TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 170, August 1972

PRODUCTS OF WEAKLY-$\aleph$-COMPACT SPACES[1]

BY

MILTON ULMER

ABSTRACT. A space is said to be weakly-$\aleph_1$-compact (or weakly-Lindelöf) provided each open cover admits a countable subfamily with dense union. We show this property in a product space is determined by finite subproducts, and by assuming that $2^{\aleph_0} = 2^{\aleph_1}$ we show the property is not preserved by finite products. These results are generalized to higher cardinals and two research problems are stated.

Introduction. Weakly-Lindelöf spaces have been studied by Frolík [F_1], by Comfort, Hindman and Negrepontis [CHN], and by Hager [H]. We extend their results by proving that for $\aleph$ regular and uncountable, weakly-$\aleph$-compactness in a product space is determined by finite subproducts. In §2 we outline a method which can possibly be used to construct, for each nonmeasurable cardinal $\mathfrak{n}$, a pair of weakly-Lindelöf spaces whose product fails to be even weakly-$\mathfrak{n}$-compact. By assuming that $2^{\aleph_0} = 2^{\aleph_1}$, in §3 we complete the construction of a pair of weakly-Lindelöf spaces whose product fails to be weakly-Lindelöf.

1. **The product theorem.** In what follows $\aleph$ will denote an infinite cardinal number (measurable or nonmeasurable).

1.1 **Definition.** A topological space Y is said to be weakly-$\aleph$-compact provided each cover of Y by open sets admits a subfamily of cardinality less than $\aleph$ with dense union. Thus a space is weakly-Lindelöf if and only if it is weakly-$\aleph_1$-compact.

1.2 **Notation.** Let $X = \Pi_{\alpha\in I} X_\alpha$ be a product space with the usual Tychonoff topology. For each nonempty set $J \subset I$ we denote by P_J the canonical projection of X onto $\Pi_{\alpha\in J} X_\alpha$. If U is a nonempty basic open set in x, we denote by $R(U)$ the finite set of coordinates on which the projection is not the entire factor. If $\mathfrak{U}$ is any family of basic open sets, we define $R(\mathfrak{U}) = \bigcup_{U\in\mathfrak{U}} R(U)$.

Received by the editors May 14, 1971.

AMS 1970 *subject classifications*. Primary 02K05, 54A25, 54B10, 54D20; Secondary 02K35, 04A30, 54C30, 54C60, 54G20.

Key words and phrases. Weakly-Lindelöf, weakly-$\mathfrak{n}$-compact, product spaces, nonmeasurable cardinals, generalized continuum hypothesis, Lusin's hypothesis.

(1) This paper contains part of the second chapter of the author's doctoral dissertation [U_1] which was written under the helpful supervision of Professor W. W. Comfort. Work on this paper was partially supported by the National Science Foundation under grants NSF-GP-8357 and NSF-GP-18825.

Presented to the North Central Section of the Mathematical Association of America October 23, 1971 under the title *A product theorem for weakly-λ-compact spaces*.

1.3 **Theorem.** *Let $\aleph$ be a regular uncountable cardinal. If each finite subproduct of X is weakly-$\aleph$-compact, then X is weakly-$\aleph$-compact.*

Proof. Let $\mathfrak{U}$ be any cover of X by basic open sets. Given any finite subset F of I, let $\mathfrak{U}_F$ be any subfamily of $\mathfrak{U}$ such that the cardinality of $\mathfrak{U}_F$ is less than $\aleph$, and $\{P_F(U)\colon U \in \mathfrak{U}_F\}$ has dense union in $\Pi_{\alpha\in F} X_\alpha$. Since P_F is an open surjection, clearly such an $\mathfrak{U}_F$ exists.

Let $C_0 \subset I$ be any nonempty subset of cardinality less than $\aleph$. Let $\mathfrak{F}_1 = \bigcup \mathfrak{U}_F$, the union being taken over all finite subsets of C_0. Set $C_1 = C_0 \cup R(\mathfrak{F}_1)$. Since $\aleph$ is regular, card $(\mathfrak{F}_1) < \aleph$, and hence card $(C_1) < \aleph$.

Inductively, let k be a positive integer, and suppose C_i and $\mathfrak{F}_i$ have been chosen for all $i \leq k$, where $\mathfrak{F}_i = \bigcup \mathfrak{U}_F$, the union being taken over all finite subsets of C_{i-1}, and where $C_i = C_{i-1} \cup R(\mathfrak{F}_i)$. By this construction the cardinality of C_k is less than $\aleph$, and hence we can repeat the above process to obtain C_{k+1} and $\mathfrak{F}_{k+1}$.

Now set $C = \bigcup_{n\in N} C_n$, and $\mathfrak{U}' = \bigcup_{n\in N} \mathfrak{F}_n$, where N denotes the set of positive integers. Since each $\mathfrak{F}_n$ is of cardinality less than $\aleph$, and since $\aleph$ is uncountable and regular, we have that card $(\mathfrak{U}') < \aleph$. Thus it suffices to show that $\mathfrak{U}'$ has dense union in X.

Let U be any nonempty basic open subset of X. Since $R(U)$ is finite, there exists an integer m such that $R(U) \cap C = R(U) \cap C_m$. Let F be the finite set $R(U) \cap C_m$. By the construction of $\mathfrak{F}_m$ there is a basic open set $V \in \mathfrak{U}_F \subset \mathfrak{F}_m$ for which $P_F(U) \cap P_F(V) \neq \emptyset$. But we also know that $R(V)$ is contained in C_{m+1}. Thus $R(U) \cap R(V)$ is a subset of F. Hence $U \cap V \neq \emptyset$. This proves the theorem.

The hypothesis that $\aleph$ be regular cannot be removed. This is evident from [NU, Example 1.7] which gives us, for each singular cardinal $\aleph$, a product space in which each finite subproduct has smaller cardinality than $\aleph$ and yet the entire product fails to be pseudo-$\aleph$-compact. Since each weakly-$\aleph$-compact space is pseudo-$\aleph$-compact, this example fulfills our need.

We do not consider weakly-$\aleph_0$-compact spaces because in the presence of regularity, weakly-$\aleph_0$-compactness is equivalent to compactness.

The next two sections tell us Theorem 1.3 cannot be improved to yield that for regular and uncountable $\aleph$, X is weakly-$\aleph$-compact whenever each coordinate space is weakly-$\aleph$-compact.

2. **The foundations of an example.** Again, let $\mathfrak{n}$ be any uncountable nonmeasurable cardinal number, We now turn our attention to the problem of constructing two spaces Y and Z which are weakly-Lindelöf but whose product fails to be even weakly-$\mathfrak{n}$-compact. The spaces used will turn out to be subspaces of the

Stone-Čech compactification of a discrete space of cardinality $\mathfrak{n}$. (Obviously, the author is indebted to the work of Comfort [C_2] and Frolík [F_3] for the inspiration of this example.)

Recall that a set S has nonmeasurable cardinality provided the following condition is satisfied: If $\mathfrak{U}$ is any ultrafilter on S and $\mathfrak{U}$ has the countable intersection property, then $\bigcap \mathfrak{U} \neq \emptyset$. It is well known that the class of nonmeasurable cardinals is extremely large; it, may, in fact, embrace all cardinals. However, for our purposes we only need notice that $\aleph_1$ is nonmeasurable. The reader is referred to [GJ] for a highly informative treatment of this concept.

Let D be a discrete space of cardinality $\mathfrak{n}$. A filter base $\mathfrak{F}$ on D is said to be strong provided it has the countable intersection property and $\bigcap \mathfrak{F} = \emptyset$.

Take βD, the Stone-Čech compactification of D, to be the set of ultrafilters on D together with the Stone topology. In this context we have the following theorem:

2.1 **Theorem.** *Let A be a subset of $\beta D - D$ and $Y = D \cup A$; then Y is weakly-Lindelöf if and only if each strong filter base is extended by some member of A.*

Proof. Suppose Y is weakly-Lindelöf and $\mathfrak{F}$ is a strong filter base. Let $\mathfrak{U} = \{\mathrm{cl}_Y (D - V): V \in \mathfrak{F}\}$. Since $\bigcap \mathfrak{F} = \emptyset$, $\mathfrak{U}$ is an open cover for D. Suppose no ultrafilter $p \in A$ extends $\mathfrak{F}$. Then for each $p \in A$ there is a $V_p \in \mathfrak{F}$ such that $V_p \notin p$ or, equivalently $p \in \mathrm{cl}_Y (D - V_p)$. Hence $\mathfrak{U}$ is an open cover for Y.

Since Y is weakly-Lindelöf there must exist a countable family $\{U_n: n \in N\} \subset \mathfrak{U}$ with dense union. Let $\{V_n: n \in N\}$ be the set of corresponding members of $\mathfrak{F}$.

Since each singleton in D is an open subset of Y, $\bigcup_{n \in N} U_n$ must contain D. Hence $\bigcap_{n \in N} V_n = \emptyset$, which contradicts that $\mathfrak{F}$ is strong. Thus some $p \in A$ must extend $\mathfrak{F}$.

To prove the converse we let $\mathfrak{U}$ be a cover for $A \cup D$ by basic open sets. For each open set $U \in \mathfrak{U}$ let $K(U)$ denote the complement of U in D. That is, let $K(U) = D - (U \cap D)$. Now if any countable subfamily of $\{K(U): U \in \mathfrak{U}\}$ has void intersection we are done, since if $\bigcap_{n \in N} K(U_n) = \emptyset$, then $D \subset \bigcup_{n \in N} U_n$ and thus D is the dense subset we are seeking.

Suppose then every countable subfamily of $\{K(U): U \in \mathfrak{U}\}$ has nonempty intersection. Since $\mathfrak{U}$ covers D, we see that $\bigcap_{U \in \mathfrak{U}} K(U) = \emptyset$ and hence $\{K(U): U \in \mathfrak{U}\}$ is a strong filter base. But then, by the choice of A, there must exist an ultrafilter $p \in A$ such that $\{K(U): U \in \mathfrak{U}\} \subset p$. But this is simply another way of saying that $p \notin \bigcup \mathfrak{U}$ which contradicts that $\mathfrak{U}$ is a cover for A. Thus $D \cup A$ is weakly-Lindelöf.

Suppose now it is possible to partition $\beta D - D$ into two disjoint sets A_1 and A_2 each with the property given A in Theorem 2.1. Since $A_1 \cap A_2 = \emptyset$, the

the diagonal in $(D \cup A_1) \times (D \cup A_2)$ is a closed discrete space of cardinality $\mathfrak{n}$. Thus $(D \cup A_1) \times (D \cup A_2)$ is not weakly-$\mathfrak{n}$-compact, while each factor is weakly-Lindelöf.

Clearly, if D were of measurable cardinality, then no such partition would be possible. However, since D is of nonmeasurable cardinality such a partition might be possible, and the use of types of ultrafilters is the obvious way to choose A_1 and A_2.

2.2 **Definition.** Two ultrafilters p_1 and p_2 on D are said to be of the same type provided there is some permutation τ of D such that $\tau^*(p_1) = p_2$, where τ^* is the continuous extension of τ to βD.

The reader may wish to refer to [F_2], [R_1], or [R_2] for insight into the concept of types of ultrafilters.

Suppose card $(D) = \aleph_1$. If it can be shown that there exist two types of ultrafilters such that each strong filter base on D can be completed to an ultrafilter of each type, then A_1 and A_2 could be taken to be the sets of all ultrafilters on D of these two types.

If card $(D) > \aleph_1$, the situation is slightly more complex.

2.3 **Definition.** Let $\mathfrak{F}$ be a filter base on D. We say that $\mathfrak{F}$ is $\mathfrak{m}$-uniform provided $\mathfrak{m}$ is the smallest cardinal such that card $(S) = \mathfrak{m}$ for some set $S \in \mathfrak{F}$.

Suppose for each uncountable cardinal $\mathfrak{m} \leq \mathfrak{n}$ there are two types of ultrafilters such that each $\mathfrak{m}$-uniform strong filter base on D can be completed to an ultrafilter of each type, then the construction of A_1 and A_2 is obvious. Thus the following problem is suggested:

2.4 *Problem.* Let $\mathfrak{n}$ be a nonmeasurable cardinal. Do there exist, for each uncountable cardinal $\mathfrak{m} \leq \mathfrak{n}$, two types of ultrafilters such that each $\mathfrak{m}$-uniform strong filter base on D can be completed to an ultrafilter of each type?

Although the example outlined above could work only for nonmeasurable cardinals, it is conceivable that the following question has a negative solution.

2.5 *Problem.* If $\mathfrak{m}$ is measurable, is the product of two weakly-$\mathfrak{m}$-compact spaces necessarily weakly-$\mathfrak{m}$-compact?

3. **Results possible if the continuum hypothesis fails.** In this section we shall use Lusin's hypothesis, that $2^{\aleph_0} = 2^{\aleph_1}$, to complete the construction of the example. Not only is Lusin's hypothesis consistent with the usual axioms of set theory, Cohen has shown in [C_1] that Lusin's hypothesis is consistent with the usual axioms of set theory together with the remainder of the generalized continuum hypothesis. That is, we may assume $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for $\alpha \geq 1$.

Let D be a discrete space of cardinality $\aleph_1$, and let $\mathfrak{F}$ be a strong filter base on D. If $\Gamma \subset \beta D$ denotes the set of ultrafilters which extend $\mathfrak{F}$, we have the following:

3.1 **Proposition.** card $(\Gamma) \geq 2^c$.

Proof. By [GJ, Corollary 9.12] if we show that Γ is closed and infinite, then Γ must contain a copy of βN and hence card $(\Gamma) \geq$ card $(\beta N) = 2^c$.

To see Γ is closed, let $y \in \beta D - \Gamma$. Then there is some set $S \in \mathfrak{F}$ such that $S \notin y$. But then $\mathrm{cl}_{\beta D}(D - S)$ is the neighborhood of y which we need.

To see that Γ is infinite, recall that $\aleph_1$ is nonmeasurable. Since $\mathfrak{F}$ has the countable intersection property and $\bigcap \mathfrak{F} = \emptyset$, $\mathfrak{F}$ must fail to be an ultrafilter. Thus there is a set $C_1 \subset D$ such that both $\mathfrak{F} \cup \{C_1\}$ and $\mathfrak{F} \cup \{D - C_1\}$ have the finite intersection property. In fact, both $\mathfrak{F} \cup \{C_1\}$ and $\mathfrak{F} \cup \{D - C_1\}$ have the countable intersection property and hence we can repeat the process inductively to find $2^{\aleph_0}$ distinct filterbases containing $\mathfrak{F}$. By the construction, if p is any ultrafilter, then p can contain at most one filterbase from this set. Since each filterbase can be extended to an ultrafilter, the proposition is proved.

It is interesting to note that up to this point we have made no use of Lusin's hypothesis.

To complete our construction, we will choose the two subsets A_1 and A_2 by transfinite induction. In order to do this we must know that there are no more than card (Γ) strong filter bases on D.

Let $\mathcal{S}$ denote the set of strong filter bases on D. Since $\mathcal{S}$ is a subset of $\mathcal{P}(\mathcal{P}(D))$, we have card $(\mathcal{S}) \leq 2^{2^{\aleph_1}}$ (it is easy to see that equality must hold). If we assume Lusin's hypothesis, then card $(\Gamma) \geq 2^c = 2^{2^{\aleph_1}} \geq$ card $(\mathcal{S})$. We can now prove the following theorem:

3.2 **Theorem.** *Assuming that Lusin's hypothesis holds, then weakly-$\aleph_1$-compactness is not preserved by finite products.*

Proof. We shall use the condition that card $(\Gamma) \geq$ card $(\mathcal{S})$ to choose the families A_1 and A_2 from $\beta D - D$.

Let $\{\mathfrak{F}_\gamma : \gamma < 2^c\}$ be a well ordering of the strong filter bases on D. Let a_0 and b_0 be distinct ultrafilters which contain $\mathfrak{F}_0$.

Inductively, let γ be an ordinal less than 2^c, and suppose that for all $\tau < \gamma$ we have chosen distinct ultrafilters a_τ and b_τ containing $\mathfrak{F}_\tau$. Furthermore, suppose that the ultrafilters were chosen to be distinct from all those previously chosen. Since each strong filter base is contained in 2^c ultrafilters, we can choose $a_\gamma \neq b_\gamma$ to be distinct from all those previously chosen.

To complete the construction we simply set $A_1 = \{a_\gamma : \gamma < 2^c\}$ and $A_2 = \{b_\gamma : \gamma < 2^c\}$.

This establishes that Theorem 1.3 cannot be improved to say that weakly-$\aleph$-compactness is determined by the individual coordinate spaces.

The following implications are well known: The $\aleph$-chain condition $\Rightarrow$ weakly-$\aleph$-compactness $\Rightarrow$ pseudo-$\aleph$-compactness. It has been shown in [NU, Corollaries 1.4 and 1.5] that, for $\aleph$ regular and uncountable, the $\aleph$-chain condition and pseudo-$\aleph$-compactness are determined by finite subproducts. Interestingly, the proof of Theorem 1.3 can be easily adapted to yield the results for the $\aleph$-chain condition.

The situation with finite products is not similar. [NU, Example 1.6] shows that pseudo-$\aleph$-compactness is not preserved by finite products, while the analogous question for the $\aleph_1$-chain condition (i.e. the countable chain condition) has been shown to be independent of the usual axioms of set theory. For a discussion of the latter problem see [C]. From this we should realize that it is by no means certain that the answer to Problem 2.4 is affirmative.

REFERENCES

[C_1] P. J. Cohen, *The independence of the continuum hypothesis*, Proc. Nat. Acad. Sci. U.S.A. 50 (1963), 1143–1148; ibid. 51 (1964), 105–110. MR 28 #2962.

[C_2] W. W. Comfort, *A nonpseudocompact product space where finite subproducts are pseudocompact*, Math. Ann. 170 (1967), 41–44. MR 35 #965.

[C_3] ———, *A survey of cardinal invariants*. General Topology and Appl. 1 (1971), 163–199.

[CHN] W. W. Comfort, N. Hindman and S. Negrepontis, *F′-spaces and their products with P-spaces*, Pacific J. Math 28 (1969), 489–502. MR 39 #3440.

[F_1] Z. Frolík, *Generalisations of compact and Lindelöf spaces*, Czechoslovak Math. J. 9 (84) (1959), 172–217. (Russian) MR 21 #3821.

[F_2] ———, *Sums of ultrafilters*, Bull. Amer. Math. Soc.73 (1967), 87–91. MR 34 #3525.

[F_3] ———, *On two problems of W. W. Comfort*, Comment. Math. Univ. Carolinae 8 (1967), 139–144. MR 35 #966.

[H] A. W. Hager, *Projections of zero-sets* (*and the fine uniformity on a product*), Trans. Amer. Math. Soc. 140 (1969), 87–94. MR 39 #3448.

[NU] N. Noble and M. Ulmer, *Factoring functions on Cartesian products*, Trans. Amer. Math. Soc. 163 (1972), 329–339.

[R_1] M. E. Rudin, *Types of ultrafilters*, Topology Seminar (Wisconsin, 1965), Ann. of Math. Studies, no. 60, Princeton Univ. Press, Princeton, N. J., 1966, pp. 147–151. MR 35 #7284.

[R_2] W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. 23 (1956), 409–419. MR 18 #324.

[U_1] M. Ulmer, *Continuous functions on product spaces*, Doctoral Dissertation, Wesleyan University, Middletown, Conn., 1970.

[U_2] ———, *The countable chain condition*, Notices Amer. Math. Soc. 17 (1970), 462–463. Abstract #70T-G24.

DEPARTMENT OF MATHEMATICS, MACALESTER COLLEGE, ST. PAUL, MINNESOTA 55105