IRREDUCIBLE REPRESENTATIONS OF THE 
C*-ALGEBRA GENERATED BY AN n-NORMAL OPERATOR

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ABSTRACT. For a n-normal operator on Hilbert space, we determine the irreducible representations of C*(A), the C*-algebra generated by A and the identity. For a binormal operator, we determine an explicit description of the topology on the space of unitary equivalence classes of irreducible representations of C*(A).

1. Introduction and preliminaries. For a bounded linear operator on a Hilbert space H, let C*(A) denote the C*-algebra generated by A and 1. The set of unitary equivalence classes of irreducible representations of a C*-algebra equipped with the hull-kernel topology is called the spectrum of the C*-algebra [2, paragraph 3]. If A is a normal operator, then the spectrum of C*(A) is merely the spectrum of the operator A. For noncommutative C*-algebras very few examples of the spectrum of the algebra have been calculated other than the paper of Fell [3]. In this paper we determine the irreducible representations of the C*-algebra generated by an n-normal operator and explicitly calculate the spectrum of the C*-algebra generated by a binormal operator. To calculate the topology on the spectrum we use the methods that Fell used to calculate the topology on the duals of the complex unimodular groups [3].

A W*-algebra R is said to be n-normal [5] if it satisfies the identity

$$\sum \text{sgn } \sigma A_{\sigma(1)}A_{\sigma(2)} \cdots A_{\sigma(2n)} = 0$$

where A_1, A_2, \ldots, A_{2n} are arbitrary elements of R and the summation is taken over all permutations \sigma of (1, 2, 3, \ldots, 2n). A bounded linear operator A on a Hilbert space H is called n-normal if the W*-algebra generated by A is n-normal. A 2-normal operator is also called binormal [1]. If A is an n-normal operator and \pi is an irreducible representation of C*(A) on a Hilbert space H_0, then the standard identity

$$\sum \text{sgn } \sigma X_{\sigma(1)}X_{\sigma(2)} \cdots X_{\sigma(2n)} = 0$$

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is satisfied on $C^*(\pi(A))$, and since $\pi$ is irreducible, the standard identity is also satisfied on $B(H_0)$, the set of all bounded linear operators on $H_0$. Hence the dimension of $H_0$ is less than or equal to $n$. We note that a representation $\pi$ of $C^*(A)$ is completely determined by the value of $\pi$ at $A$.

For $1 \leq k \leq n$ let $\mathbf{C}_k$ be a commutative $W^*$-algebra on a Hilbert space $H_k$, and let $M_k(\mathbf{C}_k)$ denote the $W^*$-algebra on $H_k \oplus \cdots \oplus H_k$ (k-times) consisting of $k \times k$ matrices with elements from $\mathbf{C}_k$. Then $M_k(\mathbf{C}_k)$ is n-normal, and $M_1(\mathbf{C}_1) \oplus M_2(\mathbf{C}_2) \oplus \cdots \oplus M_n(\mathbf{C}_n)$ is also $n$-normal. In fact, any $n$-normal $W^*$-algebra is unitarily equivalent to one of this form [4]. Thus any $n$-normal operator $A$ is unitarily equivalent to an operator of the form $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ where $A_k$ is a $k \times k$ matrix whose entries generate a commutative $W^*$-algebra; that is, whose entries are commuting normal operators. Such an $A_k$ is called a $k$-homogeneous $n$-normal operator.

Recall [4,3.1.13] that $(\beta_1, \beta_2, \ldots, \beta_1) \in \sigma(B_1, B_2, \ldots, B_l)$ is the joint spectrum of commuting normal operators $B_1, B_2, \ldots, B_l$ if and only if there is a character $\omega$ (i.e. multiplicative linear functional) on the $C^*$-algebra, $C^*((B_i)_{i=1}^l)$, generated by $B_1, B_2, \ldots, B_l$ and 1, such that $\omega(1) = 1$, $\omega(B_i) = \beta_i$ for $i = 1, 2, \ldots, l$. We will show that the spectrum of the $C^*$-algebra generated by a homogeneous $n$-normal operator is closely related to the joint spectrum of its matrix elements.

2. Irreducible representations of $n$-normal operators. If $\mathbf{C}$ is a commutative $W^*$-algebra, then there is a natural $n$-dimensional irreducible representation of $M_n(\mathbf{C})$ defined in the following manner: For $\rho$ a nonzero character on $\mathbf{C}$ define $\hat{\rho}$ on $M_n(\mathbf{C})$ by

$$\hat{\rho}(C_{ij}) = (\rho(C_{ij})).$$

Then $\hat{\rho}$ is obviously an irreducible representation of $M_n(\mathbf{C})$ in the $n \times n$ scalar matrices. The following well-known result states that every irreducible representation is of this form.

**Proposition 1** (see [7, p. 114]). If $\mathbf{C}$ is a commutative $W^*$-algebra and $\pi$ is an irreducible representation of $M_n(\mathbf{C})$ on a Hilbert space $H$, then the dimension of $H$ is $n$, and there exists a nonzero character $\rho$ on $\mathbf{C}$ such that $\pi$ is unitarily equivalent to $\hat{\rho}$.

**Proposition 2.** Let $A = (A_{ij})$ be a homogeneous $n$-normal operator, where $\{A_{ij}\}_{i,j=1}^n$ are commuting normal operators. Suppose $\pi$ is an irreducible representation of $C^*(A)$ on a Hilbert space $H$, whose dimension $k$ is necessarily less than or equal to $n$. Then there exists a nonzero character $\rho$ on $C^*(\{A_{ij}\})$ and a $k$-dimensional reducing subspace $M$ for $\hat{\rho}(A)$ such that $\pi(A)$ is unitarily equivalent to $\hat{\rho}(A)|_M$. Conversely every such character and reducing subspace
produces a representation of \( C^*(A) \), which is however not necessarily irreducible.

**Proof.** Let \( \hat{C} = C^*([A_{ij}]_{i,j=1}^{n}) \) and suppose \( \pi \) is an irreducible representation of \( C^*(A) \) on \( H \). By Proposition 2.10.2 in [2], \( \pi \) can be extended to an irreducible representation \( \hat{\pi} \) of \( M_n(\hat{C}) \) on \( K \), where \( K \supset H \), \( H \) reduces \( \hat{\pi}(C^*(A)) \), and the dimension of \( K \) is necessarily \( n \). But now Proposition 1 implies that there is a nonzero character \( \rho \) on \( \hat{C} \) such that \( \hat{\pi} \) is unitarily equivalent to \( \hat{\rho} \). Since \( H \) reduces \( \hat{\pi}(A) \) there is a \( k \)-dimensional reducing subspace \( M \) for \( \hat{\rho}(A) \) such that \( \hat{\pi}(A) = \hat{\pi}(A)|_M \) is unitarily equivalent to \( \hat{\rho}(A)|_M \). The converse is clear.

**Proposition 3.** Let \( A = A_1 \oplus A_2 \oplus \cdots \oplus A_n \) be an \( n \)-normal operator with \( A_k \) a \( k \)-homogeneous \( n \)-normal operator of the form \( (A_k)^i \) for \( 1 \leq k \leq n \), where \( [A_{ij}]_{i,j=1}^{k} \) are commuting normal operators. Suppose \( \pi \) is an irreducible representation of \( C^*(A) \) on \( H \), whose dimension is \( h \). Then there exists an integer \( k \), \( b \leq k \leq n \), and a nonzero character \( \rho_k \) on \( C^*([A_{ij}]_{i,j=1}^{k}) \) and a \( b \)-dimensional reducing subspace \( M_k \) for \( \hat{\rho}_k(A_k) \) such that \( \pi(A) \) is unitarily equivalent to \( \hat{\rho}_k(A_k)|_{M_k} \). Conversely, any such integer, nonzero character, and reducing subspace produces a \( b \)-dimensional representation of \( C^*(A) \), which is however not necessarily irreducible.

**Proof.** Let \( \hat{C}_k = C^*([A_{ij}]_{i,j=1}^{k}) \) and suppose \( \pi \) is an irreducible representation of \( C^*(A) \) on \( H \). Again by Proposition 2.10.2 in [2], \( \pi \) can be extended to an irreducible representation \( \hat{\pi} \) of \( M_1(\hat{C}_1) \oplus M_2(\hat{C}_2) \oplus \cdots \oplus M_n(\hat{C}_n) \) on \( K \), where \( K \supset H \), \( H \) reduces \( \hat{\pi}(C^*(A)) \). Each \( E_i = \hat{\pi}(0 \oplus \cdots 0 \oplus I_i \oplus 0 \cdots 0) \), for \( 0 \leq i \leq n \), is a projection in the commutant of the image of \( \hat{\pi} \), so that \( E_i \) is either 0 or \( I_i \), since \( \hat{\pi} \) is irreducible. Hence there exists an integer \( k \), \( b \leq k \leq n \), such that \( \hat{\pi}(C_1 \oplus C_2 \oplus \cdots \oplus C_n) = 0 \) whenever \( C_k = 0 \). Thus \( \hat{\pi} \) can be considered as a \( k \)-dimensional irreducible representation of \( M_k(\hat{C}_k) \) which extends \( \pi \). Thus \( \pi \) can be considered as a \( b \)-dimensional irreducible representation of \( C^*(A_k) \). Hence Proposition 2 implies that there exists a nonzero character \( \rho_k \) on \( \hat{C}_k \) and a \( b \)-dimensional reducing subspace \( M_k \) for \( \hat{\rho}_k(A_k) \) such that \( \pi(A) \) is unitarily equivalent to \( \hat{\rho}_k(A_k)|_{M_k} \). Again the converse is clear.

**Remark.** Although the previous two propositions characterize all the irreducible representations of the \( C^*- \)algebra generated by a homogeneous or general \( n \)-normal operator, they do not give us a description of the set of unitary equivalence classes of irreducible representations. For it is quite difficult to determine whether a given \( n \times n \) scalar matrix is irreducible, and it is also quite difficult to determine when two \( n \times n \) matrices are unitarily equivalent.

3. The spectrum of the \( C^*- \)algebra generated by a binormal operator. In this section we are able to give a complete description of the spectrum of a \( C^*- \)algebra generated by a binormal operator \( A \), using Brown's characterization [1] of
binormal operators. It turns out that the spectrum of $C^*(A)$ given its hull-kernel topology is homeomorphic to the quotient of a set in $C^3$, related to the joint spectrum of the matrix elements of $A$, modulo an equivalence relation. We remark that the spectrum need not be Hausdorff.

We begin by recalling a result from [1]. Let $\text{Tri}(X, Y, Z)$ denote the triangular $2 \times 2$ matrix $(A_{ij})$ where $A_{11} = X$, $A_{22} = Z$, $A_{12} = Y$, and $A_{21} = 0$. Then Brown proves that every binormal operator is unitarily equivalent to an operator of the form $B \oplus \text{Tri}(X, Y, Z)$ where $B$ is normal, $X, Y, Z$ are commuting normal operators, and $Y$ is positive and one-to-one.

Proposition 4. Let $A = B \oplus \text{Tri}(X, Y, Z)$ be a binormal operator, with $Y \geq 0$. Then

(i) If $\pi$ is a two-dimensional irreducible representation of $C^*(A)$ on $H$ then $\pi(A)$ is unitarily equivalent to $\text{Tri}(\alpha, \beta, \gamma)$ where $(\alpha, \beta, \gamma) \in \sigma(X, Y, Z)$, and $\beta > 0$. Conversely, every such triple gives rise to a two-dimensional irreducible representation in this manner.

(ii) If $\pi$ is a nonzero character on $C^*(A)$ then $\pi(A) = X$ where $X \in \sigma(B)$ or there exists a $\mu \in \mathbb{C}$ such that either $(\lambda, 0, \mu) \in \sigma(X, Y, Z)$ or $(\mu, 0, \lambda) \in \sigma(X, Y, Z)$. Conversely, every such $\lambda$ gives rise to a character in this manner.

Of course, every irreducible representation of $C^*(A)$ for $A$ binormal has dimension less than or equal to 2.

Proof. Follows immediately from Proposition 3.

Now let

$$S_0 = \{(\alpha, \beta, \gamma): \text{either} \ (\alpha, \beta, \gamma) \in \sigma(X, Y, Z) \text{ or } (\gamma, \beta, \alpha) \in \sigma(X, Y, Z)\}$$

and let

$$S = \sigma(B) \cup S_0.$$ 

Then define an equivalence relation $\sim$ on $S$ by saying $s_1 \sim s_2$ if and only if one of the following four conditions is satisfied: (1) $s_1 = s_2$, (2) $s_1 = (\alpha, \beta, \gamma)$ and $s_2 = (\gamma, \beta, \alpha)$, (3) $s_1 = (\alpha, 0, \gamma)$ and $s_2 = (\alpha, 0, \beta)$, or (4) $s_1 = \alpha \in \sigma(B)$ and $s_2 = (\alpha, 0, \gamma)$ or vice versa. We use the set $S_0$ instead of the set $\sigma(X, Y, Z)$ for two reasons. The first reason is that an element of the form $(\alpha, 0, \gamma) \in \sigma(X, Y, Z)$ yields two characters on $C^*(A)$ and the second is because the final topologies would not agree otherwise.

Let $X$ denote the set of unitary equivalence classes of irreducible representations of $C^*(A)$. We define a map $\theta: S \to X$ as follows: If $s = (\alpha, \beta, \gamma)$ with $\beta > 0$ and $(\alpha, \beta, \gamma) \in \sigma(X, Y, Z)$, let $\theta(s)$ be the two-dimensional irreducible representation given in Proposition 4(i) by $(\theta(s))(A) = \text{Tri}(\alpha, \beta, \gamma)$. If $s = (\alpha, \beta, \gamma)$ with $\beta > 0$ and $(\gamma, \beta, \alpha) \in \sigma(X, Y, Z)$, let $\theta(s)$ be the two-dimensional irreducible representation given by $(\theta(s))(A) = \text{Tri}(\gamma, \beta, \alpha)$. Notice that if $\beta > 0$ and both $(\alpha, \beta, \gamma)$ and $(\gamma, \beta, \alpha)$ are in $\sigma(X, Y, Z)$ then this does give a single
valued definition of $\theta(s)$ since $\text{Tri}(\alpha, \beta, \gamma)$ and $\text{Tri}(\gamma, \beta, \alpha)$ are unitarily equivalent. If $\alpha \in \sigma(B)$ let $\theta(\alpha)$ be the character on $C^*(A)$ given in Proposition 4(ii) by $(\theta(\alpha))(A) = \alpha$. If $s = (\alpha, 0, \beta) \in S_0$ let $\theta(s)$ be the character on $C^*(A)$ given in Proposition 4(ii) by $(\theta(s))(A) = \alpha$. Since $\text{Tri}(\alpha, \beta, \gamma)$ and $\text{Tri}(\alpha', \beta, \gamma')$ are unitarily equivalent if and only if $\{\alpha, \gamma\} = \{\alpha', \gamma'\}$ we have that $s_1 \sim s_2$ if and only if $\theta(s_1) = \theta(s_2)$. Thus $\theta$ induces a one-to-one mapping $\theta_0$ from $S/\sim$ to $X$, which is onto by Proposition 4.

Give $S_0$ the topology it inherits from $C^3$ and $\sigma(B)$ its natural topology. Let $S$ have the disjoint union topology, and let $S/\sim$ have its quotient topology. Finally let $X$ have its hull-kernel topology. Our goal is to show that $\theta_0$ is a homeomorphism of $S/\sim$ onto $X$. Recall [2, 3.3.3] that if $\{D_i\}$ is a dense subset of $C^*(A)$, then a base for the hull-kernel topology on $X$ is given by the sets $U_\epsilon = \{n \in X : \|\theta(D_i)\| > \epsilon\}$.

Proposition 5. The map $\theta : S \to X$ is continuous.

Proof. Consider the dense set in $C^*(A)$ consisting of operators of the form $D = p(A, A^*)$ where $p$ is a polynomial in two noncommuting variables. Suppose that $s_n \in S$ converges to $s \in S$. We need to show that $\theta(s_n)$ converges to $\theta(s)$ in $X$. There are three cases to be considered.

First assume that $s \in \sigma(B)$. So we may assume $s_n \in \sigma(B)$ for all $n$. Then $\rho_n = \theta(s_n)$ and $\rho = \theta(s)$ are characters on $C^*(A)$ such that $\rho_n(A) = s_n$ and $\rho(A) = s$. Thus $\|\rho_n(D)\| = \|p(\rho_n(A), \rho_n(A^*))\| = \|p(s_n, s_n^*)\|$ converges to $\|p(s, \overline{s})\| = \|\rho(D)\|$ so that $\rho_n$ converges to $\rho$ in $X$.

Second assume that $s = (\alpha, \beta, \gamma) \in S_0$ with $\beta > 0$. Then we may assume $s_n = (\alpha, \beta_n, \gamma_n) \in S_0$ with $\beta_n > 0$ for all $n$. By Proposition 4(i) there exist unitary operators $U_n$ and $V$ such that $V(\theta(S))(A)V^* = \text{Tri}(\alpha, \beta, \gamma)$ and $U_n(\theta(s_n)(A))U_n^* = \text{Tri}(\alpha, \beta_n, \gamma_n)$. Hence $U_n(\theta(s_n)(D))U_n^* = U_n(p(\theta(s_n)(A)))$, $\theta(s_n)(A^*)U_n = p(\text{Tri}(\alpha, \beta_n, \gamma_n))$, $\text{Tri}(\alpha, \beta_n, \gamma_n)^*$ which converges to $p(\text{Tri}(\alpha, \beta, \gamma), \text{Tri}(\alpha, \beta, \gamma)^*) = V(\theta(s)(D))V^*$. Thus $\|\theta(s_n)(D)\|$ converges to $\|\theta(s)(D)\|$, so that $\theta(s_n)$ converges to $\theta(s)$ in $X$.

Lastly assume that $s = (\alpha, 0, \gamma) \in S_0$. Then we may assume $s_n = (\alpha, \beta_n, \gamma_n) \in S_0$ for all $n$. Let $N_1$ be the set of integers $n$ such that $\beta_n \neq 0$, and let $N_2$ be the set of integers $n$ such that $\beta_n = 0$. For $n \in N_2$, $\theta(s_n)(D) = p(\alpha, \overline{\alpha})$. So that if $N_2$ is infinite, $\|\theta(s_n)(D)\| = \|p(\alpha, \overline{\alpha})\| \converges to \|p(\alpha, \overline{\alpha})\|$. If $n \in N_1$ then $\theta(s_n)(A)$ is unitarily equivalent to $\text{Tri}(\alpha, \beta_n, \gamma_n)$. So that if $N_1$ is infinite, $\|\theta(s_n)(D)\| = \|p(\text{Tri}(\alpha, \beta_n, \gamma_n), \text{Tri}(\alpha, \beta_n, \gamma_n)^*)\|$ converges to $\|p(\text{Tri}(\alpha, 0, \gamma), \text{Tri}(\alpha, 0, \gamma)^*)\|$, which is greater than or equal to $\|p(\alpha, \overline{\alpha})\| = \|\theta(s)(D)\|$. So that, if $\|\theta(s)(D)\| > 1$, there exists an integer $N$ such that, for all $n \in N_1 \cup N_2$, $n \geq N$, $\|\theta(s_n)(D)\| > 1$.

Proposition 6. The mapping $\theta_0$ is a homeomorphism of $S/\sim$ onto $X$. 
Proof. Because of Proposition 5 and the remarks preceding it, we need only show that \( \theta_0 \) is a closed mapping. Since \( S \) is compact, \( S/\sim \) is also compact. Let \( F \) be a closed, hence compact, subset of \( S/\sim \). Since \( \theta_0 \) is continuous, \( \theta_0(F) \) must be compact. Let \( \{ p_n \} \) be a sequence in \( \theta_0(F) \) converging to \( p_0 \in X \). We need to show that \( p_0 \in \theta_0(F) \). Since \( X \) is second countable and \( \theta_0(F) \) is compact, there is a subsequence \( \{ p_{n_k} \} \) which converges to \( p_1 \in \theta_0(F) \). Hence we may assume \( \{ p_n \} \) converges to \( p_0 \) and \( p_1 \). If \( p_0 = p_1 \), we are done. Assume \( p_0 \neq p_1 \). By Corollary 1 in [3, p. 388], we have that dim \( p_0 + \text{dim } p_1 \leq 2 \), thus dim \( p_0 \) = dim \( p_1 \), and \( p_0, p_1 \) are the only limit points of \( \{ p_n \} \). Let \( J = \bigcap \{ q^{-1}(0) : q \) is a character on \( C^*(A) \} \). Then \( J \neq \emptyset \), \( C^*(A)/J \) is commutative and the spectrum of \( C^*(A)/J \) is just the space of characters of \( C^*(A) \) [2, 3.6.3 and 3.2.1], and hence is Hausdorff. Then since \( p_n \) converges to \( p_0 \), \( p_1 \) we must have that dim \( p_n = 2 \) for large \( n \). Hence we can assume dim \( p_n = 2 \) for all \( n \). Now let \( q \) be the quotient map of \( S \) onto \( S/\sim \). Then \( p_n = q(s_n) \) where \( s_n \in q^{-1}(F) \) and \( s_n = (\alpha_n, \beta_n, \gamma_n) \in \sigma(X, Y, Z) \) with \( \beta_n > 0 \). Suppose that \( \pi_0(A) = \alpha_0 \) and \( \pi_1(A) = \alpha_1 \), since \( \sigma(X, Y, Z) \) is compact [4], there is a subsequence \( s_{n_k} = (\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) \) which converges, say to \( (\alpha, \beta, \gamma) \). Since \( \theta \) is continuous, \( \pi_{n_k} \) converges to \( \theta(\alpha, \beta, \gamma) \). But \( \pi_0 \) and \( \pi_1 \) are the only limit points of \( \pi_n \), thus \( \beta = 0 \) and \( \alpha = \alpha_0 \) or \( \alpha_1 \). Thus \( \alpha_{n_k} \) converges to either \( \alpha_0 \) or \( \alpha_1 \). If \( \alpha_{n_k} \) converges to \( \alpha_0 \), then \( (\alpha_0, 0, \gamma) \in \sigma(X, Y, Z) \) and \( (\alpha_0, 0, \gamma) \in q^{-1}(F) \) since \( q^{-1}(F) \) is closed. Thus \( \pi_0 \in \theta_0(F) \) and we are done. On the other hand, if \( \alpha_{n_k} \) converges to \( \alpha_1 \), then \( (\gamma_{n_k}, \beta_{n_k}, \alpha_{n_k}) \in S \) and converges to \( (\gamma, 0, \alpha_1) \). Since \( \theta \) is continuous, \( \pi_{n_k} \) converges to \( \theta(\gamma, 0, \alpha_1) \). Hence \( \gamma = \alpha_1 \) or \( \alpha_0 \). If \( \gamma = \alpha_0 \), then we would have \( \pi_0 \in \theta_0(F) \) since \( s_{n_k} \in q^{-1}(F) \) and we would be done. Finally, suppose \( \gamma = \alpha_1 \). Then \( (\gamma_{n_k}, \beta_{n_k}, \alpha_{n_k}) \) converges to \( (\alpha_1, 0, \alpha_1) \). Now by the lower semicontinuity of the map \( \pi \) to \( \| \pi(D) \| \) for \( D \in C^*(A) \) [2, 3.3.2], since \( p_n \) converges to \( p_0 \), we have that

\[
\| \pi_0(A - \alpha_1) \| \leq \lim \inf \| p_n(A - \alpha_1) \|
\]

\[
\leq \lim \inf \| \text{Tri} (\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) - \text{Tri}(\alpha_1, 0, \alpha_1) \| = 0,
\]

thus \( \pi_0(A) = \alpha_0 = \alpha_1 \) and \( \pi_0 \in \theta_0(F) \).

Examples and remarks. (1) Proposition 6 can be used to give a variety of examples of non-Hausdorff spaces that are the spectrums of singly generated \( C^* \)-algebras. For example let \( H \) be \( L^2[-1, 1] \) and let \( X \) be multiplication by the function \( g(t) = t \), \( Z = -X \), and \( Y \) be multiplication by a nonnegative continuous bounded function \( f \) such that \( f^{-1}(0) \) is nonempty. Then \( A = \text{Tri}(X, Y, Z) \) is a homogenous binormal operator, in fact \( A^2 \) is normal [6]. Also, \( \sigma(X, Y, Z) = \{ (t, f(t)), -1 \leq t \leq 1 \} \cup \{ (-t, f(t)), -1 \leq t \leq 1 \} \).

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algebra $C^*(A)$ is $T_0/\sim$ and is simply obtained by taking a quotient space of a subset of the complex plane. By choosing particular functions $f$, a number of examples can be obtained.

(2) We recall that if $\mathcal{A}$ is a $C^*$-algebra and $A \in \mathcal{A}$, then the mapping that sends $\pi \in X$ to $\|\pi(A)\|$ is lower semicontinuous. That is, if $\pi_n$ converges to $\pi$ then $\|\pi(A)\| \leq \lim \inf \|\pi_n(A)\|$. If $A$ is binormal and $\mathcal{A} = C^*(A)$, then the proof of Proposition 5 shows that unless $\pi$ is the image of an element of the form $(\alpha, 0, \gamma) \in S_0$ we actually have $\|\pi(D)\| = \lim \|\pi_n(D)\|$ for all $D \in C^*(A)$. If $\pi$ is the image of an element of the form $(\alpha, 0, \gamma)$, then since $|\alpha|$ may be strictly less than $|\gamma|$ we may have $\|\pi(D)\| < \lim \inf \|\pi_n(D)\|$. 

Added in revision. Carl Pearcy has kindly informed us that this paper is related to a paper of Harry Gonshor [Canad. J. Math. 10 (1958), 97–102]. For $A$ a binormal operator, Gonshor used direct integral theory to determine what Fell [Acta. Math. 26 (1961), 233–280] later called the Hausdorff compactification $Q$ of the spectrum of $C^*(A)$ and characterized those continuous functions on $Q$ that come from elements of $C^*(A)$.

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