3)[2\lambda - 2\mu(s) - 1 - M] \geq 1$ for a sufficiently large λ. Then

$$\int_{H_t} \langle U, U\rangle dx + (1 + M) \int_0^t \int_{H_s} \langle U, U\rangle dx\, ds$$

$$\leq \int_{H_0} \langle U, U\rangle dx + \int_0^t \int_{H_s} \langle D^1F, D^1F\rangle dx\, ds$$

$$+ C \int_0^t \int_{H_s} \langle \phi, \phi\rangle dx\, ds + M \int_0^t \int_{H_s} e^{-2\lambda s} \sum_{i=1}^n \left\langle \frac{\partial \phi}{\partial x_i}, \frac{\partial \phi}{\partial x_i} \right\rangle dx\, ds$$

or

$$\int_{H_t} |D^1\phi|^2 dx + e^{2\lambda t} \int_0^t \int_{H_s} e^{-2\lambda s} |D^1\phi|^2 dx\, ds$$

$$\leq e^{2\lambda t} \left\{ \int_{H_0} |D^1\psi|^2 dx + \int_0^t \int_{H_s} |D^1F|^2 dx\, ds + C \int_0^t \int_{H_s} |\phi|^2 dx\, ds \right\}.$$

Thus

$$\int_{H_t} |D^1\phi|^2 dx + \int_0^t \int_{H_s} |D^1\phi|^2 dx\, ds$$

$$\leq e^{2\lambda t} \left\{ \int_{H_0} |D^1\psi|^2 dx + \int_0^t \int_{H_s} |D^1F|^2 dx\, ds + C \int_0^t \int_{H_s} |\phi|^2 dx\, ds \right\}. \tag{15}$$

Multiplying (8) by $(1 + Ce^{2\lambda t})$ and then adding to (15), we obtain

$$\int_{H_t} (|\phi|^2 + |D^1\phi|^2) dx + \int_0^t \int_{H_s} (|\phi|^2 + |D^1\phi|^2) dx\, ds + Ce^{2\lambda t} \int_{H_s} |\phi|^2 dx$$

$$\leq e^{2\lambda t} \left\{ \int_{H_0} (|\psi|^2 + |D^1\psi|^2) dx + \int_0^t \int_{H_s} (|F|^2 + |D^1F|^2) dx\, ds \right\}$$

$$+ Ce^{4\lambda t} \left\{ \int_{H_0} |\psi|^2 dx + \int_0^t \int_{H_s} |F|^2 dx\, ds \right\}.$$

Therefore,

$$\int_{H_t} (|\phi|^2 + |D^1\phi|^2) dx + \int_0^t \int_{H_s} (|\phi|^2 + |D^1\phi|^2) dx\, ds$$

$$\leq M_t \left\{ \int_{H_0} (|\psi|^2 + |D^1\psi|^2) dx + \int_0^t \int_{H_s} (|F|^2 + |D^1F|^2) dx\, ds \right\}$$

where $M_t = \max(e^{2\lambda t}, Ce^{4\lambda t})$. The constant C can be taken as larger than 1, so M_t is chosen as $M_t = Ce^{4\lambda t}$.

4. Proof of the main theorem. The weak solution u of the equation (4) satisfies the following equality:

$$\int_0^t \int_{H_s} \left[\sum_{i=1}^n \left\langle A^i u, \frac{\partial \phi}{\partial x_i}\right\rangle + \left\langle u, \frac{\partial \phi}{\partial s}\right\rangle - \langle Bu, \phi\rangle\right] dx\, ds + \int_{H_0} \langle u, \phi\rangle dx - \int_{H_t} \langle u, \phi\rangle dx = 0 \tag{16}$$

for any $\phi \in C^\infty(\mathfrak{D})$ with compact support in $\mathfrak{D}$ and for any fixed $t \in [0, \infty)$. We have to prove that u vanishes indentically in $\mathfrak{D}$ under our assumption. It sufficies to show that $\int_0^\infty \int_{H_s} \langle u, F\rangle\, dx\, dt = 0$ for any $F \in C^\infty(\mathfrak{D})$ with compact support in $\mathfrak{D}$. Let ϕ_k be the solution of (5), (6). Then (16) holds for $\phi = \phi_k$.

By (5) and (16),

$$\begin{aligned}\int_0^t \int_{H_s} \langle u, F\rangle dx\, ds &= \int_0^t \int_{\tilde H_s} \langle u, F\rangle dx\, ds \\ &= \int_0^t \int_{\tilde H_s} \left\langle u, \sum_{i=1}^n A_k^i \frac{\partial \phi_k}{\partial x_i} + \frac{\partial \phi_k}{\partial s} - B_k^* \phi_k\right\rangle dx\, ds \\ &\quad - \int_0^t \int_{\tilde H_s} \left\langle u, \sum_{i=1}^n A^i \frac{\partial \phi_k}{\partial x_i} + \frac{\partial \phi_k}{\partial s} - B^* \phi_k\right\rangle dx\, ds \\ &\quad - \int_{\tilde H_0} \langle u, \phi_k\rangle dx + \int_{\tilde H_t} \langle u, \phi_k\rangle dx,\end{aligned}$$

where $\tilde H_t$ is the support of F in H_t. Since ϕ_k vanishes on the hyperplane H_0 and has compact support in H_t, we have

$$\begin{aligned}\left|\int_0^t \int_{H_s} \langle u, F\rangle dx\, ds\right| &\le \int_0^t \int_{H_s} \left|\left\langle u, \sum_{i=1}^n (A_k^i - A^i) \frac{\partial \phi_k}{\partial x_i}\right\rangle\right| dx\, ds \\ &\quad + \int_0^t \int_{H_s} |\langle u, (B_k^* - B^*)\phi_k\rangle|\, dx\, ds + \int_{H_s} |\langle u, \phi_k\rangle|\, dx \\ &= \int_0^t \int_{\tilde H_s} \left|\left\langle u, \sum_{i=1}^n (A_k^i - A^i) \frac{\partial \phi_k}{\partial x_i}\right\rangle\right| dx\, ds \\ &\quad + \int_0^t \int_{\tilde H_s} |\langle u, (B_k^* - B^*)\phi_k\rangle|\, dx\, ds + \int_{\tilde H_t} |\langle u, \phi_k\rangle|\, dx.\end{aligned} \tag{17}$$

It follows from Lemma 3 and Lemma 4 that

$$\int_0^t \int_{\tilde H_s} \langle \phi_k, \phi_k\rangle dx\, ds \quad \text{and} \quad \int_0^t \int_{\tilde H_s} \sum_{i=1}^n \left\langle \frac{\partial \phi_k}{\partial x_i}, \frac{\partial \phi_k}{\partial x_i}\right\rangle dx\, ds$$

are uniformly bounded (independent of k). On the other hand, $A_k^i \to A^i$ ($i = 1, 2, \cdots, n$) and $B_k^* \to B^*$ ($k \to \infty$) in L^2-norm on $\tilde{H}_t \subset \mathfrak{D}$. Therefore, for $u \in L^2_{\mathrm{loc}}(\mathfrak{D})$,

$$\int_0^t \int_{\tilde{H}_s} \left| \left\langle u, \sum_{i=1}^n (A_k^i - A_i) \frac{\partial \phi_k}{\partial x_i} \right\rangle \right| dx\, ds$$

$$\leq \|u\|_{L^2(\tilde{H}_t)} \left\{ \sum_{i=1}^n \int_0^t \int_{\tilde{H}_s} \left| (A_k^i - A^i) \frac{\partial \phi_k}{\partial x_i} \right|^2 dx\, ds \right\}^{1/2}$$

$$\to 0 \quad \text{as } k \to \infty,$$

and

$$\int_0^t \int_{\tilde{H}_s} |\langle u, (B_k^* - B^*)\phi_k \rangle| \, dx\, ds \leq \|u\|_{L^2(\tilde{H}_t)} \left(\int_0^t \int_{\tilde{H}_s} |(B_k^* - B^*)\phi_k|^2 \, dx\, ds \right)^{1/2}$$

$$\to 0 \quad \text{as } k \to \infty.$$

By (iii) the last term on the right-hand side of (17) can be estimated as follows:

$$\int_{\tilde{H}_t} |\langle u, \phi_k \rangle| \, dx \leq \|\phi_k\|_{L^2(\tilde{H}_t)} \left(\int_{\tilde{H}_t} |u|^2 \, dx \right)^{1/2} \leq \|\phi_k\|_{L^2(\tilde{H}_t)} \cdot (Ke^{-2\alpha t})^{1/2}$$

for any positive α. Therefore,

$$\left| \int_0^t \int_{H_s} \langle u, F \rangle dx\, ds \right| \leq K^{1/2} \|\phi_k\|_{L^2(\tilde{H}_t)} e^{-\alpha t},$$

which together with Lemma 3 yields

$$\lim_{t \to \infty} \left| \int_0^t \int_{H_s} \langle u, F \rangle dx\, ds \right| = 0.$$

This proves that the weak solution u vanishes identically on $\mathfrak{D}$.

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