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# SEQUENCES OF CONVERGENCE REGIONS FOR CONTINUED FRACTIONS $K(a_n/1)$(1)

BY

WILLIAM B. JONES AND R. I. SNELL

ABSTRACT. Sufficient conditions are given for convergence of continued fractions $K(a_n/1)$ such that $a_n \in E_n$, $n \geq 1$, where $\{E_n\}$ is a sequence of element regions in the complex plane. The method employed makes essential use of a nested sequence of circular disks (inclusion regions), such that the $n$th disk contains the $n$th approximant of the continued fraction. This sequence can either shrink to a point, the *limit point case*, or to a disk, the *limit circle case*. Sufficient conditions are determined for convergence of the continued fraction in the limit circle case and these conditions are incorporated in the element regions $E_n$. The results provide new criteria for a sequence $\{E_n\}$ with unbounded regions to be an admissible sequence. They also yield generalizations of certain twin-convergence regions.

1. **Introduction.** A sequence of nonempty sets $\{E_n\}$ in the complex plane will be called a *sequence of convergence regions* for continued fractions

$$\mathop{K}_{n=1}^{\infty}\left(\frac{a_n}{1}\right) = \frac{a_1}{1} + \frac{a_2}{1} + \frac{a_3}{1} + \cdots, \tag{1.1}$$

if the conditions

$$a_n \in E_n, \quad a_n \neq 0, \ n \geq 1, \tag{1.2}$$

insure the convergence of (1.1). Recent papers concerned with the problem of finding sequences of convergence regions for (1.1) include: [1], [2], [4], [5] and [7]. The purpose of this paper is to give some new results for this problem. Our main theorems are related to two special classes of sequences of convergence regions: (1) twin-convergence regions and (2) admissible sequences [1].

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If $\{E_n\}$ is a periodic sequence of convergence regions with period two, then $E_1$, $E_2$ are called *twin-convergence regions.* A summary of the known twin-convergence regions for (1.1) was given recently by [5]. The best result known prior to [5] was the theorem of Lange and Thron [7] which states that if we set $a_n = c_n^2$ then the conditions

(1.3a) $$|c_{2n} + i\Gamma| \leq \rho, \quad |c_{2n} - i\Gamma| \leq \rho,$$

(1.3b) $$|c_{2n-1} + i(1+\Gamma)| \geq \rho, \quad |c_{2n-1} - i(1+\Gamma)| \geq \rho,$$

(1.3c) $$|\Gamma| < \rho < |1 + \Gamma|,$$

where $\Gamma$ is a complex number, are sufficient for convergence of (1.1). A generalization of the Lange-Thron theorem was given by [5, Theorem 5.4], which provides a class of twin-convergence regions containing (1.3). In Corollary 3.3, we give criteria for a sequence of convergence regions, not necessarily periodic, which contains the Lange-Thron theorem as well as its generalization in [5]. In a similar manner, Theorem 3.5 contains as a special case the twin-convergence regions given by [5, Theorem 5.2] with the exception of one limiting case (see Remark (1) following Theorem 3.5).

A sequence of nonempty regions $\{E_n\}$ in the complex plane is called an *admissible sequence* [1] provided that:

(i) For $n \geq 1$, $E_n$ is either a circle with center at the origin plus its interior ($C_0$ + int), or a circle with center at the origin plus its exterior ($C_0$ + ext), and

(ii) The continued fraction (1,1) converges if for $n \geq 1$, $a_n \in E_n$, $a_n \neq 0$.

The collection of all admissible sequences is denoted by AS. Lane and Wall [6] completely settled the problem of finding all admissible sequences where each region of the sequence is bounded, by showing that if $\{E_n\} \in$ AS and $E_n$ is bounded for $n \geq 1$, it is necessary and sufficient that there exist a sequence of positive numbers $\{\kappa_n\}$ such that

(1.4a) $$0 < \kappa_n < 1, \quad n \geq 0,$$

and

(1.4b) $$E_n = \{w: |w| \leq (1 - \kappa_{n-1})\kappa_n\}, \quad n \geq 1.$$

Hayden [1, Theorem 1] proved that if $E_n$ and $E_{n+1}$ are successive elements of an admissible sequence, then at least one of them must be bounded; he also gave sufficient conditions [1, Theorem 2] for sequences with unbounded regions to be admissible (a statement of these conditions is given in remarks preceding Corollary 3.4). A new set of sufficient conditions for admissible sequences with unbounded regions is given by Corollary 3.4. It is shown that these new conditions have an overlapping relation with Hayden's result referred to above.

The general approach employed in this article is that previously used by [2], [4], [5] and [8]. By assuming the existence of a sequence of value regions $\{V_n\}$ such that $0 \in V_n$ and

$$a_n/(1 + V_n) \subseteq V_{n-1} \quad \text{if } a_n \in E_n, \tag{1.5}$$

we obtain a nested sequence of closed disks $\{S_n(V_n)\}$ which can either converge to a point, the *limit point case*, or to a disk, the *limit circle case* (see (1.6) for the meaning of the functions $S_n$). In the limit point case the continued fraction converges, since $S_n(0)$ is the $n$th approximant and $0 \in V_n$. Thus it suffices to determine sufficient conditions for convergence of the continued fraction in the limit circle case and to choose the element regions $E_n$ so as to incorporate these conditions. The method is elementary in the sense that no deep function-theoretic results are used and, by virtue of the many applications obtained thus far, it appears to provide a unified approach to the convergence region problem.

Before stating the theorems, it is helpful to have some additional terminology and definitions. An (*infinite*) *continued fraction* is an ordered pair of sequences $[\{a_n\}_{n=1}^{\infty}, \{f_n\}_{n=1}^{\infty}]$, where $a_1, a_2, \cdots$ are complex numbers, $a_n \neq 0$, $n = 1, 2, \cdots$, and where the $f_n$ are elements in the extended complex plane defined as follows: If $s_n$ denotes the linear fractional transformation (l.f.t.)

$$s_n(z) = a_n/(1 + z), \qquad n = 1, 2, \cdots, \tag{1.6a}$$

and

$$S_1(z) = s_1(z); \qquad S_n(z) = S_{n-1}(s_n(z)), \qquad n = 2, 3, \cdots, \tag{1.6b}$$

then

$$f_n = S_n(0), \qquad n = 1, 2, \cdots. \tag{1.7}$$

The $a_n$ are called *elements* of the continued fraction $[\{a_n\}, \{f_n\}]$ and $f_n$ is called the $n$th *approximant*. A continued fraction is said to *converge* if its sequence of approximants converges and, in this case, $f = \lim f_n$ is called the *value* of the continued fraction. For convenience the continued fraction $[\{a_n\}, \{f_n\}]$ is sometimes denoted by (1.1), $K_{n=1}^{\infty}(a_n/1)$ or, more simply $K(a_n/1)$.

Finally, if $f$ is a function of $k$ variables, we mean by $f(A_1, \cdots, A_k)$ the set

$$\{f(x_1, \cdots, x_k): x_m \in A_m, \; m = 1, \cdots, k\}.$$

2. **Sequences of linear fractional transformations.** Thron [8] has shown that a sequence of l.f.t.'s $\{T_n\}$ satisfying the conditions

$$T_n(U) \subseteq T_{n-1}(U) \subseteq U, \qquad n \geq 1, \tag{2.1}$$

where $U$ denotes the unit disk $\{z: |z| \leq 1\}$, can be written in the form

$$T_n(z) = C_n + R_n \frac{z + \overline{G}_n}{G_n z + 1}, \qquad n \geq 0, \tag{2.2a}$$

where

$$|R_n| = r_n \searrow r \geq 0, \quad |C_n - C_{n-1}| \leq r_{n-1} - r_n, \quad |G_n| = g_n < 1. \tag{2.2b}$$

From (2.1) and (2.2) it is clear that $\{T_n(U)\}$ is a nested sequence of closed disks; $C_n$ and $r_n$ are the center and radius, respectively, of $T_n(U)$. From (2.2b) we see that $C = \lim C_n$ exists. If $r_n \searrow r = 0$, the *limit point case* is said to occur, since $\{T_n(U)\}$ converges to the point $C$. When $r_n \searrow r > 0$, $\{T_n(U)\}$ converges to the closed disk with center $C$ and radius $r$; this is referred to as the *limit circle case*. This section contains three theorems on convergence of sequences $\{T_n\}$ satisfying (2.2) for which the limit circle case holds. These results will be used to derive convergence regions for continued fractions in the following section. Theorems 2.1 and 2.2 are more general than, but parallel to, Lemmas 4.1 and 4.2, respectively, in [5]. Corresponding proofs are almost identical and are included here for completeness.

**Theorem 2.1.** *Let $\{T_n\}$ be a sequence of l.f.t.'s of the form* (2.2) *with $r > 0$ (limit circle case). Suppose that there exist sequences of points $\{\xi_n\}$ and $\{\delta_n\}$ in the extended complex plane such that*

$$T_n(\xi_n) = T_{n-1}(\delta_n), \quad |\xi_n| \geq 1, \quad |\delta_{n-1}| \leq 1, \quad n \geq 1. \tag{2.3}$$

*If for some constant $\epsilon > 0$, either $|\xi_n| \geq 1 + \epsilon$ for all $n \geq 1$ or $|\delta_{n-1}| \leq 1 - \epsilon$ for all $n \geq 1$, then*

$$\sum_{n=1}^{\infty} (1 - g_n) < \infty. \tag{2.4}$$

**Proof.** From (2.2) and (2.3) we obtain

$$C_n + R_n \Lambda_n = C_{n-1} + R_{n-1} \lambda_{n-1}, \tag{2.5}$$

where

$$\Lambda_n = \frac{\xi_n + \overline{G}_n}{G_n \xi_n + 1}, \qquad \lambda_{n-1} = \frac{\delta_{n-1} + \overline{G}_{n-1}}{G_{n-1}\delta_{n-1} + 1}. \tag{2.6}$$

Since the transformation $w(z) = (z + \overline{G}_{n-1})/(G_{n-1}z + 1)$ maps the unit disk onto itself, it follows that $|\lambda_{n-1}| \leq 1$. That $\{\Lambda_n\}$ is a bounded sequence can be seen from (2.5) and using the fact that $|\lambda_{n-1}| \leq 1$, $r_n \searrow r > 0$ and $\{C_n\}$ converges.

Equations (2.2b) and (2.5) imply that

$$r_n|\Lambda_n| \le r_{n-1} - r_n + r_{n-1}|\lambda_{n-1}|.$$

Thus, letting $H_n = (|\Lambda_n| - |\lambda_{n-1}|)/(1 + |\Lambda_n|)$, we obtain

$$0 < r_n/r_{n-1} \le 1 - H_n \le 1$$

and hence

$$r_n \le r_0 \prod_{k=1}^{n} (1 - H_k).$$

Therefore the series $\Sigma H_n$ is convergent, since otherwise the infinite product $\Pi(1 - H_k)$ would diverge to zero, contradicting the hypothesis $r_n \searrow r > 0$. Since $\{\Lambda_n\}$ is bounded, we conclude that $\Sigma(|\Lambda_n| - |\lambda_{n-1}|)$ converges and also that both of the series

$$\sum_{n=1}^{\infty} (|\Lambda_n| - 1), \qquad \sum_{n=1}^{\infty} (1 - |\lambda_{n-1}|) \tag{2.7}$$

are convergent.

Now we assume that $|\delta_{n-1}| \le 1 - \epsilon < 1$, $n \ge 1$. It will suffice to show that

$$(1 - g_{n-1})K \le 1 - |\lambda_{n-1}|, \qquad n \ge 1, \tag{2.8}$$

for some positive constant. It can be seen that for all $K$ such that $0 < K < \frac{1}{2}$, (2.8) is equivalent to

$$|\delta_{n-1} + \overline{G}_{n-1}| \le [1 - K(1 - g_{n-1})]|G_{n-1}\delta_{n-1} + 1|, \qquad n \ge 1. \tag{2.9}$$

Squaring both sides of (2.9), collecting terms, and dividing by $(1 - g_{n-1})$, we obtain the equivalent inequality

$$K[2 - K(1 - g_{n-1})]|G_{n-1}\delta_{n-1} + 1|^2 \le (1 - |\delta_{n-1}|^2)(1 + g_{n-1}). \tag{2.10}$$

The right side of (2.10) is positive and uniformly bounded away from zero for all $n \ge 1$, since $|\delta_{n-1}| \le 1 - \epsilon < 1$. On the other hand, the left side of (2.10) is bounded above by $8K$. Hence (2.10) will hold for all $n \ge 1$, provided $K$ is sufficiently small. Thus (2.8) and (2.4) are satisfied. A similar argument can be used if we assume that $|\xi_n| \ge 1 + \epsilon > 1$. This completes the proof.

**Theorem 2.2.** *Let $\{T_n\}$ be a sequence of l.f.t.'s of the form (2.2) with $r > 0$ (limit circle case). Suppose that there exist sequences of points $\{\eta_n\}$ and $\{\zeta_n\}$ in the extended complex plane and a constant $\epsilon > 0$ such that*

$$T_n(\eta_n) = T_{n-1}(\zeta_{n-1}), \qquad ||\eta_n| - 1| \ge \epsilon, \qquad ||\zeta_{n-1}| - 1| \ge \epsilon, \qquad n \ge 1. \tag{2.11}$$

*If* $\Sigma(1-g_n)<\infty$, *then* $\{T_n(z)\}$ *converges at least for all* $z$ *such that* $|z|\neq 1$ *and*

$$\lim T_n(z)=\lim[C_n+R_n/G_n],\qquad |z|\neq 1. \tag{2.12}$$

**Proof.** By writing (2.2a) in the form

$$T_n(z)=C_n+\frac{R_n}{G_n}\left[1-\frac{1-g_n^2}{G_nz+1}\right] \tag{2.13}$$

and noting that $g_n\longrightarrow 1$, we see that it suffices to prove that $\{R_n\overline{G}_n\}$ is a convergent sequence. From (2.2) and (2.11) we obtain

$$\begin{aligned} R_k\overline{G}_k-R_{k-1}\overline{G}_{k-1}&=(C_{k-1}-C_k)-R_k\frac{1-g_k^2}{G_k+(1/\eta_k)}\\ &\quad+R_{k-1}\frac{1-g_{k-1}^2}{G_{k-1}+(1/\zeta_{k-1})}. \end{aligned} \tag{2.14}$$

Summing equations of the form (2.14) for $k=m+1,\cdots,n$ gives

$$\begin{aligned} R_n\overline{G}_n-R_m\overline{G}_m&=(C_m-C_n)-\sum_{k=m+1}^{n}R_k\frac{1-g_k^2}{G_k+(1/\eta_k)}\\ &\quad+\sum_{k=m+1}^{n}R_{k-1}\frac{1-g_{k-1}^2}{G_{k-1}+(1/\zeta_{k-1})}. \end{aligned} \tag{2.15}$$

It follows, from (2.14) and the bounds given in (2.11) for the sequences $\{\eta_n\}$ and $\{\zeta_{n-1}\}$, that $\{R_n\overline{G}_n\}$ is a Cauchy sequence. This completes the proof.

**Theorem 2.3.** *Let* $\{T_n\}$ *be a sequence of l.f.t.'s of the form* (2.2) *with* $r>0$ *(limit circle case). Suppose that there exist sequences of points* $\{j_n\}$, $\{k_n\}$ *and* $\{u_n\}$ *in the extended complex plane and a constant* $0<\epsilon<1$ *such that*

$$T_n(j_n)=T_{n-1}(k_{n-1})=T_{n-2}(u_{n-2}),\qquad n\geq 2, \tag{2.16a}$$

*and*

$$|j_n|\geq 1+\epsilon,\qquad |u_n|\leq 1-\epsilon,\qquad n\geq 1, \tag{2.16b}$$

*and*

$$||k_{n(p)}|-1|\geq\epsilon,\qquad p\geq 1, \tag{2.16c}$$

*for some infinite subsequence* $\{k_{n(p)}\}$ *of* $\{k_n\}$. *Then* $\{T_n(z)\}$ *converges at least for all* $z$ *in the extended complex plane such that* $|z|\neq 1$ *and*

$$\lim T_n(z)=\lim[C_n+R_n/G_n],\qquad |z|\neq 1, \tag{2.17}$$

**Proof.** From Theorems 2.1 and 2.2 we conclude that the two subsequences $\{T_{2n}(z)\}$ and $\{T_{2n-1}(z)\}$ converge at least for all $z$ such that $|z|\neq 1$ and

$$\lim T_{2n-1}(z) = \lim[C_{2n-1} + R_{2n-1}/G_{2n-1}], \qquad |z| \neq 1, \tag{2.18a}$$

$$\lim T_{2n}(z) = \lim[C_{2n} + R_{2n}/G_{2n}], \qquad |z| \neq 1. \tag{2.18b}$$

Furthermore, the series $\Sigma(1 - g_n)$ converges and so $g_n \rightarrow 1$. If we set $R_n = r_n \exp(i\omega_n)$ and $G_n = g_n \exp(i\gamma_n)$, then it follows from (2.18) that the two limits

$$\lim_{n\to\infty} \exp(i(\omega_{2n-1} - \gamma_{2n-1})), \qquad \lim_{n\to\infty} \exp(i(\omega_{2n} - \gamma_{2n})) \tag{2.19}$$

exist. It suffices to prove that these limits are equal. We assume that the subsequence $\{n(p)\}$ of indexes in (2.16c) contains an infinite subsequence $\{2m(p)\}$ of even integers (a similar argument will hold with a subsequence of odd integers). For these even integers (2.16a) gives

$$\begin{aligned} C_{2m(p)} &+ r_{2m(p)} \exp(i\omega_{2m(p)}) \frac{k_{2m(p)} + \bar{G}_{2m(p)}}{G_{2m(p)}k_{2m(p)} + 1} \\ &= C_{2m(p)-1} + r_{2m(p)-1} \exp(i\omega_{2m(p)-1}) \frac{u_{2m(p)-1} + \bar{G}_{2m(p)-1}}{G_{2m(p)-1}u_{2m(p)-1} + 1}. \end{aligned} \tag{2.20}$$

From (2.20) it can be seen that the two limits in (2.19) will be equal provided that

$$\begin{aligned} \lim_{p\to\infty} &\exp(i\gamma_{2m(p)}) \frac{k_{2m(p)} + \bar{G}_{2m(p)}}{G_{2m(p)}k_{2m(p)} + 1} \\ &= \lim_{p\to\infty} \exp(i\gamma_{2m(p)-1}) \frac{u_{2m(p)-1} + \bar{G}_{2m(p)-1}}{G_{2m(p)-1}u_{2m(p)-1} + 1} = 1. \end{aligned} \tag{2.21}$$

But it is easily verified that

$$\left| \frac{k_{2m(p)} \exp(i\gamma_{2m(p)}) + g_{2m(p)}}{G_{2m(p)}k_{2m(p)} + 1} - 1 \right| \leq \frac{(|k_{2m(p)}| + 1)(1 - g_{2m(p)})}{||k_{2m(p)}|g_{2m(p)} - 1|}, \tag{2.22}$$

and

$$\begin{aligned} &\left| \frac{u_{2m(p)-1} \exp(i\gamma_{2m(p)-1}) + g_{2m(p)-1}}{G_{2m(p)-1}u_{2m(p)-1} + 1} - 1 \right| \\ &\qquad \leq \frac{(|u_{2m(p)-1}| + 1)(1 - g_{2m(p)-1})}{||u_{2m(p)-1}|\, g_{2m(p)-1} - 1|}. \end{aligned} \tag{2.23}$$

From (2.16b), (2.16c) and $g_n \rightarrow 1$ it follows that the right sides of (2.22) and (2.23) both tend to zero as $p \rightarrow \infty$. This completes the proof.

**3. Convergence regions.** This section is used to derive convergence regions for continued fractions of the form $K(a_n/1)$.

**Theorem 3.1.** *Let* $\{\Gamma_n\}$ *be a sequence of complex numbers and* $\{\rho_n\}$ *a sequence of positive real numbers such that for* $n \geq 0$, $|\Gamma_n| \neq |1+\Gamma_n|$ *and* $\rho_n$ *lies in the open interval between* $|\Gamma_n|$ *and* $|1+\Gamma_n|$. *Let* $\Delta_n = \rho_n^2 - |1+\Gamma_n|^2$ *and, for each* $n \geq 1$, *let* $E_n$ *be the region in the complex plane defined as follows: If* $|\Gamma_{n-1}| < \rho_{n-1} < |1+\Gamma_{n-1}|$, *then*

$$E_n = \{w: |w(1+\bar{\Gamma}_n) + \Gamma_{n-1}\Delta_n| + \rho_n|w| \leq \rho_{n-1}|\Delta_n|\} \tag{3.1a}$$

*and if* $|1+\Gamma_{n-1}| < \rho_{n-1} < |\Gamma_{n-1}|$, *then*

$$E_n = \{w: |w(1+\bar{\Gamma}_n) + \Gamma_{n-1}\Delta_n| - \rho_n|w| \geq \rho_{n-1}|\Delta_n|\}. \tag{3.1b}$$

*Let* $K(a_n/1)$ *be a continued fraction with elements satisfying*

$$a_n \in E_n, \quad a_n \neq 0, \quad n \geq 1, \tag{3.2}$$

*and with nth approximant denoted by* $f_n$. *If there exists a positive constant* $\epsilon > 0$ *such that*

$$\frac{\rho_n}{|\bar{\Gamma}_n + |\Gamma_n|^2 - \rho_n^2|} \geq 1+\epsilon, \quad n \geq 0, \tag{3.3}$$

*then both of the sequences* $\{f_{2n-1}\}$ *and* $\{f_{2n}\}$ *are convergent. If, in addition*

$$|\rho_n/|\Gamma_n| - 1| \geq \epsilon, \quad n \geq 0, \tag{3.4}$$

*then the continued fraction* $K(a_n/1)$ *converges.*

**Lemma 3.2.** *Let* $\{\Gamma_n\}$, $\{\rho_n\}$ *and* $\{E_n\}$ *be sequences defined as in Theorem 3.1. Let* $\{V_n\}$ *be the sequence of closed regions in the extended complex plane defined by*

$$V_n = \begin{cases} \{z: |z-\Gamma_n| \leq \rho_n\}, & \text{if } |\Gamma_n| < \rho_n < |1+\Gamma_n|, \\ \{z: |z-\Gamma_n| \geq \rho_n\}, & \text{if } |1+\Gamma_n| < \rho_n < |\Gamma_n|. \end{cases} \tag{3.5}$$

*Then*

$$s(E_n, V_n) \subseteq V_{n-1}, \quad n \geq 1, \tag{3.6}$$

*where* $s(w, z) = w/(1+z)$.

**Proof.** We shall verify (3.6) in the case for which $|1+\Gamma_{n-1}| < \rho_{n-1} < |\Gamma_{n-1}|$ and $|1+\Gamma_n| < \rho_n < |\Gamma_n|$. The case for which $|\Gamma_{n-1}| < \rho_{n-1} < |1+\Gamma_{n-1}|$

and $|\Gamma_n| < \rho_n < |1 + \Gamma_n|$ was proven by [4, Lemma 2.1]; proofs for the other two cases are included in [5, Lemma 5.5]. First, it is readily shown that $s(w, V_n)$ consists of the circular disk $\{z: |z + D_n| \le q_n\}$, where $D_n = w(1 + \overline{\Gamma}_n)/\Delta_n$, $q_n = \rho_n|w|/\Delta_n$, $\Delta_n = \rho_n^2 - |1 + \Gamma_n|^2$. It follows immediately that $s(w, V_n) \subseteq V_{n-1}$ if and only if $|D_n + \Gamma_{n-1}| \ge q_n + \rho_{n-1}$, which is equivalent to the inequality in (3.1b). This completes the proof.

**Proof of Theorem 3.1.** Let $\{v_n\}$ denote the sequence of l.f.t.'s defined by

$$v_n(z) = \frac{\rho_n z}{\overline{\Gamma}_n z - |\Gamma_n|^2 + \rho_n^2}. \tag{3.7}$$

It is easily verified that the image of the region $V_n$ (defined by (3.5)) under the mapping $w = v_n(z)$ is the unit disk $U = \{z: |z| \le 1\}$; that is,

$$v_n(V_n) = U, \qquad n \ge 0. \tag{3.8}$$

Let $\{t_n\}$ and $\{T_n\}$ denote sequences of l.f.t.'s defined by

$$t_n(z) = v_{n-1}\{s_n[v_n^{-1}(z)]\}, \qquad n \ge 1, \tag{3.9a}$$

$$T_1(z) = t_1(z); \qquad T_n(z) = T_{n-1}[t_n(z)], \qquad n \ge 2, \tag{3.9b}$$

where $s_n(z) = s(a_n, z) = a_n/(1 + z)$, $a_n \in E_n$. It follows from (3.6) and (3.9) that $\{T_n\}$ satisfies (2.1) and hence can be represented in the form (2.2). From (3.9) it also follows that

$$S_n(z) = v_0^{-1}\{T_n[v_n(z)]\}. \tag{3.10}$$

Thus $f_n = S_n(0) = v_0^{-1}[T_n(0)]$, and hence the continued fraction $K(a_n/1)$ will converge if and only if the sequence $\{T_n(0)\}$ converges. In the limit point case, the sequence $\{T_n(U)\}$ converges to the point $C = \lim C_n$ and, therefore, the continued fraction converges. Hence, it remains to consider what happens if the limit circle case ($r_n \searrow r > 0$) occurs. From (1.6) it follows that

$$S_n(-1) = S_{n-1}(\infty) = S_{n-2}(0), \qquad n \ge 3, \tag{3.11}$$

and so from (3.10) we have

$$T_n[v_n(-1)] = T_{n-1}[v_{n-1}(\infty)] = T_{n-2}[v_{n-2}(0)], \qquad n \ge 3. \tag{3.12}$$

Our next step is to set

$$j_n = v_n(-1), \qquad k_n = v_n(\infty), \qquad u_n = v_n(0) = 0, \qquad n \ge 1. \tag{3.13}$$

Then by (3.3) we have $|j_n| = |v_n(-1)| \ge 1 + \epsilon$ and Theorems 2.1 and 2.2 imply that the two sequences $\{T_{2n-1}(z)\}$ and $\{T_{2n}(z)\}$ converge for all $z$ such that $|z| \ne 1$. In particular, $\{T_{2n-1}(0)\}$ and $\{T_{2n}(0)\}$ converge so that $\{f_{2n-1}\}$ and

$\{f_{2n}\}$ are convergent. If, in addition, (3.4) holds, then $||k_n| - 1| \geq \epsilon$ and so, by Theorem 2.3, $\{T_n(z)\}$ converges for all $z$ such that $|z| \neq 1$. Thus $\{T_n(0)\}$ and also $\{f_n\}$ are convergent. This completes the proof.

An important special case of Theorem 3.1 is the following:

**Corollary 3.3** (*alternating disk—complement of disk case*). *Let* $\{\Gamma_n\}$ *be a sequence of complex numbers and* $\{\rho_n\}$ *a sequence of positive real numbers such that*

$$(3.14)\qquad |1 + \Gamma_{2n}| < \rho_{2n} < |\Gamma_{2n}|, \qquad |\Gamma_{2n+1}| < \rho_{2n+1} < |1 + \Gamma_{2n+1}|, \qquad n \geq 0$$

*and let* $\Delta_n = \rho_n^2 - |1 + \Gamma_n|^2$. *Let* $K(a_n/1)$ *be a continued fraction with elements* $a_n$ *satisfying*

$$(3.15)\qquad a_n \in E_n, \quad a_n \neq 0, \quad n \geq 1,$$

*where*

$$(3.16a)\quad E_{2n+1} = \{w: |w(1 + \overline{\Gamma}_{2n+1}) + \Gamma_{2n}\Delta_{2n+1}| - \rho_{2n+1}|w| \geq \rho_{2n}|\Delta_{2n+1}|\}, \quad n \geq 0,$$

$$(3.16b)\quad E_{2n} = \{w: |w(1 + \overline{\Gamma}_{2n}) + \Gamma_{2n-1}\Delta_{2n}| + \rho_{2n}|w| \leq \rho_{2n-1}|\Delta_{2n}|\}, \quad n \geq 1,$$

*and with nth approximant denoted by* $f_n$. *If* (3.3) *holds for some positive constant* $\epsilon > 0$, *then both* $\{f_{2n-1}\}$ *and* $\{f_{2n}\}$ *converge. If, in addition,* (3.4) *holds, then the continued fraction* $K(a_n/1)$ *is convergent.*

**Remarks.** (1) By taking $\Gamma_{2n+1} = \Gamma_1, \rho_{2n+1} = \rho_1, \Gamma_{2n} = \Gamma_2$ and $\rho_{2n} = \rho_2$ in Corollary 3.3, we obtain part of the result proved by [5, Theorem 5.4]. The further special case with $\rho_1 = \rho_2 = \rho$ and $\Gamma_1 = -(1 + \Gamma_2) = \Gamma$ is the result of Lange and Thron [7] stated in the introduction.

(2) Corollary 3.3 is referred to as the *alternating disk-complement of disk case,* since the $V_n$ of (3.5) are alternately disks and complements of disks.

For admissible sequences that contain unbounded regions, Hayden [1, Theorem 2] gave the following sufficient conditions:

*Suppose* $\{E_n^*\}$ *is a sequence such that for* $n \geq 1$,

(i) *either* $E_n^*$ *is a* $C_0$ + int *or* $E_n^*$ *is a* $C_0$ + ext,

(ii) *at least one of* $E_n^*$ *or* $E_{n+1}^*$ *is a* $C_0$ + int, *and*

(iii) *there exists a number* $\kappa_{n-1}$ *and a number* $r_n$ *such that*

(a)

$$(3.17)\qquad 0 < \kappa_{n-1} < 1, \qquad 0 < r_n \leq 1,$$

$$(3.18)\qquad E_n^* = \begin{cases} \{w: |w| \leq r_n(1 - \kappa_{n-1})\kappa_n\}, & \text{if } E_n^* \text{ is bounded,} \\ \{w: |w| \geq (1 + \kappa_{n-1})(2 - \kappa_n)\}, & \text{if } E_n^* \text{ is unbounded,} \end{cases}$$

*and*

(b) *if* $p$ *is an integer such that* $E^*_{p+1}$ *is unbounded, and if* $M$ *is the collection of all such integers, then either* $M$ *is finite or* $\Pi_{k\in M} r_k = 0$. *Then* $\{E^*_n\} \in$ AS.

The following corollary of Theorem 3.1 is comparable with the sufficient conditions of Hayden stated above.

**Corollary 3.4.** *Let* $\{E_n\}$ *be a sequence of regions in the complex plane such that for each* $n \geq 1$ *the following conditions are satisfied*:

(i) *at least one of the regions* $E_n$ *or* $E_{n+1}$ *is bounded,*

(ii) *there exists a sequence of positive numbers* $\{\kappa_n\}$ *and a positive constant* $0 < \epsilon < 1$ *such that*

$$0 < \epsilon \leq \kappa_{n-1} < 1, \quad \text{if } E_n \text{ is bounded,} \tag{3.19a}$$

$$0 < \kappa_{n-1} \leq 1 - \epsilon < 1, \quad \text{if } E_n \text{ is unbounded,} \tag{3.19b}$$

*and*

$$E_n = \begin{cases} \{w: |w| \leq (1-\kappa_{n-1})\kappa_n\}, & \text{if } E_n \text{ is bounded,} \\ \{w: |w + (2-\kappa_n)\kappa_n| - (1-\kappa_n)|w| \geq \kappa_{n-1}\kappa_n(2-\kappa_n)\}, & \\ & \text{if } E_n \text{ is unbounded.} \end{cases} \tag{3.20}$$

*If* $a_n \in E_n$, $a_n \neq 0$, $n \geq 1$, *then the continued fraction* $K(a_n/1)$ *is convergent.*

**Proof.** First we establish the relationship between the element regions (3.20) and those defined by (3.1). Let sequences $\{\Gamma_n\}$ and $\{\rho_n\}$ be defined as follows:

$$\Gamma_n = \begin{cases} 0, & \text{if } E_{n+1} \text{ is bounded,} \\ -1, & \text{if } E_{n+1} \text{ is unbounded.} \end{cases} \tag{3.21a}$$

$$\rho_n = \begin{cases} 1-\kappa_n, & \text{if } E_{n+1} \text{ is bounded,} \\ \kappa_n, & \text{if } E_{n+1} \text{ is unbounded.} \end{cases} \tag{3.21b}$$

Now it is easily checked that when $E_n$ is bounded, (3.1a) reduces to the bounded set in (3.20) and when $E_n$ is unbounded, (3.1b) reduces to the unbounded set in (3.20). Moreover, (3.3) and (3.4) are implied by (3.19). Hence our corollary is an immediate consequence of Theorem 3.1.

**Remarks.** (1) When $E_n$ is bounded the expression given by (3.20) is of the same form as Hayden's expression (3.18) with $r_n = 1$.

(2) The unbounded region $E_n$ defined by (3.20) contains the unbounded region $E^*_n$ of (3.18). $E_n$ is connected and symmetric with respect to the real axis. The *boundary* of $E_n$ is contained in the annular region

$$\kappa_n(1-\kappa_{n-1}) \le |w| \le (2-\kappa_n)(1+\kappa_{n-1}),$$

and its real intercepts are at

$$w = -\kappa_n(1-\kappa_{n-1}) \quad \text{and} \quad w = -(2-\kappa_n)(1+\kappa_{n-1}).$$

An illustration of the unbounded regions $E_n$ and $E_n^*$ is shown in Figure 1, for the case with $\kappa_{n-1} = 1/\sqrt{2}$ and $\kappa_n = 1-(1/\sqrt{2})$.

(3) In view of Remarks (1) and (2), it can be seen that Corollary 3.4 has an overlapping relation with the sufficient conditions of Hayden stated above. Conditions (3.19) are more restrictive than (3.18a). However, when the $\kappa_n$ satisfy (3.19), the element region $E_n$ contains $E_n^*$ for $n \ge 1$. Moreover, Hayden's condition (iiib) is not required in Corollary 3.4.

In proving Theorem 3.1 we have not made use of the results of § 2 in their greatest generality. In particular, (2.16c) allows an infinite subsequence of the $k_n$ to be equal to one. If we set $k_n = v_n(\infty)$ as in (3.13), then $\infty$ must lie on the boundary of the region $V_n$, a situation realized when $V_n$ is a half-plane. The following theorem is an example of a result which can be proved when an infinite subsequence of the $V_n$ are half-planes. The case in which all of the $V_n$ are half-planes leads to element regions $E_n$ with parabolic boundaries; this case has already been extensively treated (see, for example: [3], [4] and [9, Theorem 31.3]).

**Theorem 3.5.** *Let $\{\Gamma_{2n+1}\}$ be a sequence of complex numbers such that for $n \ge 0$, $|\Gamma_{2n+1}| \ne |1+\Gamma_{2n+1}|$. Let $\{\rho_{2n+1}\}$ be a sequence of positive numbers such that for $n \ge 0$, $\rho_{2n+1}$ lies in the open interval between $|\Gamma_{2n+1}|$ and $|1+\Gamma_{2n+1}|$. Let $\{P_{2n}\}$ be a sequence of complex numbers such that*

$$(3.22) \qquad 0 < \rho_{2n} < \cos\psi_{2n}, \qquad \rho_{2n} = |P_{2n}|, \qquad \psi_{2n} = \arg P_{2n}, \qquad n \ge 0.$$

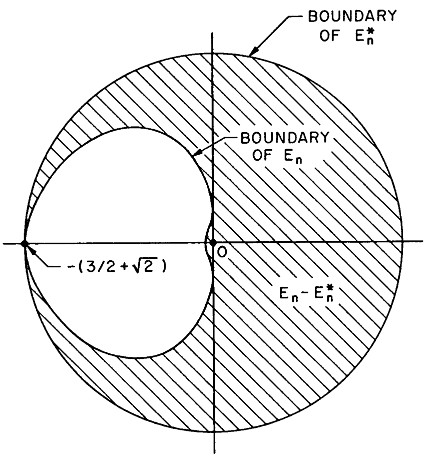


FIGURE 1. Comparison of unbounded region $E_n$ of Corollary 3.4 with unbounded region $E_n^*$ of (3.18), with $\kappa_{n-1} = 1/\sqrt{2}$, $\kappa_n = 1-(1/\sqrt{2})$.

*Let* $\overset{\bullet}{K}(a_n/1)$ *be a continued fraction with elements* $a_n$ *satisfying*

(3.23) $$a_n \in E_n, \quad a_n \neq 0, \quad n \geq 1,$$

*where*

(3.24a)

$$E_{2n+1} = \left\{ w: |w| \leq \frac{p_{2n}|\Delta_{2n+1}|}{\rho_{2n+1} + (\operatorname{sgn} \Delta_{2n+1})|1 + \Gamma_{2n+1}|\cos(\arg w - \arg(1 + \Gamma_{2n+1}) - \psi_{2n})} \right\},$$

(3.24b)

$$E_{2n} = \left\{ w: |w| \leq \frac{(\cos \psi_{2n} - p_{2n})|\delta_{2n-1}|}{\rho_{2n-1} + (\operatorname{sgn} \delta_{2n-1})|\Gamma_{2n-1}|\cos(\arg w - \arg \Gamma_{2n-1} - \psi_{2n})} \right\},$$

*where* $\Delta_n = \rho_n^2 - |1 + \Gamma_n|^2$ *and* $\delta_n = |\Gamma_n|^2 - \rho_n^2$.

*Let* $f_n$ *denote the nth approximant of* $K(a_n/1)$. *If there exist positive constants* $\epsilon$ *and* $M$ *such that*

(3.25) $$\left|P_{2n} - \frac{1}{2}\right| \leq M < \frac{1}{2}, \quad \frac{\rho_{2n+1}}{|\overline{\Gamma}_{2n+1} + |\Gamma_{2n+1}|^2 - \rho_{2n+1}^2|} \geq 1 + \epsilon, \quad n \geq 0,$$

*then the sequences* $\{f_{2n-1}\}$ *and* $\{f_{2n}\}$ *both converge. If, in addition,*

(3.26) $$|(\rho_{2n+1}/\Gamma_{2n+1}) - 1| \geq \epsilon, \quad n \geq 0,$$

*then the continued fraction* $K(a_n/1)$ *is convergent.*

**Lemma 3.6.** *Let* $\{\Gamma_{2n+1}\}$, $\{\rho_{2n+1}\}$, $\{P_{2n}\}$ *and* $\{E_n\}$ *be sequences defined as in Theorem* 3.5. *Let* $\{V_n\}$ *be the sequence of closed regions in the extended complex plane defined by*

(3.27a) $$V_{2n} = \{z: \operatorname{Re}(z \exp(-i\psi_{2n})) \geq -p_{2n}\}, \quad n \geq 0,$$

(3.27b) $$V_{2n+1} = \begin{cases} \{z: |z - \Gamma_{2n+1}| \leq \rho_{2n+1}\} & \text{if } |\Gamma_{2n+1}| < \rho_{2n+1} < |1 + \Gamma_{2n+1}|, \\ \{z: |z - \Gamma_{2n+1}| \geq \rho_{2n+1}\} & \text{if } |1 + \Gamma_{2n+1}| < \rho_{2n+1} < |\Gamma_{2n+1}|, \end{cases} \quad n \geq 0.$$

*Then*

(3.28) $$s(E_n, V_n) \subseteq V_{n-1}, \quad n \geq 1,$$

*where* $s(w, z) = w/(1 + z)$.

**Proof.** It is readily shown that $s(w, V_{2n}) = \{\zeta: |\zeta - D_{2n}| \leq |D_{2n}|\}$, where $D_{2n} = w \exp(-i\Psi_{2n})/[2(\cos\Psi_{2n} - p_{2n})]$. Therefore, for $s(w, V_{2n}) \subseteq V_{2n-1}$ it is necessary and sufficient that (a) $|D_{2n} - \Gamma_{2n-1}| + |D_{2n}| \leq \rho_{2n-1}$ if $|\Gamma_{2n-1}| < \rho_{2n-1} < |1 + \Gamma_{2n-1}|$, or (b) $|D_{2n} - \Gamma_{2n-1}| \geq |D_{2n}| + \rho_{2n-1}$ if $|1 + \Gamma_{2n-1}| < \rho_{2n-1} < |\Gamma_{2n-1}|$. In either case, it can be seen that $s(w, V_{2n}) \subseteq V_{2n-1}$ if and only if $w \in E_{2n}$ given by (3.24b). Similarly, $s(w, V_{2n-1}) = \{\zeta: |\zeta - D_{2n-1}| \leq q_{2n-1}\}$, where $D_{2n-1} = -w(1 + \overline{\Gamma}_{2n-1})/\Delta_{2n-1}$, $q_{2n-1} = |w|\rho_{2n-1}/|\Delta_{2n-1}|$ and $\Delta_{2n-1} = \rho^2_{2n-1} - |1 + \Gamma_{2n-1}|^2$. It follows that $s(w, V_{2n-1}) \subseteq V_{2n-2}$ if and only if $q_{2n-1} \leq p_{2n-2} + |D_{2n-1}| \cos(\arg D_{2n-1} - \Psi_{2n-2})$. But this condition is equivalent to the statement $w \in E_{2n-1}$, where $E_{2n-1}$ is given by (3.24a). This completes the proof.

**Proof of Theorem 3.5.** Let $\{V_n\}$ denote the sequence of l.f.t.'s defined by

$$v_{2n}(z) = z/(z - 2P_{2n}), \qquad n \geq 0, \tag{3.29a}$$

$$v_{2n+1}(z) = \begin{cases} \dfrac{-\rho_{2n+1}z}{\overline{\Gamma}_{2n+1}z + \rho^2_{2n+1} - |\Gamma_{2n+1}|^2}, \\ \qquad \text{if } |\Gamma_{2n+1}| < \rho_{2n+1} < |1 + \Gamma_{2n+1}|, \\ \dfrac{\rho_{2n+1}z}{\overline{\Gamma}_{2n+1}z + \rho^2_{2n+1} - |\Gamma_{2n+1}|^2}, \\ \qquad \text{if } |1 + \Gamma_{2n+1}| < \rho_{2n+1} < |\Gamma_{2n+1}|. \end{cases} \tag{3.29b}$$

It is readily shown that $v_n(V_n) = U \equiv \{z: |z| \leq 1\}$, $n \geq 0$. The remainder of the proof is now completely analogous to the proof of Theorem 3.1 and hence is omitted.

**Remarks.** (1) Theorem 3.5 reduces to a result proved by [5, Theorem 5.2] in the special case for which $\Gamma_{2n+1} = \Gamma$, $\rho_{2n+1} = \rho$, $P_{2n} = P = pe^{i\psi}$, and $|\Gamma| < \rho < |1 + \Gamma|$, except that our theorem does not permit $p = 0$.

(2) When $|\Gamma_{2n+1}| < \rho_{2n+1} < |1 + \Gamma_{2n+1}|$, the boundary of $E_{2n+1}$ is an hyperbola and the boundary of $E_{2n+2}$ is an ellipse. On the other hand, when $|1 + \Gamma_{2n+1}| < \rho_{2n+1} < |\Gamma_{2n+1}|$, the boundary of $E_{2n+1}$ is an ellipse and the boundary of $E_{2n+2}$ is an hyperbola. In each case a focus of the conic is at the origin and the axes are easily determined from the polar form (3.24).

(3) Conditions (3.22) imply that

$$|P_{2n} - ½| < ½ \tag{3.30a}$$

and the condition that $\rho_{2n+1}$ lies in the open interval between $|\Gamma_{2n+1}|$ and $|1 + \Gamma_{2n+1}|$ implies that

$$\rho_{2n+1}/|\bar{\Gamma}_{2n+1} + |\Gamma_{2n+1}|^2 - \rho^2_{2n+1}| > 1. \tag{3.30b}$$

Thus we see that conditions (3.25) uniformly bound the quantities on the left side of (3.30) away from their limiting values.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, BOULDER, COLORADO 80302 (Current address of William B. Jones)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PUGET SOUND, TACOMA, WASHINGTON 98416 (Current address of R. I. Snell)