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BOUNDARY LINKS AND AN UNLINKING THEOREM(1)

BY

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ABSTRACT. This paper gives a homotopic theoretic criterion for a higher dimensional link to be trivial.

An m-link is an embedding of m disjoint copies of the n-sphere into the $(n+2)$-space. There are various equivalence relations amongst links such as isotopy and cobordism. Some results and definitions are found in [21], [7] and [9].

Generally, it is useful to compare, via our equivalence relations, any link to the standard trivial link. Any link isotopic to it is called trivial; if the link is cobordant to the trivial one, we say it is a slice.

An interesting concept, weaker than triviality, is that of boundary link: an m-link is trivial if the embedding extends to one of m disjoint copies of the $(n+1)$-disk; an m-link is boundary if it extends to an embedding of m compact $(n+1)$-manifolds with boundary the sphere. These are called Seifert manifolds.

The purpose of this paper is to give some homotopic conditions for a link to be (i) boundary, (ii) trivial. These conditions are reflected on the homotopy type of X, the complement of the image of the link in the ambient space, as follows:

(i) An m-link is boundary if, and only if, there is an epimorphism from $\pi_1(X)$ onto the free group in m generators which sends meridians to generators.

(ii) Let $\mathfrak{L}$ be an m-link of dimension ≥ 4 and $\bigvee^m S^1$, the wedge of m circles; suppose

$$\pi_i(X) = \pi_i\left(\bigvee^m S^1\right) \quad \text{for } i \leq q \leq \tfrac{1}{2}(n+1)$$

and that for $i = 1$ meridians are sent on to generators.

Then $\mathfrak{L}$ is a boundary link where the Seifert manifolds can be chosen to be $(q-1)$-connected. In particular for $q = \frac{1}{2}(n+1)$, $\mathfrak{L}$ is trivial.

As by-products, we obtain some results about cobordism of links which are presented in §3.

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0. **Notation.** In this paper, all manifolds, mappings and isotopies belong to the smooth or PL, locally flat, categories, consistently. All manifolds are oriented and if M_i $(i = 1, \cdots, m)$ are manifolds, $\Sigma_{i=1}^m M_i$ stands for the disjoint union of them. As usual S^n and D^{n+1} are the n-sphere and the $(n+1)$-disk. In particular, mS^n is a disjoint union of m copies of S^n.

An m-link is an embedding $\mathfrak{L}$: $mS^n \longrightarrow S^{n+2}$ $(m, s \geq 1)$; we say that $\mathfrak{L}$ is boundary if it extends to an embedding $\Sigma_{i=1}^m V_i \longrightarrow S^{n+2}$, where V_i is a compact, orientable manifold with boundary S_i^n. The collection $\{V_i\}$ is called a collection of Seifert manifolds for $\mathfrak{L}$. In particular, if the V_i are disks, we say that $\mathfrak{L}$ is trivial.

Consider $\mathfrak{L}_i$: $S_i^n \longrightarrow S^{n+2}$, the restriction of $\mathfrak{L}$ to S_i^n $(i = 1, \cdots, m)$; we can find tubular neighborhoods T_i of $\mathrm{Im}(\mathfrak{L}_i) = L_i$, which are mutually disjoint [20]. Let X be $S^{n+2} - \bigcup T_i$, a compact manifold with boundary $m(S^1 \times S^m)$ and of the same homotopy type as $S^{n+2} - \mathrm{Im}(\mathfrak{L})$.

Consider arcs in X joining the $S^1 \times S^n$ to a common basepoint; the union of these arcs and X is called A and, for $n \geq 2$, its fundamental group is free in m generators called meridians. The group is denoted by $F(m)$ or by $F[a_1, \cdots, a_m]$ when the generators are specified.

Finally, if G is any group, G_n indicates the nth member of the lower central series [18]; G_ω is by definition, $\bigcap G_i$. In particular, if $F = F(m)$, $F_\omega = 1$ [10].

1. **Ambient surgery.** Let $\mathfrak{L}$: $mS^n \longrightarrow S^{n+2}$ be a boundary link and $\{V_i\}$ a collection of Seifert manifolds for it. Let f_i: $V_i \longrightarrow I$ be a map with $f_i^{-1}(0) = \partial V_i \cong S^n$ and $W_i = \{(x, t) \in V_i \times I |\ f_i(x) \leq t\}$. With the aid of the (trivial) normal bundle of the V_i, we can find embeddings ψ_i: $W_i \longrightarrow S^{n+2}$; notice that $\partial W_i = V_{i0} \cup V_{i1}$, where $V_{it} \cong V_i$ and the two copies are attached by the boundary. The manifold

$$Y = \overline{S^{n+2} - \bigcup \psi_i(W_i)}$$

is called S^{n+2} cut along the V_i, a compact manifold with boundary $\Sigma_{i=1}^m (V_{i0} \cup V_{i1})$. The composition $V_i \cong V_{it} \subset Y$ is called ν_{it}. See [20].

Let ϕ: $S^k \times D^{n+1-k} \longrightarrow V_i$ be an embedding; define $\theta_t(V_i, \phi)$ as $W_i \cup D^{k+1} \times D^{n+1-k}$ where the handle is attached to V_{it} by ϕ; $\partial\theta_t(V_i, \phi)$ is equal to $V_{is} \cup \chi(V_i, \phi)$, $s = 1 - t$.

Suppose that for some $k \geq 1$, the V_i are k-connected. By Alexander duality $H_q(Y; \mathbf{Z}) = \Sigma H_q(V_i; \mathbf{Z})$, $q \leq k - 1$; let $Y_\alpha = S^{n+2} - \mathrm{Int} \bigcup_{j \leq \alpha} W_j$, then by [7, §4], Y_1 is 1-connected. If Y_r is 1-connected, then by the van Kampen theorem we have

$$\begin{tikzcd}[column sep=small] & \pi_1(Y_{r+1}) \arrow[dr] & \\ 0 = \pi_1(\partial W_{r+1}) \arrow[ur] \arrow[dr] & & \pi_1(Y_r) = 0 \\ & \pi_1(W_{r+1}) \arrow[ur] \arrow[d, equal] & \\ & 0 & \end{tikzcd}$$

therefore Y_{r+1} is 1-connected and, by induction, so is Y. By the Hurewicz theorem, Y is then $(k-1)$-connected.

Lemma (1). *If* (i) $n \geq 2k+1$ *or* (ii) $n = 2k$ *or* $2k-1$, $n \leq 4$, *and* $\alpha \in \pi_k(V_i)$ *is in* $\ker \nu_{it*}$, *the embedding* $\psi_i: W_i \to Y \cup W_i$ *extends to* $\theta_t(V_i, \phi)$, *where* ϕ *is an embedding representing* α.

Proof. We refer to the proof of Lemma (3) of [7]. Let $\alpha' \in \pi_k(V_i)$ corresponding to α under $V_i \cong V_{it} \subset \partial Y$, since $\nu_{it*}\alpha = 0$, β' is the boundary of an element β' of $\pi_{k+1}(Y, W_i)$. This latter pair is k-connected, hence by (1) of [6], there is an embedding $g: D^{k+1} \to Y$ representing β' which is transversal to the boundary. Since V_{it} is 1-connected, g can be isotoped by (2) of [6] and (3) of [7] so that $g(S^k)$ represents α. By using a tubular neighborhood of $\mathrm{Im}(g)$ we get the desired extension.

2. **The Fundamental group.** Let $\pi = \pi_1(X)$; in [4] it is proven that $H_1(\pi) = \mathbf{Z}^m$ and $H_2(\pi) = 0$, where $H_q(\pi)$ is the qth homology group of π with trivial integral coefficients. The inclusion $A \subset X$ induces a homomorphism $i: F(m) \to \pi$ of fundamental groups.

Lemma (2). *For* $n \geq 2$, i *induces*

$$i_*: F(m)/F(m)_j \cong \pi/\pi_j, \qquad j \text{ finite},$$
$$i_*: F(m) \subset \pi/\pi_\omega.$$

Proof. In fact, i induces an isomorphism on the first and second homology groups of $F(m)$ and π. The result follows now from [18, Theorem 3.4].

In particular, the meridians of $\mathfrak{L}$ generate a free subgroup of π/π_ω.

Proposition (3). $\mathfrak{L}$ *is a boundary link if, and only if, the inclusion* $i_*: F(m) \subset \pi/\pi_\omega$ *is an isomorphism.*

Proof. If $\mathfrak{L}$ is a boundary link, $\pi/\pi_\omega = F(m)$ via i_* as in [16]. Suppose now that i_* is an isomorphism. Let $\bigvee^m S^1$ be the wedge of m circles, we can map A onto $\bigvee^m S^1$ by projection to get a diagram

$$\begin{tikzcd} X \arrow[dr, dashed] & \\ \cup & \\ A \arrow[r, "p"] & \bigvee^m S^1 \end{tikzcd}$$

which by [3, p. 194] can be completed if and only if the corresponding diagram of fundamental groups

$$\begin{tikzcd} \pi \arrow[drr, dashed] & & \\ F(m) \arrow[u, "i_*"] \arrow[rr, "p_*"] & & F(m) \end{tikzcd}$$

can be completed, that is, if i_* is an isomorphism.

So there is a map $q: X \longrightarrow \bigvee^m S^1$ which can be approximated by a smooth (resp. PL, locally flat) map. Choose points x_i in the ith circle of $\bigvee^m S^1$; the manifolds $V_i' = q^{-1}(x_i)$ can be deformed to $V_i \subset S^{n+2}$ with $\partial V_i = L_i$.

Let $\mathfrak{L}$ and $\mathfrak{L}'$ be two m-links of dimension n. $\mathfrak{L}'$ arises from $\mathfrak{L}$ by a simple F-isotopy on the ith component if there is a torus $V = D^2 \times S^n$ contained in $S^{n+2} - \bigcup_{j \neq i} L_j$ and an orientation preserving homeomorphism $f: S^{n+2} \longrightarrow S^{n+2}$ such that $f(L_j) = L_j'$ for $j \neq i$ and either

(i) L_i is the core of V (i.e. $L_i = (0) \times S^n$) and $H_n(L_i') \cong H_n(S_i^n) \longrightarrow H_n(f(V))$ is an isomorphism, or

(ii) L_i' is the core of $f(V)$ and $H_n(L_i) \cong H_n(S_i^n) \longrightarrow H_n(V)$ is an isomorphism.

Suppose (i) is the case; let π be the group of $\mathfrak{L}$ and ρ that of $\mathfrak{L}'$.

Theorem (4) (Smythe). *The group π is a retract of ρ under a map $\psi: \rho \longrightarrow \pi$ inducing an isomorphism $\pi/\pi_2 \cong \rho/\rho_2$ and preserving meridians.*

For a proof see [17].

Corollary (5). *Every link F-isotopic to a boundary link is itself a boundary link.*

Proof. In fact, if $\mathfrak{L}'$ is a boundary, $i: F(m) \cong \rho/\rho_\omega$ and the generators are meridians. On the other hand by [18], $\psi: \rho/\rho_\omega \cong \pi/\pi_\omega$ and so $\psi i: F \approx \pi/\pi_\omega$ and the generators are meridians; hence $\mathfrak{L}$ is boundary.

Let $\mathfrak{L}$ be a boundary link, $\{V_i\}$ a collection of Seifert manifolds for it. Let Y be the complement of $\mathfrak{L}$ cut along the V_i with fundamental group G. Call $H_i = \pi_1(V_i)$ and $\nu_{it}: H_i \longrightarrow G$ the obvious homomorphisms. We can find an explicit construction for the cover $\hat{X}$ of X associated to π_ω. To motivate its construction recall $\bigvee^m S^1 \subset X$ as the meridians; the cover $\hat{X}$ should contain the universal cover of this wedge of circles. Let in fact, $Y(w)$, for $w \in F[a_1, \cdots, a_m]$, be a copy of Y with boundary $\Sigma_{i=1}^m (V_{i0}(w) \cup V_{i1}(w))$; $\hat{X}$ is obtained by identifying Int $V_{i1}(w)$ to Int $V_{i0}(wa_i)$. Suppose G is presented by $\langle y_1, \cdots, y_\alpha; R_1, \cdots, R_\beta \rangle$, then

Proposition (6). *We have the following group presentations*:

(1) $\pi_\omega = \langle y_{\sigma w}: R_\tau(y_{\sigma w}),\ \nu_{i1}^{(w)} = \nu_{i0}^{(wa_i)},\ 1 \le \sigma \le \alpha,\ 1 \le \tau \le \beta\rangle$,

(2) $\pi = \langle w, y_\sigma: R_\tau(y_\sigma),\ wy_\sigma w^{-1} = y_{\sigma w},\ \nu_{i1}^{(w)} = \nu_{i0}^{(wa_i)}\rangle$. *Here* $\nu_{it}^{(w)}: H_i \to G^{(w)} = \pi_1(Y(w))$, $w \in F(m)$.

Proof. (1) follows from Neuwirth's theorem [13]. We can simply assume that the groups $G^{(w)}$ are in the vertices of the universal cover of $\bigvee^m S^1$ and that the amalgamations are performed along the edges of such cover. Assertion (2) follows from considering the group extension

$$1 \longrightarrow \pi_\omega \longrightarrow \pi \longrightarrow F(m) \longrightarrow 1$$

which splits since $F(m)$ is free [11, Chapter IV].

3. **Cobordism.** A link $\mathfrak{L}$ is split if the L_i can be separated from each other by $(n+1)$-spheres $\Sigma \subset X$. Given $\mathfrak{L}_0$ and $\mathfrak{L}_1$, m-links of dimension n, we say that they are cobordant if there exists an embedding $\mathcal{H}: mS^n \times I \to S^{n+2} \times I$ where $\operatorname{Im}(\mathcal{H})$ meets $\partial(S^{n+2} \times I)$ tranversally and $\mathcal{H}|mS^n \times \{t\} = \mathfrak{L}_t$ for $t = 0, 1$. A link cobordant to a splitted link is called split-cobordant.

Let $\mathfrak{L}$ be an m-boundary link of dimension $n \ge 2$, $\{V_i\}$ a collection of Seifert manifolds for it.

Theorem (7). *Every boundary link of dimension* $n \ge 2$ *is split-cobordant.*

Proof. As in III.6 of [4], we can add handles to the V_i^{n+1} in D^{n+3} up to one dimension below the middle and, by general position [19], these handles can be taken to be disjoint. The result is a collection of manifolds W_i^{n+2} in D^{n+3} and embeddings $j_i: V_i \times I \to W_i$ satisfying

(i) $W_i \cap S^{n+2} = V_i$ and $j_i(x, t) = \frac{1}{2}(t+1)x$ in D^{n+3}.

(ii) $\partial W_i = V_i \cup j_i(\partial V_i \times I) \cup V_i'$, where $V_i \cap V_i' = \emptyset$, $V_i' \cap j_i(\partial V_i \times I) = \partial V_i' = j_i(\partial V_i \times 0)$.

(iii) V_i' is connected up to the middle dimension. (In particular for n even, V_i' is a disk.)

(iv) W_i is obtained from $j_i(V_i \times I)$ by adding handles of index $\le \frac{1}{2}n$.

We use now the engulfing argument of Lemma 4 of [9] for W_1; in the notation of [2, Theorem 2], $X = V_i'$ and V (of [2]) $= D^{n+3}$ with cuts along the W_i $(1 \le i \le m)$. The hypothesis of the engulfing theorem is verified as in [9] so we can find a ball B_1^{n+3} in V with $B_1 \cap W_1 = V_1'$. Now we repeat the argument for $V = D^{n+3}$ with cuts along $W_1 \cup B_1, W_2, \cdots, W_m$ and to the engulfing process. By induction we get $B_1, \cdots, B_m$ with $B_i \cap W_i = V_i'$. Then $B_1 \# \cdots \# B_m$ is a ball B and $D^{n+3} - B$ contains a cobordism of $\mathfrak{L}$ to a link split by the spheres ∂B_i.

4. **Unlinking spheres in codimension two.** The following is a homotopy theoretic criterion for determining whether an m-link of dimension $n \ge 4$ is trivial or not.

Theorem (8). *Let $\mathfrak{L}$ be an m-link of dimension $n \geq 4$ with complement X; if X is homotopy equivalent to the complement of the trivial link, where $\pi_1(X) = F(m)$ is generated by the meridians, then $\mathfrak{L}$ is itself trivial.*

This result has been found by Levine [7] for the case $m = 1$, $n \geq 4$ and by Shaneson [15] for $m = 1$, $n = 3$. Simultaneous to this work, Lee [5] proved the result for $m \geq 2$ and odd dimensions ≥ 7. Cappell [1] in his thesis obtained, in a more general setting, results similar to ours.

In this paragraph all homology is to be taken with integral coefficients. Recall the construction of the cover $\hat{X}$ of X made in §2: $\hat{X}$ is obtained by pasting copies of X cut along the Seifert manifolds. These copies are called $Y(w)$, $w \in F(m)$, and have boundary $\Sigma(V_{i0}(w) \cup V_{i1}(w))$.

Lemma (9). *Let $\mathfrak{L}$ be a boundary link; the Seifert manifolds can be chosen to be 1-connected if, and only if, $\pi = \pi_1(X)$ is free generated by the meridians.*

Proof. In fact, if the manifolds, V_i are simply connected, $\pi = F(m)$ by Proposition (6), assertion (2). Conversely, if $\pi = F[a_1, \cdots, a_m]$ the cover $\hat{X}$ of X is in this case the universal cover since $F(m)_\omega = 1$. Then, the map $\pi_1(\mathrm{Int}\, V_i) \to \pi = F$ is zero because the map $\mathrm{Int}\, V_i \subset X$ factors though $\hat{X}$ by construction. By a result of Serre [14], $\pi_1(\mathrm{Int}\, V_i) = \pi_1(V_i)$ is finitely generated, say by $\alpha_1^i, \cdots \alpha_r^i$. Let $f_j^i\colon D^2 \to X$ $(1 \leq i \leq m,\ 1 \leq j \leq r)$ be transverse regular to the V_i such that $f_j^i(S^1) \subset \mathrm{Int}\, V_i$ represents α_j^i. The f_j^i exist because of the remark about the inclusion map above; by general position [19] the images of the f_j^i are disjoint. The technique of [7, §5] allows us to make V_1 simply connected. Suppose that, by induction, $V_1, \cdots, V_k$ are 1-connected; choose α_j^{k+1} and let $\alpha \in \pi_1(V_i)$ represent an innermost component of $f_j^{k+1}(D^2) \cap (\bigcup_i V_i)$. If $i < k$, $\alpha = 0$; if $i \geq k + 1$, $\alpha \in \ker \nu_{it*}$ and we can do surgery on V_i as in Lemma (1) to eliminate α. So, without altering $V_1, \cdots, V_k$, we can make V_{k+1} 1-connected. The result now follows by induction.

Lemma (10). *Suppose the Seifert manifolds of Lemma* (9) *are* $(k-1)$*-connected and* (i) $\nu_{it*}\colon \pi_k(V_i) \to \pi_k(Y)$ *is a monomorphism for* $t = 0, 1$, *all* i, (ii) $\pi_k(\mathrm{Int}\, V_i) \to \pi_k(X)$ *is zero for all* i; *then* $\pi_k(V_i) = 0$ *for all* i.

Proof. Let $\alpha \in \pi_k(V_{i0})$; by (ii) there is a map $f\colon D^{k+1} \to X$ such that $f(S^k) \subset \mathrm{Int}\, V_{i0}$ and represents α. We may assume that f is t-regular to all V_i so that the inverse image by f of $\bigcup_{i=1}^m V_i$ is a not necessarily connected k-manifold in D^{k+1}; let M be an innermost component of it, i.e. such that there exists a connected submanifold W of D^{k+1} with $\partial W = M$ and suppose $f|M$ maps M to V_j (some j). As in Lemma (4) of [7], $f|M$ extends to W and we can eliminate M in the manner described in [7]. By a sequence of such modifications we will have $f^{-1}(\bigcup V_i) = S^k$ so that $\nu_{i_0t*}\alpha = 0$ for some t and, by (i), $\alpha = 0$.

We can now prove the following

Proposition (11). *Let $n \geq 2k+1$ and $\mathfrak{L}$ an m-link of dimension $n \geq 4$ whose complement X verifies*

$$(U_k) \qquad \pi_i(X) \cong \pi_i(\overset{m}{\vee} S^1), \quad i \leq k.$$

(U_1 includes the assumption that the fundamental group of X is generated by the meridians.) Then $\mathfrak{L}$ is a boundary link and we can find a collection of k-connected Seifert manifolds for it.

Proof. Following an application of Lemma (9) suppose, in the notation of Lemma (10), that the V_i are all $(k-1)$-connected $(k \geq 2)$, and that $\nu_{it*}: \pi_k(V_i) \to \pi_k(Y)$ are not monomorphisms. By [14], ker ν_{it*} is a finitely generated abelian group, so by Lemma (1) it can be eliminated by surgery; by Lemma (10) the V_i are now connected.

We must kill $\pi_k(V_i)$ when $n = 2k$ or $2k-1$ under assumption (U_k). If $n = 2k-1$, ker ν_{it*} is generated by primitive elements [12] because $\pi_k(V_i) = H_k(V_i)$ is free abelian; therefore, by Lemma (1) ν_{it*} can be made monomorphic and then, by Lemma (10), the V_i can be exchanged for $2k$-disks.

For $n = 2k$, notice that $H_k(V_i) = H_{k+1}(V_i)$ has torsion T_i. By Lemma (5) of [7], we can make the $\nu_{it*}|T_i$ monomorphic. With the notation of §2 and [8] and by the Mayer-Vietoris theorem, $H_k\hat{X} = \pi_k\hat{X} = \pi_k X = 0$ is presented by

$$\begin{array}{c} 0 = H_{k+1}\hat{X} \longrightarrow \sum H_k(V_i)\otimes \Theta_m \xrightarrow{d} H_k(Y)\otimes\Theta_m \longrightarrow H_k\hat{X} = 0 \\ \hfill \| \\ \hfill 0 \end{array}$$

where Θ_m is the integral group ring of $F[a_1, \cdots, a_m]$ viewed as the ring of integral Laurent polynomials in m noncommuting variables $t_1, \cdots, t_m$ and where d is given by the formula

$$d(\alpha \otimes 1) = \nu_{i0*}\alpha \otimes t_i - \nu_{i1*}\alpha \otimes 1 \quad \text{for } \alpha \in H_k(V_i).$$

From the sequence, d is an isomorphism.

Lemma (12). *Under the present hypothesis,* ker ν_{it*} *is generated by primitive elements.*

Proof. Assume $i = 1$ and let $\alpha \in \ker \nu_{1t*}$. Then $\alpha = p\alpha'$ and α' is a primitive element. Now, $p\nu_{1t*}\alpha' = 0$ so that $\nu_{1t*}\alpha' \in H_k(Y)$ is a torsion element. Represent $\nu_{1t*}\alpha'$ by an embedding $\beta: S^k \to Y(1) \subset \hat{X}$. Since $\hat{X}$ is contractible, β extends to $\rho': D^{k+1} \to \hat{X}$; β' can be assumed to be transversal to the $V_{it}(w)$ $(w \in F(m))$, contained in X, so $\beta'^{-1}(\bigcup V_{it}(w))$ is a k-dimensional manifold in D^{k+1}. Consider the components $\beta'^{-1}(\bigcup V_{it}(1))$; they, together with S^k, bound a manifold W and hence in $H_k(Y)\otimes\Theta_m$, W establishes a homology

$$\nu_{1t^*}\alpha'\otimes 1=\sum_{i=1}^{m}(\nu_{i0}\sigma_i\otimes t_i-\nu_{i1}\sigma_i'\otimes 1) \tag{3}$$

where σ_i (resp. σ_i') represents the intersection of $\beta'(W)$ with $V_{i0}(1)$ (resp. $V_{i1}(1)$). Since d is an isomorphism, from (3) we conclude that $\sigma_i=\sigma_i'\in T_i$. Notice that each σ_i (σ_i') can be represented by a connected submanifold. In fact, if $\sigma_i=\xi_1+\xi_2$ and the ξ_i represent nonconnected submanifolds M_i of D^{k+1}, we can join M_1 and M_2 by an arc γ_1 in $V_{i0}(1)$ and by another arc γ_2 in $\beta'(W)$. The resulting loop $\gamma_1\gamma_2^{-1}$ is nullhomotopic in Y, hence it bounds a 2-disk; by a Whitney process we can alter β' so that $\beta'^{-1}(V_{i0}(1))=M_1\ \#\ M_2$.

Now consider each component of $D^{k+1}-W$ and repeat the above reasoning: the components of $\beta'^{-1}(\bigcup V_{it}(w))$ all represent torsion elements, in particular those that are innermost components. Let $\gamma\in H_k(V_{it}(w))=H_k(V_i)$ be represented by an innermost component. Then $\gamma\in T_i$ and $\nu_{is}\gamma=0$ for some s. Since $\nu_{is}|T_i$ is a monomorphism, $\gamma=0$ and the intersection that represents γ can be eliminated by the method of [7, p. 13]. In such a way we can assume $\beta'(D^{k+1})\subset Y(1)$ and so $\nu_{1t}(\alpha')=0$. Thus, the ker ν_{it} can be eliminated by surgery and by Lemma (10), the V_i can be exchanged by $2k+1$ disks and the theorem is proven. The present proof is a direct generalization of Levine's proof which is remarkably simple and geometrical.

Theorem (8) is equivalent to the following statement:

(Lee's form of the Unlinking Theorem). *Let M^{n+1}, $n\geq 3$, be a closed manifold, homotopy equivalent to $K_{nm}=S^1\times S^n\ \#\cdots\#\ S^1\times S^n$* (*m* times); *then M is isomorphic to K_{nm} in the category PL.*

As a last remark, both in Proposition (3) and in Theorem (8), the condition that the fundamental group be generated by meridians is essential. In fact, without it both results are false. See [22].

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