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GROUP ACTIONS ON SPIN MANIFOLDS(1)

BY

G. CHICHILNISKY

ABSTRACT. A generalization of the theorem of V. Bargmann concerning unitary and ray representations is obtained and is applied to the general problem of lifting group actions associated to the extension of structure of a bundle. In particular this is applied to the Poincaré group $\mathcal{P}$ of a Lorentz manifold M. It is shown that the topological restrictions needed to lift an action in $\mathcal{P}$ are more stringent than for actions in the proper Poincaré group $\mathcal{P}_{\uparrow}^{+}$. Similar results hold for the Euclidean group of a Riemannian manifold.

1. **Introduction.** We are concerned with some general techniques related to the question of lifting group actions and with the application of these results to the Poincaré group of a general Lorentz manifold admitting a spin structure and the Euclidean group of a similar Riemannian manifold.

Bargmann [1] considered the general case of lifting a group action from a projective Hilbert space $\hat{H}$ to a Hilbert space H. We shall prove a result which includes this as a special case (provided we accept a result of Wigner [21]). Our method was inspired by the exposition of Simms [19].

Using this result we prove a lifting theorem for a general bundle obtained from an extension of the structural group of a bundle. In doing this we use a theorem of Haefliger [10], who considers conditions for the existence of extensions of structures, and in particular for the existence of spin structures. We establish the number of such possible extensions. Our result includes also the Lorentz case (cf. Bichteler [2]). As far as we know, the problem of lifting group actions on spin manifolds was first suggested by Marsden [13].

We conclude, for example, that if $\mathcal{P}$ is the Poincaré group of a Lorentz manifold M and $\mathcal{P}_{\uparrow}^{+}$ is the component of the identity then there is no topological obstruction to lifting an action of isometries of $\mathcal{P}^{+}$ to an action of corresponding spin transformations (other than M admits a spin structure, i.e., w_2, the second Stiefel-Whitney class should vanish). However, to include the full group $\mathcal{P}$ we require additional topological restrictions, namely that M have a unique spin structure, that is $H^1(M, Z_2) = 0$.

We shall prove the results in a very general context. However, simpler proofs

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are possible in the special case of spin manifolds; cf. Chichilnisky [4] and Chernoff-Marsden [3].

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2. **Lifting group actions.** Let G and F be Lie groups with Lie algebras LG and LF respectively. A *factor set* for (LG, LF) is a bilinear skew symmetric map $\omega: LG \times LG \longrightarrow LF$ such that $\omega(x, [y, z]) + \omega(z, [x, y]) + \omega(y, [z, x]) = 0$ for all $x, y, z \in LG$. The factor set ω is *trivial* if $\omega(x, y) = T([x, y])$ for a linear map T: $LG \longrightarrow LF$. The quotient group of the additive group of factor sets by the trivial factor sets is, by definition, $H^2(LG, LF)$. Observe that other definitions are possible and are not all the same (cf. Jacobson [11]). As usual, we denote the homotopy groups of a space A by $\pi_i(A)$ and the cohomology groups by $H^i(A, C)$, where C is the coefficient group.

Theorem 1. *Let $\mathcal{S}$ and $\mathcal{G}$ be topological groups and $\Pi: \mathcal{S} \longrightarrow \mathcal{G}$ a continuous homomorphism. Let $F = \Pi^{-1}(e)$, e = identity, and suppose that F is an abelian Lie group. Assume further that $\Pi: \mathcal{S} \longrightarrow \mathcal{G}$ has local cross sections. Let ϕ be a continuous homorphism of a (connected and) simply connected finite dimensional Lie group G into $\mathcal{G}$. Assume that $\Pi_i(F) = 0$, $i \geq 2$.*

If $H^2(LG, LF) = 0$ then there exists a homorphism $\bar{\phi}: G \longrightarrow \mathcal{S}$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{S} \arrow[d, "\pi"] \\ G \arrow[ur, "\bar{\phi}"] \arrow[r, "\phi"'] & \mathcal{G} \end{tikzcd}$$

(We call $\bar{\phi}$ a lifting of ϕ.)

If F is discrete then $\bar{\phi}$ is unique. More generally if F lies in the center of $\mathcal{S}$, the number of liftings of ϕ is the number of continuous homomorphisms of G into F.

Remark. The condition of the existence of the cross section is not necessary when $\mathcal{S}$ and $\mathcal{G}$ are Lie groups (cf. Steenrod [21, p. 33]). However, if $\mathcal{S}$ and $\mathcal{G}$ are topological groups, the condition is necessary so that $\mathcal{S}$ will be a bundle over $\mathcal{G}$ and so that obstruction theory may be applied. In the case that $\mathcal{S}$ is finite dimensional this condition is automatically fulfilled (cf. Steenrod [21, p. 218]). There are examples in the infinite dimensional case where there are no local cross sections. On the other hand, D. G. Ebin [6] has proved, for example the existence of local cross sections for some interesting infinite dimensional manifolds related to the topological group of diffeomorphisms of a manifold.

One obtains Bargmann's result in case that $\mathcal{S}$ is the unitary group of a Hilbert space H and $\mathcal{G}$ that of the corresponding projective space $\hat{H}$, $\mathcal{S}$ and $\mathcal{G}$ being equipped with the strong operator topology. These are easily seen to be topological groups (Simms [19] asserts, incorrectly, that they are not: it is the full general linear group of H which is not a topological group). In this case it is easy to see the existence of a local cross section. By Wigner's theorem F is the circle S^1, so $LF = \mathbf{R}$. This case corresponds to that of quantum mechanics.

Proof of Theorem 1. Since $\pi_1(G) = 0$ and $\pi_1(F) = 0$, $i \geq 2$, there is no obstruction to lifting ϕ to a continuous map, say g (cf. Steenrod [20]). Now write, for a, $b \in G$,

$$g(ab) = g(a)g(b)j(a, b)$$

where $j(a, b) = g(b)^{-1}g(a)^{-1}g(ab)$. Thus $j\colon G \times G \longrightarrow F$ and j is continuous. Moreover, j is a factor set for (G, F) as is easily checked. By Simms [19, Theorem 3, p. 15], there is a continuous map $k\colon G \longrightarrow F$ such that $k(xy) = j(x, y)k(x)k(y)$.

If we set $\overline{\phi}(a) = g(a)k(a)$ then it is easily checked that $\overline{\phi}$ is a homomorphism. The assertion on the number of liftings is easy to check. □

We would also like to consider the case in which G is not connected but the component of the identity G_0 of G is simply connected. In subsequent applications we will have $F = \mathbf{Z}_2$ so for simplicity we shall consider only the case of discrete F. However, some additional algebraic assumptions are required.

We shall illustrate the procedure when G has two components. The general case is similar but the hypotheses become more complicated. Let $\pi\colon \mathcal{S} \longrightarrow \mathcal{G}$ be as above. We have

Corollary. *Assume F is a finite abelian group and let $G = G_0 \cup gG_0$ with G_0, the component of the identity, simply connected. Assume that $g^2 = e$. Let $\phi\colon G \longrightarrow \mathcal{G}$ be a continuous homomorphism and suppose that for some $\alpha \in F_{\phi(g)} = \pi^{-1}(\phi(g))$, $\alpha^2 = e$.*

Then there is a unique lifting $\overline{\phi}$ of ϕ which is a homomorphism of G into $\mathcal{S}$ and $\overline{\phi}(g) = \alpha$.

Remark. The cohomology condition on G disappears since $LF = 0$ in this case. One should think of g and α as spatial reflections or time reversal.

Proof. As in Theorem 1 we get a unique lifting $\overline{\phi}$ such that $\overline{\phi}$ on G_0 is a homomorphism and $\overline{\phi}(g) = \alpha$. To show ϕ is a homomorphism, consider

$$\psi\colon G_0 \times G_g \longrightarrow F, \quad \psi(a, b) = \overline{\phi}(b)^{-1}\overline{\phi}(a)^{-1}\overline{\phi}(ab).$$

For $a = e$, $b = g$ we obtain $\psi(e, g) = e$ so ψ is always e by continuity. Next consider ψ on $G_g \times G_g$,

$$\psi\colon G_g \times G_g \longrightarrow F, \qquad \psi(a, b) = \overline{\phi}(b)^{-1}\overline{\phi}(a)^{-1}\overline{\phi}(ab).$$

For $a = b = g$ we obtain $\psi(g, g) = \alpha^{-1}\alpha^{-1}\phi(e) = e$. Thus ψ is constant on $G \times G$ and so $\overline{\phi}$ is a homomorphism. □

If $F = \mathbf{Z}_2$ and β is the other element in $F_{\phi(g)}$ then if, as above, F lies in the center of $\mathcal{S}$, then $\beta^2 = e$ also, so we have two possible liftings of ϕ.

For a corresponding result in the quantum mechanical case, see Wigner [22].

3. **Extension of structure.** Let B be a principal fiber bundle over a manifold M with structural group a Lie group G. Let $p: H \longrightarrow G$ be a homomorphism of a Lie group H onto G. By a p *extension of structure* we mean a bundle $\overline{B}$ over M with structural group H and a map $S: \overline{B} \longrightarrow B$ such that the following diagram is commutative:

$$\begin{tikzcd} B \times H \arrow[r, "S \times p"] \arrow[d, "r"'] & B \times G \arrow[d, "r"] \\ B \arrow[r, "S"] \arrow[dr] & B \arrow[d] \\ & M \end{tikzcd}$$

where r denotes right multiplication. This means, in effect, that for $s \in \overline{B}$, both s and $S(s)$ lie over the same point, and that S restricted to a fiber is equivalent to p. Two reductions are equivalent if the corresponding bundles over B are equivalent (cf. Steenrod [21]).

In case B is the principal tangent bundle of an oriented Riemannian manifold with structural group $G = \mathrm{SO}(n)$, and $H = \mathrm{Spin}(n)$ is the universal covering group $(n > 2)$, we call $\overline{B}$ a *spin structure*. This is the definition in Milnor [16]. See also Crumeyrolle [5] and Chernoff-Marsden [3] for equivalent definitions. The definition may also be given in terms of associated vector bundles as in Palais [18, p. 92]. Thus if $\mathrm{Spin}(n)$ acts faithfully on a complex space $\mathbf{C}^k$ (i.e., we have a faithful representation), we can construct an associated vector bundle of spinors. (For example for $n = 3$, $\mathrm{Spin}(3) = \mathrm{SU}(2)$ acts on $\mathbf{C}^2$ for "spin ½".) See Steenrod [21]. The existence of a spin structure then means that we can find a system of spin coordinate changes which cover the corresponding coordinate changes of tangent vectors. To include reflections we can, if we wish, enlarge the structural group to $\mathrm{O}(n) = \mathrm{SO}(n) \times \mathbf{Z}_2$ with the replacement of $\mathrm{Spin}(n)$ by $\mathrm{Spin}(n) \times \mathbf{Z}_2$.

In case M is an oriented time oriented Lorentz manifold the structural group for the tangent bundle is the restricted Lorentz group $L_+^{\uparrow}$ with universal covering group $\mathrm{SL}(2, \mathbf{C})$. Two associated vector bundles are $\mathbf{C}^2$ (Weyl spinors) and $\mathbf{C}^4$ (dirac spinors). This case is topologically very similar to the Riemannian case.

Mathematically it is most convenient to work with the principal bundles. At the end, however, we will translate the result into the language of vector bundles.

Haefliger in [10] found necessary and sufficient conditions for the existence of a spin structure. P. Chernoff has given an elementary proof; cf. [3]. The same results are valid in the Lorentz case, cf. Bichteller [2].

Theorem 2 (Haefliger). *Let B be a principal fiber bundle over a manifold M with structural group G, a connected Lie group.*

Let $p: H \longrightarrow G$ be the universal covering group of G.

Let $F = p^{-1}(e)$. The obstruction to having a p-extension of structure is a certain element of $H^2(M, F)$.

If one examines the definitions we see that finding an extension means lifting f to $\bar{f}: M \longrightarrow C_H$ such that the following diagram commutes:

$$\begin{tikzcd} & C_H \arrow[d] \\ M \arrow[ur, "\bar{f}"] \arrow[r, "f"'] & C_G \end{tikzcd}$$

where C_H and C_G are the classifying spaces for H and G. This is an obstruction problem and the obstruction lies in $H^{n+1}(M, \pi_n(\widetilde{F}))$, $n > 1$, where $\widetilde{F}$ is the fiber of $C_H \longrightarrow C_G$. But since $\pi_n(F) = 0$, $n > 2$, the obstruction lies in $H^2(M, \pi_1(\widetilde{F})) = H^2(M, F)$. Also, the number of liftings equals the number of elements in $H^1(M, \pi_1(F)) = H^1(M, F)$. (Cf. Steenrod [21].)

So we have that: *The number of inequivalent extensions equals the number of elements in $H^1(M, F)$.*

4. **Lifting bundle maps.** Let now G, H be Lie groups (not necessarily connected) and $p: H \longrightarrow G$ a homomorphism with $F = p^{-1}(e)$ commutative. Let B be a bundle over M with structural group G and $S: \overline{B} \longrightarrow B$ an extension of structure.

Using Čech cohomology one sees that we can represent the bundle $S: \overline{B} \longrightarrow B$ as an element c of $H^1(B, F)$ (see the proof of the following lemma).

Lemma. *A bundle map $f: B \longrightarrow B$ lifts to a bundle map $\bar{f}: \overline{B} \longrightarrow \overline{B}$ iff $f^*(c) = c$. The number of such liftings equals the number of elements of F.*

Proof. Let $\{U_i\}$ be a trivializing open cover of B for the bundle $S: \overline{B} \longrightarrow B$; thus $c \in H^1(B, F)$ assigns to $U_i \cap U_j$ a coordinate transition map $g_{ij}: U_i \cap U_j \longrightarrow F$. Locally $\overline{B} \mid U_i = U_i \times F$. Then $\bar{f}$ is defined by $\bar{f}(x, a)_{U_i} = (f(x), a)_{U_i}$. The fact that $f^*(c) = c$ means just that $\bar{f}$ is well-defined.

To show that $\bar{f}$ is a bundle map we must show that, on the fiber $\overline{B}_x$ over $x \in M$, $\bar{f}$ acts as an element of H. Consider $S^{-1}(B_x \cap U_i)$ which will be part of the fiber $\overline{B}_x$. Now $\bar{f}$ acts on $S^{-1}(B_x \cap U_i)$ by some $h_i \in H$ since f acts on B_x by a specific $g \in G$. By continuity, $\bar{f}$ acts on $\overline{B}_x$ by a specific $h \in H$ (note $\overline{B}_x$ need not be connected).

We could as well have set $\overline{f}(x, a)_{U_i} = (f(x), b \cdot a)_{U_i}$ for fixed $b \in F$. This would give another bundle map since F is abelian (Steenrod [21, p. 40]). The lemma follows.

Thus we obtain the following result.

Theorem 3. *Let B, $\overline{B}$ be as above, and let $\mathcal{G}$ be a topological group of bundle mappings $f: B \longrightarrow B$ such that $f^*(c) = c$. Then the bundle mappings $\overline{f}: \overline{B} \longrightarrow \overline{B}$ covering elements of $\mathcal{G}$ form a group and there is a projection $\pi: \mathcal{S} \longrightarrow \mathcal{G}$ with fiber F.*

If there exists a local cross section of $\pi: \mathcal{S} \longrightarrow \mathcal{G}$, we can apply the results of §1 to this situation. Theorem 3 plays the same role in Theorem 1 that Wigner's theorem plays in Bargmann's theorem. In applications below, $\mathcal{G}$ will often be a Lie group and F discrete, so that $\mathcal{S}$ will be a Lie group as well as local cross sections will be automatic.

If $\mathcal{G}$ lies in the (path) connected component of all (homeomorphic) bundle maps $f: B \longrightarrow B$ then $f^*(c) = c$ because f is homotopic to the identity. In general, $f^*(c)$ need not be c as simple examples show (see Chichilnisky [4]). A sufficient condition is that $H^1(B, F)$ has exactly one element representing a p-extension of structure which means that the p-extension is unique. By Theorem 2 (in case G is connected) this is the same as $H^1(M, F) = 0$. ($c \in H^1(B, F)$ represents a p-extension of structure if, for each $x \in M$, $i^*(c) \in H^1(B_x, F)$ represents the covering $p: H \longrightarrow G$ where $i: B_x \longrightarrow B$ is the inclusion).

Thus, *if B has a unique p-extension of structure then any bundle map $f: B \longrightarrow B$ may be lifted to a bundle map $\overline{f}: \overline{B} \longrightarrow \overline{B}$.*

5. Spin manifolds, and the Euclidean and Poincaré groups. Let us now transcribe the above results to the special case of a spin manifold. We treat the Euclidean case first. Thus, let M be an oriented Riemannian manifold. Let Spin(n) (where $n = \dim M$) act faithfully on a complex vector space $\mathbf{E}$. If M is a spin manifold then we obtain an associated vector bundle $\pi: E \longrightarrow M$ with fiber (locally) equal to $\mathbf{E}$. In this case $F = \mathbf{Z}_2$ and $\mathcal{G}$ is a Lie group, by [17].

Let $f: M \longrightarrow M$ be an orientation preserving isometry, so Tf, the tangent of f, induces a bundle map on the principal tangent bundle. The lift $\overline{Tf}$ to the principal Spin(n) bundle induces a map $\tilde{f}: E \longrightarrow E$ such that $\tilde{f}$ covers f:

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & M \end{tikzcd}$$

and, moreover, in local vector bundle charts about $m \in M$ and $f(m)$, respectively, the restriction $\tilde{f} | E_m$ is an element of $\mathrm{Spin}(n)$ covering $Tf(m) \in \mathrm{SO}(n)$. Conversely, if we have such an $\tilde{f}$ then $\tilde{f}$ induces a lift of Tf on the principal level. Let us call $\tilde{f}$ a spinor transformation of f.

We briefly mention that this problem arises naturally when symmetry groups of quantum mechanical systems with spin are studied; the Hilbert space $\mathbf{H}$ being the L_2 sections of the bundle E. Namely, $\tilde{f}$ represents the transformation of spinors corresponding to the coordinate transformation f; the transformation of states becomes $\psi \longrightarrow \tilde{f} \circ \psi \circ f^{-1}$, a unitary transformation, $\psi \in \mathbf{H}$. See Chernoff-Marsden [3] for more details.

If E is replaced by $\hat{E}$, the bundle obtained by taking the projective space $\hat{E}_x$ over each point, then it is not hard to see that any such f will have a lifting. (In this case, however, Theorem 3 is more special than Wigner's theorem and does not follow from it.) Also we cannot use Bargmann's theorem to lift actions of isometries because sections of $\hat{E}$ are not the same as elements of $\hat{\mathbf{H}}$.

Thus we have proven the following: *if ϕ is an action* (*i.e.*, *representation*) *of a simply connected and connected group G on a spin manifold M by isometries, then ϕ lifts uniquely to an action $\tilde{\phi}$ consisting of spinor transformations.*

Note that each $\phi(g)$ for $g \in G$ is necessarily orientation preserving and $T\phi(g)^*c = c$ since G is connected.

Similar results hold for isometries of a Lorentz manifold. It is easy to see that the set of isometries $\mathfrak{I}$ of a Lorentz manifold form a Lie group. Using canonical coordinates, we see that $\{X | \, L_X g = 0\}$, L_X = Lie derivative with respect to the vector field X is contained in the Lie algebra of $\mathfrak{G} = \mathrm{SO}(3, 1)$, where g is the Lorentz metric. $\mathfrak{G}$ is finite dimensional, and so using a result of Palais (cf. Kobayashi-Nomizu [12, p. 307]) it follows immediately that $\mathfrak{I}$ is a Lie group.

It can be argued (cf. Geroch [8]) that a physically acceptable Lorentz manifold M should admit a spin structure; $w_2 = 0$. Geroch [8] has shown (and it is not hard) that this is equivalent to parallelizability of M.

We now examine a theorem that requires a unique spin structure. Consider first the Riemannian case. If we have an extension from $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$ we also have an extension from $\mathrm{O}(n) \approx \mathrm{SO}(n) \times \mathbf{Z}_2$ to $\mathrm{Spin}(n) \times \mathbf{Z}_2$. We shall thus enlarge the structural groups so that orientation reversing isometries may be considered. The element of the cohomology group representing the reduction does not change. By the *Euclidean group* $\mathfrak{E}$ of M we mean the group of all isometries of M. It is a Lie group [17]. Let $\mathfrak{E}_0$ be the identity component of $\mathfrak{E}$. We have for a spin manifold M:

Any $f \in \mathfrak{E}_0$ may be lifted to exactly two spinor transformations $\tilde{f}$.

Note. If G is not simply connected this is not true. Consider, for example, the action of $\mathrm{SO}(3)$ on $M = \mathbf{R}^3$. Here the spin bundle is $\mathbf{R}^3 \times \mathbf{C}^2$, or as a principal bundle, $\mathbf{R}^3 \times \mathrm{SU}(2)$.

If M has a unique spin structure, i.e., $H^1(M, \mathbf{Z}_2) = 0$, then any $f \in G$ lifts to exactly two spinor transformations. The corollary to Theorem 1 also applies. For example, suppose $\mathrm{SO}(n) \times \mathbf{Z}_2 = \mathrm{O}(n)$ acts on M and M has a unique spin structure. Fix $g \in \mathrm{O}(n)$, $g^2 = e$. Thus $\phi(g)^2 = e$ and if $\phi(g)$ is like a reflection we would expect that the two liftings of $\phi(g)$ to spinor transformations also have squares $=$ e. Then the action lifts as in the corollary (to two different actions of spinor transformations).

In the Lorentz case we similarly enlarge the structural group from $L_+^\uparrow$ to L and define the Poincaré group $\mathcal{P}$ as the group of all isometries with $\mathcal{P}_\uparrow^+$, the component of the identity. As above, it follows that $\mathcal{P}$ is a finite dimensional Lie group. Elements of $\mathcal{P}_\uparrow^+$ always induce spinor transformations on M but if we want every $f \in \mathcal{P}$ to induce a spinor transformation, we need to require that M have a unique spin structure; $H^1(M, \mathbf{Z}_2) = 0$. Similarly we have results for actions of groups as above (there is a more or less obvious analogue of the corollary to Theorem 1 which would hold for four component groups like the standard Poincaré group).

BIBLIOGRAPHY

1. V. Bargmann, *On unitary ray representations of continuous groups*, Ann. of Math. (2) 59 (1954), 1–46. MR 15, 397.

2. K. Bichteler, *Global existence of spin structures for gravitational fields*, J. Mathematical Phys. 9 (1968), 813–815.

3. P. Chernoff and J. Marsden, *Hamiltonian systems and quantum mechanics* (in preparation).

4. G. Chichilnisky, *Group actions on spin manifolds*, Thesis, Berkeley, 1970.

5. A. Crumeyrolle, *Structures spinorielles*, Ann. Inst. H. Poincaré Sect. A 11 (1969), 19–55. MR 42 #6737.

6. D. G. Ebin, *The manifold of Riemannian metrics*, Proc. Sympos. Pure Math., vol. 15, Amer. Math. Soc., Providence, R. I., 1970, pp. 11–40. MR 42 #2506.

7. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956. MR 17, 1040.

8. R. Geroch, *Spinor structure of space-times in general relativity*. I, J. Mathematical Phys. 9 (1968), 1739–1744. MR 38 #3019.

9. M. J. Greenberg, *Lectures on algebraic topology*, Benjamin, New York, 1967. MR 35 #6137.

10. A. Haefliger, *Sur l'extension du groupe structural d'un espace fibré*, C. R. Acad. Sci. Paris 243 (1956), 558–560. MR 18, 920.

11. N. Jacobson, *Lie algebras*, Interscience Tracts in Pure and Appl. Math., no. 10, Interscience, New York, 1962. MR 26 #1345.

12. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*. Vol. I, Interscience Tracts in Pure and Appl. Math., no 15, Interscience, New York, 1969. MR 38 #6501.

13. J. Marsden, *Hamiltonian systems with spin*, Canad. Math. Bull. 12 (1969), 203–208. MR 40 #1109.

14. J. Milnor, *The representation rings of some classical groups*, Mimeographed notes, Princeton University, Princeton, N. J., 1963.

15. ———, *Notes on characteristic classes*, Mimeographed notes, Princeton University, Princeton, N. J., 1957.

16. ———, *Spin structures on manifolds*, Enseignement Math. (2) 9 (1963), 198–203. MR 28 #622.

17. S. B. Myers and N. Steenrod, *The group of isometries of a Riemannian manifold*, Ann. of Math. (2) 40 (1939), 400–416.

18. R. S. Palais, *Seminar on the Atiyah-Singer index theorem*, Ann. of Math. Studies, no. 57, Princeton Univ. Press, Princeton, N. J., 1965. MR 33 #6649.

19. D. J. Simms, *Lie groups and quantum mechanics*, Lecture Notes in Math., no. 52, Springer-Verlag, Berlin and New York, 1968. MR 38 #904.

20. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966. MR 35 #1007.

21. N. E. Steenrod, *The topology of fibre bundles*, Princeton Math. Series, vol. 14, Princeton Univ. Press, Princeton, N. J., 1951. MR 12, 522.

22. E. P. Wigner, *On unitary representations of the inhomogeneous group*, Ann. of Math. (2) 40 (1939), 149–204.

23. ———, *Unitary representations of the inhomogeneous Lorentz group including reflections*, Group Theoretical Concepts and Methods in Elementary Particle Physics (Lectures Istanbul Summer School Theoret. Phys., 1962), Gordon and Breach, New York, 1964, pp. 37–80. MR 30 #1210.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

Current address: Department of Economics, University of California, Berkeley, California 94720