BOUNDARY LINKS AND AN UNLINKING THEOREM

BY

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ABSTRACT. This paper gives a homotopic theoretic criterion for a higher dimensional link to be trivial.

An $m$-link is an embedding of $m$ disjoint copies of the $n$-sphere into the $(n + 2)$-space. There are various equivalence relations amongst links such as isotopy and cobordism. Some results and definitions are found in [21], [7] and [9].

Generally, it is useful to compare, via our equivalence relations, any link to the standard trivial link. Any link isotopic to it is called trivial; if the link is cobordant to the trivial one, we say it is a slice.

An interesting concept, weaker than triviality, is that of boundary link: an $m$-link is trivial if the embedding extends to one of $m$ disjoint copies of the $(n + 1)$-disk; an $m$-link is boundary if it extends to an embedding of $m$ compact $(n + 1)$-manifolds with boundary the sphere. These are called Seifert manifolds.

The purpose of this paper is to give some homotopic conditions for a link to be (i) boundary, (ii) trivial. These conditions are reflected on the homotopy type of $X$, the complement of the image of the link in the ambient space, as follows:

(i) An $m$-link is boundary if, and only if, there is an epimorphism from $\pi_1(X)$ onto the free group in $m$ generators which sends meridians to generators.

(ii) Let $\mathcal{L}$ be an $m$-link of dimension $\geq 4$ and $\vee^m S^1$, the wedge of $m$ circles; suppose

$$\pi_i(X) = \pi_i\left(\vee^m S^1\right) \quad \text{for} \quad i \leq q \leq \frac{1}{2} \,(n + 1)$$

and that for $i = 1$ meridians are sent on to generators.

Then $\mathcal{L}$ is a boundary link where the Seifert manifolds can be chosen to be $(q - 1)$-connected. In particular for $q = \frac{1}{2}(n + 1)$, $\mathcal{L}$ is trivial.

As by-products, we obtain some results about cobordism of links which are presented in §3.

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(1) This paper was written in México under the auspices of the O. A. S. Multinational Plan.
0. Notation. In this paper, all manifolds, mappings and isotopies belong to the smooth or PL, locally flat, categories, consistently. All manifolds are oriented and if $M_i$ ($i = 1, \ldots, m$) are manifolds, $\Sigma_{i=1}^m M_i$ stands for the disjoint union of them. As usual $S^n$ and $D^{n+1}$ are the $n$-sphere and the $(n+1)$-disk. In particular, $mS^n$ is a disjoint union of $m$ copies of $S^n$.

An $m$-link is an embedding $\mathcal{L} : mS^n \to S^{n+2}$ ($m, s \geq 1$); we say that $\mathcal{L}$ is boundary if it extends to an embedding $\Sigma_{i=1}^m V_i \to S^{n+2}$, where $V_i$ is a compact, orientable manifold with boundary $S^n$. The collection $\{V_i\}$ is called a collection of Seifert manifolds for $\mathcal{L}$. In particular, if the $V_i$ are disks, we say that $\mathcal{L}$ is trivial.

Consider $\mathcal{L}_i : S^n \to S^{n+2}$, the restriction of $\mathcal{L}$ to $S^n$ ($i = 1, \ldots, m$); we can find tubular neighborhoods $T_i$ of $\text{Im}(\mathcal{L}_i) = L_i$, which are mutually disjoint [20]. Let $X$ be $S^{n+2} - \bigcup T_i$, a compact manifold with boundary $m(S^1 \times S^m)$ and of the same homotopy type as $S^{n+2} - \text{Im}(\mathcal{L})$.

Consider arcs in $X$ joining the $S^1 \times S^n$ to a common basepoint; the union of these arcs and $X$ is called $A$ and, for $n \geq 2$, its fundamental group is free in $m$ generators called meridians. The group is denoted by $F(m)$ or by $F(a_1, \ldots, a_m)$ when the generators are specified.

Finally, if $G$ is any group, $G_i$ indicates the $i$th member of the lower central series [18]; $G_\omega$ is by definition, $\bigcap G_i$. In particular, if $F = F(m)$, $F_\omega = 1$ [10].

1. Ambient surgery. Let $\mathcal{L} : mS^n \to S^{n+2}$ be a boundary link and $\{V_i\}$ a collection of Seifert manifolds for it. Let $f_i : V_i \to I$ be a map with $f_i^{-1}(0) = \partial V_i \cong S^n$ and $W_i = \{(x, t) \in V_i \times I \mid f_i(x) \leq t\}$. With the aid of the (trivial) normal bundle of the $V_i$, we can find embeddings $\psi_i : W_i \to S^{n+2}$; notice that $\partial W_i = V_{i0} \cup V_{i1}$, where $V_{ii} \cong V_i$ and the two copies are attached by the boundary. The manifold

$$Y = S^{n+2} - \bigcup \psi_i(W_i)$$

is called $S^{n+2}$ cut along the $V_i$, a compact manifold with boundary $\Sigma_{i=1}^m (V_{i0} \cup V_{i1})$. The composition $V_i \cong V_{it} \subset Y$ is called $\nu_{it}$. See [20].

Let $\phi : S^k \times D^{n+1-k} \to V_i$ be an embedding; define $\theta_i(V_i, \phi)$ as $W_i \cup D^k \times D^{n+1-k}$ where the handle is attached to $V_{it}$ by $\phi$; $\partial \theta_i(V_i, \phi)$ is equal to $V_{is} \cup \chi(V_i, \phi)$, $s = 1 - t$.

Suppose that for some $k \geq 1$, the $V_i$ are $k$-connected. By Alexander duality $H_q(Y, Z) = \Sigma H_q(V_i, Z)$, $q \leq k - 1$; let $Y_\alpha = S^{n+2} - \text{Int} \bigcup_{j \leq \alpha} W_j$, then by [7, §4], $Y_1$ is 1-connected. If $Y_\tau$ is 1-connected, then by the van Kampen theorem we have
therefore \( Y_{r+1} \) is 1-connected and, by induction, so is \( Y \). By the Hurewicz theorem, \( Y \) is then \((k - 1)\)-connected.

**Lemma (1).** If (i) \( n \geq 2k + 1 \) or (ii) \( n = 2k \) or \( 2k - 1 \), \( n < 4 \), and \( \alpha \in \pi_k(V_i) \) is in ker \( \nu_{it*} \), the embedding \( \psi_i: W_i \to Y \cup W_i \) extends to \( \theta_i(V_i, \phi) \), where \( \phi \) is an embedding representing \( \alpha \).

**Proof.** We refer to the proof of Lemma (3) of [7]. Let \( \alpha' \in \pi_k(V_i) \) corresponding to \( \alpha \) under \( V_i \cong V_{it} \subset \partial Y \), since \( \nu_{it*} \alpha = 0 \), \( \beta' \) is the boundary of an element \( \beta' \in \pi_{k+1}(Y, W_i) \). This latter pair is \( k \)-connected, hence by (1) of [6], there is an embedding \( g: D^{k+1} \to Y \) representing \( \beta' \) which is transversal to the boundary. Since \( V_{it} \) is 1-connected, \( g \) can be isotoped by (2) of [6] and (3) of [7] so that \( g(S^k) \) represents \( \alpha \). By using a tubular neighborhood of \( \text{Im}(g) \) we get the desired extension.

2. The Fundamental group. Let \( \pi = \pi_1(X) \); in [4] it is proven that \( H_1(\pi) = \mathbb{Z}^m \) and \( H_2(\pi) = 0 \), where \( H_q(\pi) \) is the \( q \)th homology group of \( \pi \) with trivial integral coefficients. The inclusion \( A \subset X \) induces a homomorphism \( i: F(m) \to \pi \) of fundamental groups.

**Lemma (2).** For \( n \geq 2 \), \( i \) induces

\[
i_*: F(m)/F(m)_j \cong \pi/\pi_j, \quad j \text{ finite},
\]

\[
i^*_\omega: F(m) \subset \pi/\pi_\omega.
\]

**Proof.** In fact, \( i \) induces an isomorphism on the first and second homology groups of \( F(m) \) and \( \pi \). The result follows now from [18, Theorem 3.4].

In particular, the meridians of \( \mathcal{L} \) generate a free subgroup of \( \pi/\pi_\omega \).

**Proposition (3).** \( \mathcal{L} \) is a boundary link if, and only if, the inclusion \( i_*: F(m) \subset \pi/\pi_\omega \) is an isomorphism.

**Proof.** If \( \mathcal{L} \) is a boundary link, \( \pi/\pi_\omega = F(m) \) via \( i_* \) as in [16]. Suppose now that \( i_* \) is an isomorphism. Let \( \vee^m S^1 \) be the wedge of \( m \) circles, we can map \( A \) onto \( \vee^m S^1 \) by projection to get a diagram
which by [3, p. 194] can be completed if and only if the corresponding diagram of fundamental groups can be completed, that is, if $i_*$ is an isomorphism.

So there is a map $q: X \to \bigvee^m S^1$ which can be approximated by a smooth (resp. PL, locally flat) map. Choose points $x_i$ in the $i$th circle of $\bigvee^m S^1$; the manifolds $V_i' = q^{-1}(x_i)$ can be deformed to $V_i \subset S^{n+2}$ with $\partial V_i = L_i$.

Let $\mathcal{L}$ and $\mathcal{L}'$ be two $m$-links of dimension $n$. $\mathcal{L}'$ arises from $\mathcal{L}$ by a simple $F$-isotopy on the $i$th component if there is a torus $V = D^2 \times S^n$ contained in $S^{n+2} - \bigcup_{j \neq i} L_j$ and an orientation preserving homeomorphism $f: S^{n+2} \to S^{n+2}$ such that $f(L_i) = L_i'$ for $j \neq i$ and either

(i) $L_i$ is the core of $V$ (i.e. $L_i = (0) \times S^n$) and $H_n(L_i') \cong H_n(S^n) \to H_n(f(V))$ is an isomorphism, or

(ii) $L_i'$ is the core of $f(V)$ and $H_n(L_i') \cong H_n(S^n) \to H_n(V)$ is an isomorphism.

Suppose (i) is the case; let $\pi$ be the group of $\mathcal{L}$ and $\rho$ that of $\mathcal{L}'$.

**Theorem (4) (Smythe).** The group $\pi$ is a retract of $\rho$ under a map $\psi: \rho \to \pi$ inducing an isomorphism $\pi/\pi_2 \cong \rho/\rho_2$ and preserving meridians.

For a proof see [17].

**Corollary (5).** Every link $F$-isotopic to a boundary link is itself a boundary link.

**Proof.** In fact, if $\mathcal{L}'$ is a boundary, $i: F(m) \cong \rho/\rho_\omega$ and the generators are meridians. On the other hand by [18], $\psi: \rho/\rho_\omega \cong \pi/\pi_\omega$ and so $\psi i: F \cong \pi/\pi_\omega$ and the generators are meridians; hence $\mathcal{L}$ is boundary.

Let $\mathcal{L}$ be a boundary link, $\{V_i\}$ a collection of Seifert manifolds for it. Let $Y$ be the complement of $\mathcal{L}$ cut along the $V_i$ with fundamental group $G$. Call $H_i = \pi_1(V_i)$ and $\nu_i: H_i \to G$ the obvious homomorphisms. We can find an explicit construction for the cover $X$ of $X$ associated to $\pi_\omega$. To motivate its construction recall $\bigvee^m S^1 \subset X$ the meridians; the cover $\hat{X}$ should contain the universal cover of this wedge of circles. Let in fact, $Y(w)$, for $w \in F(a_1, \ldots, a_m)$, be a copy of $Y$ with boundary $\Sigma_{i=1}^m (V_{i0}(w) \cup V_{i1}(w))$; $\hat{X}$ is obtained by identifying $\text{Int} V_{i1}(w)$ to $\text{Int} V_{i0}(wa_i)$. Suppose $G$ is presented by $\langle y_1, \ldots, y_a; R_1, \ldots, R_\beta \rangle$, then
Proposition (6). We have the following group presentations:

(1) \[ \pi_\omega = \langle y_\sigma w; R_\sigma(y_\sigma w), v^{(w)}_0, v^{(w)}_1 \rangle, \quad 1 \leq \sigma \leq \alpha, \quad 1 \leq r \leq \beta \rangle, \]
(2) \[ \pi = \langle w, y_\sigma; R_\sigma(y_\sigma), w y_\sigma w^{-1} = y_\sigma w, v^{(w)}_0, v^{(w)}_1 \rangle. \quad \text{Here} \quad v^{(w)}_i : H_i \rightarrow G^{(w)} = \pi_1(Y(w)), w \in F(m). \]

Proof. (1) follows from Neuwirth's theorem [13]. We can simply assume that the groups \( G^{(w)} \) are in the vertices of the universal cover of \( \sqrt{m} S^1 \) and that the amalgamations are performed along the edges of such cover. Assertion (2) follows from considering the group extension

\[ 1 \rightarrow \pi_\omega \rightarrow \pi \rightarrow F(m) \rightarrow 1 \]

which splits since \( F(m) \) is free [11, Chapter IV].

3. Cobordism. A link \( L \) is split if the \( L_i \) can be separated from each other by \( (n + 1) \)-spheres \( \Sigma \subset X \). Given \( L_0 \) and \( L_1 \), \( m \)-links of dimension \( n \), we say that they are cobordant if there exists an embedding \( H : mS^n \times I \rightarrow S^{n+2} \times I \) where \( \text{Im}(H) \) meets \( \partial(S^{n+2} \times I) \) transversally and \( H([mS^n \times \{t\}]) = \Sigma_t \) for \( t = 0, 1 \). A link cobordant to a splitted link is called split-cobordant.

Let \( L \) be an \( n \)-boundary link of dimension \( n \geq 2 \), \( \{V_i\} \) a collection of Seifert manifolds for it.

Theorem (7). Every boundary link of dimension \( n \geq 2 \) is split-cobordant.

Proof. As in III.6 of [4], we can add handles to the \( V_i^{n+1} \) in \( D^{n+3} \) up to one dimension below the middle and, by general position [19], these handles can be taken to be disjoint. The result is a collection of manifolds \( W_i^{n+2} \) in \( D^{n+3} \) and embeddings \( j_i : V_i \times I \rightarrow W_i \) satisfying

(i) \( W_i \cap S^{n+2} = V_i \) and \( j_i(x, t) = \frac{1}{2}(t + 1)x \) in \( D^{n+3} \).
(ii) \( \partial W_i = V_i \cup j_i(\partial V_i \times I) \cup V_i', \) where \( V_i \cap V_i' = \emptyset, V_i' \cap j_i(\partial V_i \times I) = \partial V_i' = j_i(\partial V_i \times 0) \).
(iii) \( V_i' \) is connected up to the middle dimension. (In particular for \( n \) even, \( V_i' \) is a disk.)
(iv) \( W_i \) is obtained from \( j_i(V_i \times I) \) by adding handles of index \( \leq \frac{1}{2}n \).

We use now the engulfing argument of Lemma 4 of [9] for \( W_i \); in the notation of [2, Theorem 2], \( X = V_i' \) and \( V \) (of [2]) = \( D^{n+3} \) with cuts along the \( W_i \) (\( 1 \leq i \leq m \)). The hypothesis of the engulfing theorem is verified as in [9] so we can find a ball \( B_i^{n+3} \) in \( V \) with \( B_i^{n+3} \cap W_i = V_i' \). Now we repeat the argument for \( V = D^{n+3} \) with cuts along \( W_1 \cup B_1, W_2, \ldots, W_m \) and to the engulfing process. By induction we get \( B_1, \ldots, B_m \) with \( B_1 \cap W_i = V_i' \). Then \( B_1 \# \cdots \# B_m \) is a ball \( B \) and \( D^{n+3} - B \) contains a cobordism of \( L \) to a link split by the \( \partial S \) spheres \( \partial B_i \).

4. Unlinking spheres in codimension two. The following is a homotopy theoretic criterion for determining whether an \( m \)-link of dimension \( n \geq 4 \) is trivial or not.
Theorem (8). Let $\mathcal{L}$ be an $m$-link of dimension $n \geq 4$ with complement $X$; if $X$ is homotopy equivalent to the complement of the trivial link, where $\pi_1(X) = F(m)$ is generated by the meridians, then $\mathcal{L}$ is itself trivial.

This result has been found by Levine [7] for the case $m = 1$, $n \geq 4$ and by Shaneson [15] for $m = 1$, $n = 3$. Simultaneous to this work, Lee [5] proved the result for $m \geq 2$ and odd dimensions $\geq 7$. Cappell [1] in his thesis obtained, in a more general setting, results similar to ours.

In this paragraph all homology is to be taken with integral coefficients. Recall the construction of the cover $\hat{X}$ of $X$ made in §2: $\hat{X}$ is obtained by pasting copies of $X$ cut along the Seifert manifolds. These copies are called $Y_{w_i}$, $w_i \in F(m)$, and have boundary $\Sigma(V_i(w)) = V_i(w)$.

Lemma (9). Let $\mathcal{L}$ be a boundary link; the Seifert manifolds can be chosen to be 1-connected if, and only if, $\pi_1(X)$ is free generated by the meridians.

Proof. In fact, if the manifolds, $V_i$, are simply connected, $\pi = F(m)$ by Proposition (6), assertion (2). Conversely, if $\pi = F[a_1, \ldots, a_m]$ the cover $\hat{X}$ of $X$ is in this case the universal cover since $F(m) = 1$. Then, the map $\pi_1(\text{Int } V_i) \to \pi_1 = F$ is zero because the map $\text{Int } V_i \subset X$ factors though $\hat{X}$ by construction. By a result of Serre [14], $\pi_1(\text{Int } V_i) = \pi_1(V_i)$ is finitely generated, say by $\alpha_1, \ldots, \alpha_r$. Let $f_i : D^2 \to X$ be transverse regular to the $V_i$ such that $f_i(S^1) \subset \text{Int } V_i$ represents $\alpha_i$. The $f_i$ exist because of the remark about the inclusion map above; by general position [19] the images of the $f_i$ are disjoint.

The technique of [7, §5] allows us to make $V_1$ simply connected. Suppose that, by induction, $V_1, \ldots, V_k$ are 1-connected; choose $\alpha_j^{k+1}$ and let $\alpha \in \pi_1(V_i)$ represent an innermost component of $f_j^{k+1}(D^2) \cap (\bigcup_i V_i)$. If $i < k$, $\alpha = 0$; if $i > k + 1$, $\alpha \in \ker \nu_{i*}$ and we can do surgery on $V_i$ as in Lemma (1) to eliminate $\alpha$. So, without altering $V_1, \ldots, V_k$, we can make $V_{k+1}$ 1-connected. The result now follows by induction.

Lemma (10). Suppose the Seifert manifolds of Lemma (9) are $(k - 1)$-connected and (i) $\nu_{i*} : \pi_k(V_i) \to \pi_k(Y)$ is a monomorphism for $t = 0, 1$, all $i$, (ii) $\pi_k(\text{Int } V_i) \to \pi_k(X)$ is zero for all $i$; then $\pi_k(V_i) = 0$ for all $i$.

Proof. Let $\alpha \in \pi_k(V_{i_0})$: by (ii) there is a map $f : D^{k+1} \to X$ such that $f(S^k) \subset \text{Int } V_{i_0}$ and represents $\alpha$. We may assume that $f$ is $t$-regular to all $V_i$ so that the inverse image by $f$ of $\bigcup_{i=1}^n V_i$ is a not necessarily connected $k$-manifold in $D^{k+1}$; let $M$ be an innermost component of it, i.e. such that there exists a connected submanifold $W$ of $D^{k+1}$ with $\partial W = M$ and suppose $f[M]$ maps $M$ to $V_j$ (some $j$). As in Lemma (4) of [7], $f[M]$ extends to $W$ and we can eliminate $M$ in the manner described in [7]. By a sequence of such modifications we will have $f^{-1}(\bigcup_i V_i) = S^k$ so that $\nu_{i_0t*} \alpha = 0$ for some $t$ and, by (i), $\alpha = 0$. 

We can now prove the following

**Proposition (11).** Let \( n \geq 2k + 1 \) and \( \mathcal{O} \) an \( m \)-link of dimension \( n \geq 4 \) whose complement \( X \) verifies

\[
(U_k) \quad \pi_i(X) \cong \pi_i(\bigvee S^1), \quad i \leq k.
\]

\((U_1\) includes the assumption that the fundamental group of \( X \) is generated by the meridians.) Then \( \mathcal{O} \) is a boundary link and we can find a collection of \( k \)-connected Seifert manifolds for it.

**Proof.** Following an application of Lemma (9) suppose, in the notation of Lemma (10), that the \( V_i \) are all \((k - 1)\)-connected \((k \geq 2)\), and that \( \nu_{it*}: \pi_k(V_i) \to \pi_k(Y) \) are not monomorphisms. By [14], \( \ker \nu_{it*} \) is a finitely generated abelian group, so by Lemma (1) it can be eliminated by surgery; by Lemma (10) the \( V_i \) are now connected.

We must kill \( \pi_k(V_i) \) when \( n = 2k \) or \( 2k - 1 \) under assumption \((U_1)\). If \( n = 2k - 1 \), \( \ker \nu_{it*} \) is generated by primitive elements [12] because \( \pi_k(V_i) = H_k(V_i) \) is free abelian; therefore, by Lemma (1) \( \nu_{it*} \) can be made monomorphic and then, by Lemma (10), the \( V_i \) can be exchanged for \( 2k \)-disks.

For \( n = 2k \), notice that \( H_k(V_i) = H_{k+1}(V_i) \) has torsion \( T_i \). By Lemma (5) of [7], we can make the \( \nu_{it*}[T_i] \) monomorphic. With the notation of \( \S 2 \) and [8] and by the Mayer-Vietoris theorem, \( H_k(\tilde{X}) = \pi_k(\tilde{X}) = \pi_k(X) = 0 \) is presented by

\[
0 = H_{k+1}(X) \to \sum H_k(V_i) \otimes \Theta_m \xrightarrow{d} H_k(Y) \otimes \Theta_m \to H_k(\tilde{X}) = 0
\]

where \( \Theta_m \) is the integral group ring of \( F[a_1, \ldots, a_m] \) viewed as the ring of integral Laurent polynomials in \( m \) noncommuting variables \( t_1, \ldots, t_m \) and where \( d \) is given by the formula

\[
d(\alpha \otimes 1) = \nu_{i0*} \alpha \otimes t_i - \nu_{i1*} \alpha \otimes 1 \quad \text{for} \quad \alpha \in H_k(V_i).
\]

From the sequence, \( d \) is an isomorphism.

**Lemma (12).** Under the present hypothesis, \( \ker \nu_{it*} \) is generated by primitive elements.

**Proof.** Assume \( i = 1 \) and let \( \alpha \in \ker \nu_{1*} \). Then \( \alpha = p \alpha' \) and \( \alpha' \) is a primitive element. Now, \( p \nu_{1*} \alpha' = 0 \) so that \( \nu_{1*} \alpha' \in H_k(Y) \) is a torsion element.

Represent \( \nu_{1*} \alpha' \) by an embedding \( \beta: S^k \to Y(1) \subset \tilde{X} \). Since \( \tilde{X} \) is contractible, \( \beta \) extends to \( \beta': D^{k+1} \to \tilde{X} \); \( \beta' \) can be assumed to be transversal to the \( V_{it}(w) \) \((w \in F(m))\), contained in \( X \), so \( \beta'^{-1}(\bigcup V_{it}(w)) \) is a \( k \)-dimensional manifold in \( D^{k+1} \). Consider the components \( \beta'^{-1}(\bigcup V_{it}(1)) \); they, together with \( S^k \), bound a manifold \( W \) and hence in \( H_k(Y) \otimes \Theta_m \), \( W \) establishes a homology
(3) \[ \nu_{1*} \alpha' \otimes 1 = \sum_{i=1}^{m} (\nu_{i0} \sigma_i \otimes \tau_i - \nu_{i1} \sigma_i' \otimes 1) \]

where \( \sigma_i \) (resp. \( \sigma_i' \)) represents the intersection of \( \beta'(W) \) with \( V_{i0}(1) \) (resp. \( V_{i1}(1) \)). Since \( d \) is an isomorphism, from (3) we conclude that \( \sigma_i = \sigma_i' \in T_i \). Notice that each \( \sigma_i \) (resp. \( \sigma_i' \)) can be represented by a connected submanifold. In fact, if \( \sigma_i = \xi_i + \xi_1 \) and the \( \xi_i \) represent nonconnected submanifolds \( M_i \) of \( D^{k+1} \), we can join \( M_1 \) and \( M_2 \) by an arc \( \gamma_1 \) in \( V_{i0}(1) \) and by another arc \( \gamma_2 \) in \( \beta'(W) \). The resulting loop \( \gamma_1 \gamma_2^{-1} \) is nullhomotopic in \( Y \), hence it bounds a 2-disk; by a Whitney process we can alter \( \beta' \) so that \( \beta'^{-1}(V_{i0}(1)) = M_1 \# M_2 \).

Now consider each component of \( D^{k+1} - W \) and repeat the above reasoning: the components of \( \beta'^{-1}(U_{V_{it}(w)}) \) all represent torsion elements, in particular those that are innermost components. Let \( \gamma \in H_k(V_{it}(w)) = H_k(V_i) \) be represented by an innermost component. Then \( \gamma \in T_i \) and \( \nu_{is} \gamma = 0 \) for some \( s \). Since \( \nu_{is} \big| T_i \) is a monomorphism, \( \gamma = 0 \) and the intersection that represents \( \gamma \) can be eliminated by the method of [7, p. 13]. In such a way we can assume \( \beta'(D^{k+1}) \subset Y(1) \) and so \( \nu_{1*} (\alpha') = 0 \). Thus, the ker \( \nu_{it} \) can be eliminated by surgery and by Lemma (10), the \( \nu_{it} \) can be exchanged by 2\( k + 1 \) disks and the theorem is proven. The present proof is a direct generalization of Levine’s proof which is remarkably simple and geometrical.

Theorem (8) is equivalent to the following statement:

(Lee’s form of the Unlinking Theorem). Let \( M^{n+1}, \ n \geq 3, \) be a closed manifold, homotopy equivalent to \( K_{nm} = S^1 \times S^n \cdots \times S^1 \times S^n \) (\( m \) times); then \( M \) is isomorphic to \( K_{nm} \) in the category PL.

As a last remark, both in Proposition (3) and in Theorem (8), the condition that the fundamental group be generated by meridians is essential. In fact, without it both results are false. See [22].

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