THE ACTION OF THE AUTOMORPHISM GROUP OF $F_2$
UPON THE $A_6$- AND $PSL(2, 7)$-DEFINING SUBGROUPS OF $F_2$

BY

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ABSTRACT. In this paper is described a graphical technique for determining
the action of the automorphism group $\Phi_2$ of the free group $F_2$ of rank 2 upon
those normal subgroups of $F_2$ with quotient groups isomorphic to $G$, where $G$
is a group represented faithfully as a permutation group. The procedure is applied
with $G = PSL(2, 7)$ and $A_6$ (the case $G = A_5$ having been treated in an earlier
paper) with the following results:

Theorem 1. $\Phi_2$ acts upon the 57 subgroups of $F_2$ with quotient isomorphic
to $PSL(2, 7)$ with orbits of lengths 7, 16, 16, and 18. The action of $\Phi_2$ is that
of $A_6$ in one orbit of length 16, and of symmetric groups of appropriate degree
in the other three orbits.

Theorem 2. $\Phi_2$ acts upon the 53 subgroups of $F_2$ with quotients isomorphic
to $A_6$ with orbits of lengths 10, 12, 15, and 16. The action is that of full sym-
metric groups of appropriate degree in all orbits.

1. Introduction. In [1], the author calculated the action of $\Phi_2$, the automorphism
group of $F_2$, (the free group of rank 2) upon the 19 $A_5$-defining subgroups of $F_2$
(i.e., the normal subgroups $N$ of $F_2$ with $F_2/N \cong A_5$), where $A_5$ is the alternat-
ing group of degree 5. The action of $\Phi_2$ upon these subgroups was found to have
orbits of lengths 9 and 10, and $\Phi_2$ acts as the full symmetric group (of appropriate
degree) upon each orbit.

In this paper, the technique of [1] is applied, with some modifications, to the
$PSL(2, 7)$- and $A_6$-defining subgroups of $F_2$ (of which there are respectively 57 and
53) to arrive at the following results:

Theorem 1. The action of $\Phi_2$ upon the $PSL(2, 7)$-defining subgroups of $F_2$ has
orbits of lengths 7, 16, 16, and 18. $\Phi_2$ acts as a full alternating group upon one
orbit of length 16 and as a full symmetric group upon the other three.

Theorem 2. The action of $\Phi_2$ upon the $A_6$-defining subgroups of $F_2$ has orbits
of lengths 10, 12, 15, and 16. $\Phi_2$ acts as a full symmetric group upon each orbit.

2. Calculation technique. In this section, we describe without proofs the cal-
culation technique developed in [1].
Let $G$ be a transitive permutation group of degree $j$ such that $G$ can be generated by the ordered pair of permutations $(r, t)$. Assume that $G$ acts upon the set $\{1, \ldots, j\}$. Let $F_2$ have free generators $a$ and $b$. Define $H_i = \{\text{all words } W(a, b) \in F_2\}$ the permutation $W(r, t)$ fixes $i$, $i = 1, \ldots, j$. Then $F_2/\bigcap_{i=1}^j H_i \cong G$; that is, $N(r, t) = \bigcap_{i=1}^j H_i$ is a $G$-defining subgroup of $F_2$.

**Proposition 1.** Let $\phi \in \Phi_2$. Write $W_a(a, b) = \phi(a)$, $W_b(a, b) = \phi(b)$. Then $W_a(r, t)$ and $W_b(r, t)$ generate $G$ and $N(r, t) \subseteq \phi(N(W_a(r, t), W_b(r, t)))$.

(See [1] for proof.)

Draw a graph $\Gamma(r, t)$ as follows: Pick two different colors, and call one "color $a$" and the other "color $b$". Label $j$ points with the numbers $1, \ldots, j$. Connect vertex $i$ to vertex $k$ by an edge of color $a$ oriented from vertex $i$ to vertex $k$ if and only if $r(i) = k$. Form $r$-connections with color $b$ similarly. Call $\Gamma(r, t)$ a (2-color) $G$-defining graph. Two such graphs are said to be isomorphic if there is a 1-1 correspondence between the sets of vertices of the two graphs and between their sets of edges which preserves color and the relations "is the initial vertex of" and "is the terminal vertex of".

**Proposition 2.** Let $\Gamma(r, t)$ and $\Gamma(r_1, t_1)$ be isomorphic $G$-defining graphs. Then $N(r, t) = N(r_1, t_1)$.

(See [1].)

Let $\Phi_2$ act upon the 2-color $G$-defining graphs in this fashion: if $\phi(a) = W_a(a, b)$ and $\phi(b) = W_b(a, b)$, then $\phi(\Gamma(r, t)) = \Gamma(W_a(r, t), W_b(r, t))$. From Propositions 1 and 2, if $\phi(\Gamma(r, t)) = \Gamma(r_1, t_1)$, then $N(r, t) = \phi(N(r_1, t_1))$.

The converse to Proposition 2 is not in general true; however, the converse fails only if the action of $\Phi_2$ upon the set of $G$-defining graphs is imprimitive.

Proposition 1 is used with respect to a generating set of $\Phi_2$. $\Phi_2$ has generators $U(a \rightarrow ab, b \rightarrow b)$, $P(a \rightarrow b, b \rightarrow a)$, and $\sigma(a \rightarrow a^{-1}, b \rightarrow b)$. The objects we are permuting in this paper are normal subgroups of $F_2$, so $\sigma$ is superfluous, since it differs by an inner automorphism from a word in $U$ and $P$: namely, $\sigma = PUPU^{-1}PU \cdot P \sigma U \sigma P$.

3. General computing plan. It is desired to compute the action of $\Phi_2$ upon the $G$-defining subgroups of $F_2$.

First, pick a generating pair $(r, t)$ for $G$. Then systematically apply words in $P$ and $U$ to $\Gamma(r, t)$ until it is not possible to produce $G$-defining graphs which are not isomorphic to previously produced graphs. Such a collection of $G$-defining graphs is an orbit of the action of $\Phi_2$ upon the full set of $G$-defining graphs.

If possible, find a generating pair $(r_1, t_1)$ for $G$ such that $\Gamma(r_1, t_1)$ is not isomorphic to any graph previously produced and generate its orbit with respect to the action of $\Phi_2$ as described in the previous paragraph.
Continue this procedure until all orbits of $G$-defining graphs, together with the effect of $P$ and $U$ upon them, have been found. If $m$ $G$-defining graphs have been found, consider $P$ and $U$ as permutations of degree $m$ and draw $\Gamma(U, P)$. It is a graph with as many connected components as there are orbits in the action of $\Phi_2$ upon the $G$-defining graphs. What can be done with these graphs will be described in the following sections.

4. Application to $\text{PSL}(2,7)$. Here we sketch the application of §3 with $G = \text{PSL}(2,7)$, omitting details of calculation.

$\text{PSL}(2,7)$ is a simple group of order 168 which can be represented as the group of linear fractional transformations $x \mapsto (ax + b)/(cx + d)$ of GF(7) with $a, b, c, d$ in GF(7) and $ad - bc = 1$. As such, $\text{PSL}(2,7)$ is a transitive permutation group on the set GF(7) U {\infty}.

Any pair of noncommuting elements of order 7 in $\text{PSL}(2,7)$ is a generating set. (Order in $\text{PSL}(2,7)$ is determined by the value of $(a + d)^2$; if $g: x \mapsto (ax + b)/(cx + d) \in \text{PSL}(2,7)$, then $o(g) = 2$ iff $a + d = 0$, $o(g) = 3$ iff $(a + d)^2 = 1$, $o(g) = 4$ iff $(a + d)^2 = 2$, and $o(g) = 7$ iff $(a + d)^2 = 4$.) Thus, $r: x \mapsto x + 1$ and $t: x \mapsto x/(x + 1)$ generate $\text{PSL}(2,7)$. In permutation form, these elements are $(0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6)$ and $(1 \ 4 \ 5 \ 2 \ 3 \ 6 \ \infty)$, respectively. The length of the orbit of $\Gamma(r, t)$ is 16.

$\text{PSL}(2,7)$ is also generated by any choice of an element of order 7 for $r$ and an element of order 2 for $t$. Only one graph of this type, namely $\Gamma((1 \ 6 \ 3 \ 2 \ 5 \ 4), (0 \ 3)(1 \ 5)(2 \ 6)(4 \ 5))$, appeared in the first orbit. The pair $r_1: x \mapsto x + 1$, $t_1: x \mapsto (x - 1)/(2x - 1)$, correspond to a graph, $\Gamma((0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6), (0 \ 1)(2 \ 5)(3 \ 6)(4 \ \infty))$, not isomorphic to that one. The orbit of $\Gamma(r_1, t_1)$ has length 18.

The third orbit, of length 16, was generated starting with another pair $x \mapsto x + 1, x \mapsto (x + 4)/(3x - 1)$ of the "7 and 2" type.

Two elements of order 4 generate $\text{PSL}(2,7)$ if their squares do not commute. A search for pairs of this type for which the corresponding graphs had not been previously produced resulted in a generating pair for the fourth orbit, which is of length 7.

The effect of $P$ and $U$ is recorded in Table 1. The numbers were assigned to the graphs in order of their production.

The action of $\Phi_2$ on each orbit is primitive. (§6 contains a discussion of primitivity.) Further, the graphs for the two orbits of length 16 are not isomorphic. Thus, if $U$ is replaced by $U^{-1}$ in the heading of Table 1, the table describes the action $\Phi_2$ upon the $\text{PSL}(2,7)$-defining subgroups of $F_2$, since there are exactly 57 $\text{PSL}(2,7)$-defining subgroups in $F_2$ (P. Hall [2]).

A theorem of Jordan (as quoted in Wielandt [3, p. 40]) states that if a transitive, primitive permutation group $G$ of degree $n$ contains a $p$-cycle, where $p$ is a prime $\leq n - 3$, then $G$ contains $A_n$. In each of the four transitive groups generated by $U$ and $P$ indicated in Table 1, the hypotheses are satisfied (the last part by $U^{12}, U^{12}$,
$U^{12}$, and $U^4$, respectively). Hence, $U$ and $P$ generate $A_{16}$ in orbit 1 and $S_{18}$, $S_{16}$, and $S_7$ in the remaining orbits.

5. Application to $A_6$. Here we sketch the application of §3 with $G = A_6$, omitting details of calculation.

The theorem of Jordan quoted in the previous section was used to find starting pairs for the orbits. The starting pairs used were \(((1 \ 2 \ 3 \ 4 \ 5), (1 \ 2 \ 6)), ((1 \ 2 \ 3 \ 4 \ 5), (1 \ 3 \ 6)), ((1 \ 2)(3 \ 4 \ 5 \ 6), (1 \ 3)(5 \ 6)), and ((1 \ 2 \ 3 \ 4 \ 5), (1 \ 6)(2 \ 3)) for orbits of lengths 24, 30, 32, and 20, respectively. The action of $\Phi_2$ upon each of these orbits is imprimitive and reduces to action upon 12, 15, 16, and 10 blocks, all of length 2. The reduced action is primitive in all cases, and the theorem of Jordan shows that the action is that of full symmetric groups in all cases. Since $12 + 15 + 16 + 10 = 53$, all $A_6$-defining subgroups of $F_2$ have been treated. (See Tables 2–4 for the results of calculations.)

6. Primitivity. The imprimitivity of the action of $\Phi_2$ upon the $A_6$-defining graphs is easy to recognize once the graphs $\Gamma(U, P)$ are drawn, since these graphs have obvious nontrivial self-isomorphisms, which are indicated in Table 3. (However, the absence of nontrivial self-isomorphisms does not imply primitivity. For example, if $r = (1 \ 2)(3 \ 4 \ 5 \ 6), t = (2 \ 3 \ 5)$, the group generated by $r$ and $t$—a metacyclic group of order 36—is primitive on \{1, 2, 3, 4, 5, 6\}, but $\Gamma(r, t)$ has no nontrivial self-isomorphism.)

In the first orbit of the action of $\Phi_2$ upon the $PSL(2,7)$-defining graphs, the cyclic subgroup generated by $U$ acts intransitively with orbits of length 2, 3, 4, and 7. Since a block of $\Phi_2$ is a block of this subgroup, it is helpful to find the nontrivial blocks of $U$. They are the orbits, all possible unions of orbits, and the sets \{8, 10\}, \{9, 11\}, \{8, 10, 15\}, \{9, 11, 16\}, \{8, 10, 16\}, \{9, 11, 15\}. Since the size of a block of a transitive group must divide the degree of the group, the only possibilities for nontrivial blocks of $\Phi_2$ are the sets \{8, 10\}, \{9, 11\}, \{15, 16\}, and \{8, 9, 10, 11\}. A complete block system cannot be formed from among these, so the action is primitive.

The primitivity of the action for the remaining three orbits for $PSL(2,7)$ and the four reduced orbits for $A_6$ can be shown in a similar fashion.

7. Conclusion. A question suggested by the above results is whether there are groups $G$ for which the action of $\Phi_2$ on the $G$-defining subgroups of $F_2$ is not described by full alternating or symmetric groups. The answer is affirmative; $F_2$ contains 6 $C_4$-defining subgroups ($C_4 =$ cyclic group of order 4) and $\Phi_2$ acts transitively upon this set. However, the permutation group of degree 6 which describes this action faithfully has $S_3$ as a quotient group and therefore cannot be $A_6$ or $S_6$. 
<table>
<thead>
<tr>
<th>Orbit</th>
<th>$U$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>(1 2 3 4)(5 6 7)</td>
<td>(2 5)(3 7)</td>
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Table 2. The action of $\Phi_2$ on the $A_6$-defining graphs

<table>
<thead>
<tr>
<th>Orbit</th>
<th>$U$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1 2 3)(4 5 6 7 8)(9 10 11 12)</td>
<td>(1 4)(2 9)(3 13)(5 10)(6 18)(7 22)</td>
</tr>
<tr>
<td></td>
<td>(28 29 30)</td>
<td>(15 17)(18 28)(21 30)(26 29)</td>
</tr>
<tr>
<td></td>
<td>(27 28 29 30)(31 32)</td>
<td>(18 31)(19 29)(26 30)</td>
</tr>
<tr>
<td></td>
<td>(17 18 19 20)</td>
<td>(11 20)(14 18)</td>
</tr>
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</table>
Table 3. Blocks of the action of $\Phi_2$ upon $A_6$-defining graphs

\[
\begin{array}{c|cccc}
\text{orbit} & 1 & 2 & 3 & 4 \\
\hline
A & 1, 23 & 1, 11 & 1, 32 & 1, 2 \\
B & 2, 24 & 2, 9 & 2, 31 & 3, 8 \\
C & 3, 22 & 3, 10 & 3, 28 & 4, 9 \\
D & 4, 15 & 4, 25 & 4, 29 & 5, 10 \\
E & 5, 16 & 5, 26 & 5, 30 & 6, 11 \\
F & 6, 17 & 6, 27 & 6, 27 & 7, 12 \\
G & 7, 13 & 7, 23 & 7, 18 & 13, 17 \\
H & 8, 14 & 8, 24 & 8, 19 & 14, 18 \\
I & 9, 19 & 12, 22 & 9, 20 & 15, 19 \\
J & 10, 20 & 13, 19 & 10, 21 & 16, 20 \\
K & 11, 21 & 14, 20 & 11, 17 & \\
L & 12, 18 & 15, 21 & 12, 26 & \\
M & 16, 29 & 13, 22 & \\
N & 17, 30 & 14, 23 & \\
O & 18, 28 & 15, 24 & \\
P & & & 16, 25 & \\
\end{array}
\]

Table 4. Action of $\Phi_2$ upon $A_6$-defining subgroups

\[
\begin{array}{c}
\text{orbit} \\
1 & (A B C)(D E F G H)(I J K L) \\
2 & (A B C)(D E F G H)(I J K L) \\
3 & (A B)(C D E F)(G H I J K) \\
4 & (B C D E F)(G H I J) \\
\hline
U^{-1} & (A D)(B I)(C G)(E J)(F L) \\
& (A D)(C I)(E M)(F J)(H K) \\
& (M N O) \\
& (L M N O P) \\
& (A B)(D G)(E J) \\
\end{array}
\]
We then ask: For which $G$ is the action of $\Phi_2$ described by full alternating or symmetric groups? The results for $A_5 = \operatorname{PSL}(2,5)$, $\operatorname{PSL}(2,7)$, and $A_6$ suggest a search through the sequences $A_n$ and $\operatorname{PSL}(2, p)$ ($p$ prime) of simple groups, particularly the latter, since the number of $\operatorname{PSL}(2, p)$-defining subgroups of $F_2$ is given for all $p$ in [2]. Since the number of $\operatorname{PSL}(2, 11)$-defining subgroups of $F_2$ is already 254, hand calculation should give way to the use of an electronic computer.

Also, is orbit size predictable without such extensive calculation? What is the structure of the intransitive groups whose transitive constituents have been calculated?

The above questions provide material for further research.

REFERENCES


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