RADIAL LIMIT SETS ON THE TORUS

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ABSTRACT. Let $U^N$ denote the unit polydisc and $T^N$ the unit torus in the space of $N$ complex variables. A subset $A$ of $T^N$ is called an (RL)-set (radial limit set) if to each positive continuous function $\rho$ on $T^N$, there corresponds a function $f$ in $H^\infty(U^N)$ such that the radial limit $|f|^*$ of the absolute value of $f$ equals $\rho$, a.e. on $T^N$ and everywhere on $A$. If $N > 1$, the question of characterizing (RL)-sets is open, but two positive results are obtained. In particular, it is shown that $T^N$ contains an (RL)-set which is homeomorphic to a cartesian product $K \times T^{N-1}$, where $K$ is a Cantor set. Also, certain countable unions of "parallel" copies of $T^{N-1}$ are shown to be (RL)-sets in $T^N$. In one variable, every subset of $T$ is an (RL)-set; in fact, there is always a zero-free function $f$ in $H^\infty(U)$ with the required properties. It is shown, however, that there exist a circle $A \subset T^2$ and a positive continuous function $\rho$ on $T^2$ to which correspond no zero-free $f$ in $H^\infty(U^2)$ with $|f|^* = \rho$ a.e. on $T^2$ and everywhere on $A$.

1. Introduction. To each bounded, nonnegative function $\rho$ on the unit circle $T$ with $\log \rho \in L^1(T)$, there corresponds a bounded holomorphic function $f$ on the unit disc $U$ for which the radial limit $|f|^*$ of the absolute value of $f$ equals $\rho$ a.e. on $T$ [3, p. 54]. It is known that this result does not generalize to the unit polydisc $U^N$ in the space of $N$ complex variables. However, one positive result due to Rudin [3, p. 55] asserts that if $\rho$ is positive, bounded and lower semicontinuous on the unit torus $T^N$, there exists a function $f$ in $H^\infty(U^N)$ with $|f|^* = \rho$ a.e. on $T^N$. In this paper, a modification of Rudin's construction will be used to obtain more precise information about the sets on which the equality $|f|^* = \rho$ is satisfied. In particular, the following related class of sets will be considered.

Definition. A subset $A$ of $T^N$ is called an (RL)-set (radial limit set) if to each positive continuous function $\rho$ on $T^N$, there corresponds a function $f$ in $H^\infty(U^N)$ with $|f|^* = \rho$ a.e. on $T^N$ and everywhere on $A$.

In one variable, every subset of $T$ is an (RL)-set. Indeed, if $f$ is the outer function

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the continuity of $\rho$ implies that $|f|^2(w) = \rho(w)$ for all $w \in T$. In several variables the question of characterizing $(RL)$-sets is open. However, in §2 of this paper, two types of $(RL)$-sets will be identified. In §3, some differences between one and several variables are discussed regarding the possibility of choosing a zero-free function $f$ in the definition of $(RL)$-sets.

2. Construction of radial limit sets. The purpose of this section is to prove the following theorems which identify two types of $(RL)$-sets.

Theorem 1. Let $a = (a_1, \ldots, a_N)$ be a point in $Z^N$ with $a_j > 0$ for $1 \leq j \leq N$, and $\{p_k\}$ a sequence of complex numbers with $|p_k| = 1$.

If $A_k = \{w \in T^N: wa = p_k\}$, for $k = 1, 2, \ldots$, then $A = \bigcup A_k$ is an $(RL)$-set in $T^N$.

(As usual, $Z^N$ denotes the space of lattice points $a = (a_1, \ldots, a_N)$ where each $a_j$ is an integer. If $w = (w_1, \ldots, w_N)$ is in $T^N$, $wa$ stands for the monomial $w_1^{a_1} \ldots w_N^{a_N}$.)

Theorem 2. Suppose $K$ is the usual "middle-third" Cantor set on $[0, 1]$, $a^2z^2 = \varphi(K)$ where

$$\varphi(t) = -\exp(2\pi i t), \quad (0 \leq t \leq 1).$$

If $A = \{w \in T^N: w_1 w_2 \ldots w_N \in S\}$, then $A$ is an $(RL)$-set in $T^N$.

In Theorem 1, each set $A_k$ consists of a finite number of "parallel" $(N - 1)$-dimensional tori. Hence, the theorem asserts that any countable union of copies of $T^{N-1}$ which are parallel to $\{w \in T^N: w^a = 1\}$ is an $(RL)$-set in $T^N$. Theorem 2 says that $T^N$ contains an $(RL)$-set which is topologically the cartesian product of a Cantor set and an $(N - 1)$-dimensional torus.

The first lemma is essentially Rudin’s "modification theorem" [3, Theorem 2.4.2] upon which the construction of $(RL)$-sets will be based. Since the conclusion of the lemma is somewhat more detailed than Rudin’s original version, a proof will be sketched.

As in [3, Chapter 2], $RP(T^N)$ will be the class of all complex Borel measures $\mu$ on $T^N$ whose Poisson integral $P[d\mu]$ is the real part of a holomorphic function in $U^N$. $RP$-measures are characterized by the vanishing of their Fourier coefficients outside the positive and negative cones of $Z^N$.

If $Q$ is a trigonometric polynomial on $T^N$, $\deg(Q)$ will denote the smallest positive integer $d$ such that the Fourier coefficient $\hat{Q}(a)$ vanishes whenever $a = (a_1, \ldots, a_N) \in Z^N$ with $|a_j| > d$ for some $j$. 
Lemma 1. Suppose $\beta \in \mathbb{T}^N$ and $s \in \mathbb{Z}^N$ with $s_j > 0$ for $1 \leq j \leq N$. Let $E = \{ w \in \mathbb{T}^N : w^s = 1 \}$, and $F = \beta E = \{ (\beta_1 w_1, \ldots, \beta_N w_N) : w \in E \}$. Let $\nu$ denote the Haar measure for the compact topological group $E$, and let $\mu$ be the translation of $\nu$ to the coset $F$; i.e., $\mu(\Lambda) = \nu(\beta \Lambda)$. If $Q$ is a nonnegative trigonometric polynomial on $\mathbb{T}^N$ with $\deg(Q) < s_j$ for $1 \leq j \leq N$, then

(a) $Q - Qd\mu \in \mathcal{R}(\mathbb{T}^N)$,
(b) $\hat{Q}(0) = (Qd\mu)_{\gamma}(0)$, and
(c) $\|Qd\mu\| = \|Q\|_1$.

Proof. For $a \in \mathbb{Z}^N$, the Fourier coefficients of $\mu$ and $\nu$ are related by

$$\hat{\mu}(a) = \overline{\beta}^a \hat{\nu}(a).$$

The function $\overline{w}^a$ is a character on $E$ and is identically 1 on $E$ if and only if $a = ks$ for some integer $k$. Since $\nu$ is the Haar measure for $E$, it follows from (2) that

$$\hat{\mu}(a) = \begin{cases} \overline{\beta}^a & \text{if } a = ks \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y_N = \mathbb{Z}^N_+ \cup (-\mathbb{Z}^N_+)$ where $\mathbb{Z}^N_+$ is the positive cone of all $a \in \mathbb{Z}^N$ with $a_j > 0$ for $1 \leq j \leq N$. If $Q$ is a nonnegative trigonometric polynomial on $\mathbb{T}^N$, $\hat{Q}(a) = 0$ except for $a$ in some finite set $X \subset \mathbb{Z}^N$. Thus

$$0 \not\in X + ks \subset Y_N, \quad \text{for } k = \pm 1, \pm 2, \ldots,$$

whenever $s_j > \deg(Q)$ for $1 \leq j \leq N$.

It follows from (3) that

$$\hat{Q}(a) = \sum_{n \in X} \hat{\mu}(n) \hat{\beta}(a - n) = \sum_{k = -\infty}^{\infty} \hat{\beta}(a - ks) \hat{\beta}^{ks}.$$

If $a \not\in Y_N$ and $k \neq 0$, (4) implies $a - ks \not\in X$ so that $\hat{Q}(a - ks) = 0$. Hence, by (5),

$$\hat{Q}(a) = \hat{Q}(a)$$

for all $a \not\in Y_N$, which says that $Q - Qd\mu$ is in $\mathcal{R}(\mathbb{T}^N)$.

Finally, it follows from (4) and (5) that

$$\hat{Q}(0) = \hat{Q}(0),$$

while $Q \geq 0$ implies $\|Qd\mu\| = (Qd\mu)_{\gamma}(0)$ and $\hat{Q}(0) = \|Q\|_1$. Hence, by (6), $\|Qd\mu\| = \|Q\|_1$, and the proof is complete.

Lemma 2. Let $\beta$, $s$, and $\mu$ be defined as in Lemma 1. If $r = (r_1, \ldots, r_N)$ with $0 < r_j < 1$, and $w = (w_1, \ldots, w_N) \in \mathbb{T}^N$, then the Poisson integral of $\mu$ is given by

$$P[d\mu](rw) = \mathcal{P}((rw, \overline{\beta})^s),$$

where $\mathcal{P}$ is the Poisson kernel in one variable,

$$\mathcal{P}(\zeta) = \Re \left[ \frac{1 + \zeta}{1 - \zeta} \right] \quad (\zeta \in \mathcal{U}),$$
Proof. The familiar series expansion \([3, \text{p. 17}]\) for the Poisson kernel is combined with (3) to give
\[
P[\mu](rw) = \sum_{k=-\infty}^{\infty} \beta^k s^k r^1 s^1 \ldots r^N s^N w^k s
= \text{Re} \{[1 + (rw\beta)^s]/[1 - rw\beta]^s]\}.
\]

The next lemma follows immediately from Lemma 2 and well-known properties of the Poisson kernel in one variable \([2, \text{p. 224}]\).

**Lemma 3.** Suppose \(\beta, s, \mu, r, w\) are defined as before, and let
\[
\Gamma(\delta) = \{e^{i\theta} : 2\pi\delta < \theta < 2\pi(1 - \delta)\},
\]
and
\[
M(\delta) = \sup \{\text{Re} e^{i\theta} : 0 < R < 1 \text{ and } e^{i\theta} \in \Gamma(\delta)\}.
\]
Then,
(a) \(P[\mu](rw) \leq M(\delta) < \infty\) if \(0 < \delta < \frac{1}{2}\) and \((\omega\beta)^s \in \Gamma(\delta)\), and
(b) \(\lim_{r \to 1} P[\mu](rw) = 0\) whenever \((w\beta)^s \neq 1\).

**Lemma 4.** Suppose \(p_1, \ldots, p_n\) and \(q_1, q_2, \ldots\) are points of \(T\), and let \(\Gamma\) be a nondegenerate arc on \(T\). To each number \(\eta > 0\), there corresponds an integer \(d > \eta\) and a point \(y \in T\) for which
(a) \((p_k y)^d \in \Gamma\) for \(1 \leq k \leq n\), and
(b) \(1 \notin \{(q_j y)^d : 1 \leq j < \infty\}\).

**Proof.** A classical theorem of Dirichlet \([4, \text{Volume I, p. 235}]\) implies that for \(\epsilon > 0\), there exists an integer \(d > \eta\) for which \(|1 - (p_k y)^d| < \epsilon\) for \(1 \leq k \leq n\). It follows that if \(\epsilon\) is sufficiently small, there is an open arc \(\Lambda\) on \(T\) such that \((p_k y)^d \lambda \in \Gamma\) whenever \(\lambda \in \Lambda\). Since there are uncountably many points in \(T\) with \(\gamma^d \in \Lambda\), such a point can be chosen so that none of the points \((q_j y)^d\) (for \(1 \leq j < \infty\)) is equal to 1.

The proof of Rudin’s boundary value theorem \([3, \text{Theorem 3.5.3}]\) can now be modified to establish Theorem 1.

**Proof of Theorem 1.** Let \(\rho\) be a positive continuous function on \(T^n\) and assume without loss of generality that \(\log \rho > 0\). Choose nonnegative trigonometric polynomials \(Q_n\) on \(T^n\) such that \(\log \rho = \sum_{n=1}^{\infty} Q_n\) on \(T^n\), and, for \(n = 1, 2, \ldots\),
\[
\|Q_n\|_{\infty} \leq 2^{1-n} \|\log \rho\|_{\infty}.
\]

Fix \(\delta\) with \(0 < \delta < \frac{1}{2}\). For each \(n = 1, 2, \ldots\), Lemma 4 implies that there exist an integer \(n > \deg(Q_n)\) and a point \(\gamma_n \in T\) such that
(8) \((p_k y_n)^d_n \in \Gamma(\delta)\) for \(1 \leq k \leq n\),

and

(9) \(1 \notin \{(p_j y_n)^d_n; 1 \leq j < \infty\}\).

Let \(s_n = d_n a \in \mathbb{Z}^N\) and choose \(\beta_n \in T^N\) with \((\beta_n)^a = y_n\). As in Lemma 1, let \(E_n = \{w \in T^N: w s_n = 1\}\), \(F_n = \beta_n^{-1} E_n\), \(\nu_n\) the Haar measure for \(E_n\), and \(\mu_n\) the translation of \(\nu_n\) to the coset \(F_n\). Since \(d_n > \deg(Q_n)\), it follows from Lemma 1 that

\[ Q_n - Q_n \mu_n \in RP(T^N) \quad \text{for} \quad n = 1, 2, \ldots, \]

and

\[ \|Q_n \mu_n\| = \|Q_n\|_1 \quad \text{for} \quad n = 1, 2, \ldots. \]

Let \(d\alpha_n = Q_n \mu_n\). The trigonometric polynomials \(Q_n\) are nonnegative so that

\[ \sum \|\sigma_n\| = \sum \|Q_n\|_1 = \int \sum Q_n = \int \log \rho < \infty, \]

and the series \(\sum \sigma_n\) converges in total variation norm to a positive measure \(\sigma\). Since each \(\sigma_n\) is singular (with respect to the Haar measure of \(T^N\)), so is \(\sigma\). Moreover, if \(\alpha\) lies outside the union of the positive and negative cones of \(\mathbb{Z}^N\), then

\[ \hat{\sigma}(\alpha) = \sum \hat{\sigma}_n(\alpha) = \sum \hat{Q}_n(\alpha) = (\log \rho)^{\gamma}(\alpha), \]

so that \(\log \rho - d\sigma\) is in \(RP(T^N)\). In particular, there exists a holomorphic function \(g\) on \(U^N\) with

\[ \text{Re}\ [g] = P[\log \rho - d\sigma] \]

Define \(f = e^g\). Clearly, \(f\) is in \(H^\infty(U^N)\) since \(\log \rho\) is bounded above and \(\sigma > 0\). Also, \(|f|^* = \rho\ a. e.\ on\ T^N\). In fact, the continuity of \(\rho\) implies

\[ \lim_{r \to 1} P[\log \rho](rw) = \log \rho(w), \quad \text{for all} \quad w \in T^N, \]

hence \(|f|^*(w) = \rho(w)\) if and only if

(10) \(\lim_{r \to 1} P[d\sigma](rw) = 0.\)

Thus, it remains to show that \(s_n\) and \(\beta_n\) have been chosen so that (10) holds for all \(w \in A\).

If \(w \in A_k\), then \(w^a = p_k\) and it follows from the choice of \(s_n\) and \(\beta_n\) that

(11) \((w p_k) s_n = (p_k y_n)^d_n\)

for \(n = 1, 2, \ldots\). Hence (8) implies
(12) \((w^\beta_n)^n \in \Gamma(\delta)\) for \(w \in A_k\) and \(n \geq k\),
while (9) gives

(13) \((w^\beta_n)^n \neq 1\) for \(w \in A\) and \(n = 1, 2, \ldots\).

Since

(14) \(P[\sigma_n](rw) \leq \|Q_n\|_\infty P[\mu_n](rw)\),

it now follows from (7), (12), and Lemma 3 that

\[
P[\sigma_n](rw) \leq 2^{1-n} M(\delta) \log \|\rho\| \to_0
\]

for all \(w \in A_k\) and \(n \geq k\). Hence, for each \(w \in A_k\), the series \(\sum_{n=1}^{\infty} P[\sigma_n](rw)\) converges uniformly in \(r\) for \(0 < r < 1\), and so

\[
\lim_{r \to 1} P[\sigma_n](rw) = \lim_{r \to 1} P[\mu_n](rw) = \lim_{r \to 1} P[\alpha_n](rw)
\]

(15)

\[
< 2^{1-n} M(\delta) \to_0
\]

Finally, (13) and Lemma 3 imply that for each \(k = 1, 2, \ldots\), \(\lim_{r \to 1} P[\mu_n](rw) = 0\) if \(w \in A_k\) and \(n = 1, 2, \ldots\), so that by (15), \(\lim_{r \to 1} P[\sigma_n](rw) = 0\) for all \(w \in A\), and the proof is complete.

Proof of Theorem 2. Let \(\rho\) be continuous on \(T^N\) with \(\log \rho > 0\), and choose nonnegative trigonometric polynomials \(Q_n\) such that \(\log \rho = \Sigma Q_n\), and \(\|Q_n\|_\infty \leq 2^{1-n} \log \|\rho\|_\infty\). For each \(n = 1, 2, \ldots\), choose an integer \(k_n\) such that \(3^{k_n} > \deg(Q_n)\) and let \(d_n = 3^{k_n}\). Let \(E_n = \{w \in T^N : (w_1w_2 \cdots w_N)^{d_n} = 1\}, n = 1, 2, \ldots\) be a Haar measure for \(E_n\), and \(d\sigma_n = Q_n dv_n\). If \(\sigma\) and \(f\) are now defined as in the proof of Theorem 1, it remains to show only that \(\lim_{r \to 1} P[\sigma](rw) = 0\) for all \(w \in A\). This will follow exactly as in Theorem 1 from the following estimate:

(16) \(P[\nu_n](rw) \leq M(1/6)\) for \(w \in A, 0 < r < 1,\) and \(n = 1, 2, \ldots\),

where \(M(1/6)\) is the supremum defined in Lemma 3.

To verify (16), observe that \(\lambda \in S\) if and only if

(17) \(\lambda^{3k} \in \Gamma(1/6)\) for each \(k = 0, 1, 2, \ldots\).

If \(w\) is in \(A\), then \(w_1w_2 \cdots w_N\) is in \(S\); in particular, by (17), \((w_1w_2 \cdots w_N)^{d_n} \in \Gamma(1/6)\) for \(n = 1, 2, \ldots\), and (16) follows from Lemma 3.

3. Zero-free functions. In one variable, the unit circle is an (RL)-set. In fact, the function (1) corresponding to the positive continuous function \(\rho\) on \(T\) has the additional property that it never vanishes in \(U\). Whether the torus \(T^N\) is also an (RL)-set when \(N > 1\) is an open question. However, the next theorem shows that in general the possibility of choosing a zero-free function in the definition
Theorem 3. Suppose \( \rho \) is a positive continuous function on \( T^N \) and \( f \) a zero-free function in \( H^\infty(U^N) \) with \( |f|^* = \rho \) for all \( w \in T^N \). Then \( \log \rho \) is in \( RP(T^N) \).

Definition. If \( f \) is a function on \( U^N \) and \( w \in T^N \), the "slice function" \( f_w \) is defined on the unit disc by

\[
f_w(\lambda) = f(\lambda w) \quad (\lambda \in U).
\]

Proof of Theorem 3. Let \( f \) be any function in \( H^\infty(U^N) \) with \( |f|^* \) identically equal to \( \rho \) on \( T^N \). For each \( w \in T^N \), the slice function \( f_w \) is in \( H^\infty(U) \) and \( |f_w| \) has radial limits satisfying

\[
|f_w|^*(\lambda) = \rho(\lambda w) > 0 \quad \text{for all } \lambda \in T.
\]

Since the radial limit of a nonconstant singular inner function on \( U \) must vanish at some point of \( T \), it follows that the inner factor of \( f_w \) is a Blaschke product [1, Chapter 5]. Hence, for each \( w \in T^N \), the least harmonic majorant of \( \log |f_w| \) is the Poisson integral \( P[\log |f_w|^*] \). This implies, by [3, Theorem 3.3.6], that \( P[\log \rho] \) is the least \( N \)-harmonic majorant of \( \log |f| \) in \( U^N \).

Now if \( f \) is never zero in \( U^N \), \( \log |f| \) is its own least \( N \)-harmonic majorant. Hence \( \log |f| = P[\log \rho] \) and it follows [3, p. 73] that \( \rho \in RP(T^N) \).

The final theorem illustrates more dramatically the difference between the situations in one and several variables. In particular, it implies that if \( A \) is the circle \( \{ (\zeta, \zeta) : |\zeta| = 1 \} \) in \( T^2 \), there exists a positive contiguous function \( \rho \) on \( T^2 \) to which there corresponds no zero-free \( f \in H^\infty(U^2) \) with \( |f|^* = \rho \) a.e. on \( T^2 \) and everywhere on \( A \).

Theorem 4. Suppose \( f \) is a function in \( H^\infty(U^2) \) with

\[
|f|^*(\zeta, \zeta) = 1 \quad \text{for all } \zeta \in T,
\]

and such that \( f(\lambda, \lambda) \) never vanishes for \( \lambda \in U \). Then \( \|f^*_w\|_\infty \geq 1 \) for all \( w \in T^N \) (where \( \|f^*_w\|_\infty \) is the essential supremum of \( |f_w|^* \) on \( T \)).

Proof. Let \( F(\lambda) = f(\lambda, \lambda) \) for \( \lambda \in U \). Then \( F \in H^\infty(U) \), \( F \) has no zeros in \( U \), and by (18), \( |F|^* = 1 \) everywhere on \( T \). In particular, \( F \) is a singular inner function. But, the radial limit of a nonconstant singular inner function must vanish at some point of \( T \). So \( F \) must be constant and, in particular,

\[
|(0, 0)| = |F(0)| = 1.
\]

Now suppose \( \|f^*_w\|_\infty < 1 \) for some \( w \in T^N \). Since \( f_w \in H^\infty(U) \), it follows that \( |f_w(\lambda)| < 1 \) for all \( \lambda \in U \). In particular, \( |f(0, 0)| = |f_w(0)| < 1 \), which contradicts (19).
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