PIECEWISE MONOTONE POLYNOMIAL APPROXIMATION

BY

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ABSTRACT. Given a real function \( f \) satisfying a Lipschitz condition of order 1 on \([a, b]\), there exists a sequence of approximating polynomials \( \{P_n\} \) such that the sequence \( E_n = \|P_n - f\| \) (sup norm) has order of magnitude \( 1/n \) (D. Jackson). We investigate the possibility of selecting polynomials \( P_n \) having the same local monotonicity as \( f \) without affecting the order of magnitude of the error. In particular, we establish that if \( f \) has a finite number of maxima and minima on \([a, b]\) and \( S \) is a closed subset of \([a, b]\) not containing any of the extreme points of \( f \), then there is a sequence of polynomials \( P_n \) such that \( E_n \) has order of magnitude \( 1/n \) and such that for \( n \) sufficiently large \( P_n \) and \( f \) have the same monotonicity at each point of \( S \). The methods are classical.

Let \( C[a, b] = \{ f: f \text{ continuous on } [a, b], \|f\| = \max_{x \in [a, b]} |f(x)| \} \). \( \omega(f, h) = \sup_{x, y \in [a, b]} |f(x) - f(y)| \) will denote the modulus of continuity of \( f \). \( P_n \) and \( T_n \) are, respectively, the spaces of algebraic and trigonometric polynomials of degree less than or equal to \( n \). \( C, C_1, C_2, \ldots \) denote absolute positive constants.

\( f \in C[a, b] \) will be called piecewise monotone if it has a finite number of maxima and minima on \([a, b]\). \( a, b \) and the local maxima and minima of \( f \) will be called the peaks of \( f \). It follows from a result proved independently by W. Wolibner [6] and S. W. Young [7] that if \( f \) is a piecewise monotone function on \([a, b]\) and \( \epsilon > 0 \), then there exists an algebraic polynomial \( p \) which increases and decreases simultaneously with \( f \) on \([a, b]\) and satisfies \( \|f - p\| < \epsilon \). We are concerned here with the accuracy of this type of approximation as a function of the degree of the approximating polynomial. Jackson’s classic theorem [2, p. 56] states that for any function \( f \in C[a, b] \) there exists \( p \in P_n \) such that \( \|f - p\| < C_1 \omega(f, 1/n) \). O. Shisha [5], J. Roulier [4], and G. G. Lorentz and K. L. Zeller [1] have obtained results on the accuracy of the approximation of an increasing function by increasing polynomials of degree less than or equal to \( n \). Lorentz and Zeller have shown that for each increasing function \( f \) on \([a, b]\) and for each \( n > 0 \) there exists an increasing \( p \in P_n \) such that \( \|f - p\| < C_2 \omega(f, 1/n) \). They have also shown that for any function \( f \), increasing on \([-\pi, 0]\) and even on \([0, \pi]\), there exists an even polynomial \( t \in T_n \), increasing on \([-\pi, 0]\), which satisfies \( \|f - t\| < C_3 \omega(f, 1/n) \). Our main results are Theorem 3 and Theorem 4.
in which we extend, in some sense, the result of Lorentz and Zeller to piecewise monotone functions.

We will let $J_n(x)$ be the Jackson kernel; i.e.,

$$J_n(x) = \frac{1}{\lambda_n} \left( \frac{\sin nx/2}{\sin x/2} \right)^4,$$

where $\lambda_n = \int_{-\pi}^{\pi} \left( \frac{\sin nx/2}{\sin x/2} \right)^4 dx.$

It is known [2, p. 54] that $n^3 \leq \lambda_n \leq 2n^3$.

**Lemma.** $J_n(x)$ has the following properties:

(i) If $n$ is even, $0 < \delta < \pi$, then

$$\int_0^{\delta} J_n(x) \, dx > C \int_0^{\pi-\delta} J_n(x) \, dx.$$

(ii) If $0 < \delta \leq \pi/2$, then

$$\int_{\delta}^{\pi} J_n(x) \, dx \leq C/n^3\delta^3.$$

**Proof.** (i) Case 1. $0 < \delta < \pi/2$.

Since $J_n(x)$ is an even periodic function,

$$\int_{\pi-\delta}^{\pi} J_n(x) \, dx = \int_{\pi}^{\pi+\delta} J_n(x) \, dx.$$

Thus,

$$\int_0^{\delta} J_n(x) \, dx - \int_{\pi-\delta}^{\pi} J_n(x) \, dx = \int_0^{\delta} [J_n(x) - J_n(x+n)] \, dx.$$

Now, when $n$ is even,

$$J_n(x) - J_n(x+n) = \frac{1}{\lambda_n} \left[ \left( \frac{\sin nx/2}{\sin x/2} \right)^4 \right].$$

Since $(\tan x/2)^4 \leq 1$ for $0 \leq x \leq \pi/2$, the integrand in (3) is nonnegative, and, hence, (3) is nonnegative for $0 < \delta < \pi/2$.

Case 2. $\pi/2 \leq \delta \leq \pi$.

Here

$$\int_0^{\delta} J_n(x) \, dx - \int_{\pi-\delta}^{\pi} J_n(x) \, dx = \int_0^{\pi-\delta} J_n(x) \, dx - \int_{\delta}^{\pi} J_n(x) \, dx,$$

and the result follows from Case 1.

(ii) $\int_{\delta}^{\pi} J_n(x) \, dx \leq \frac{1}{n^3} \int_{\delta}^{\pi} \left( \frac{\sin nx/2}{\sin x/2} \right)^4 dx \leq \frac{1}{n^3} \int_{\delta}^{\pi} \frac{dx}{(\sin x/2)^4}$.

Now $\sin(x/2) \geq x/\pi$ for $0 \leq x \leq \pi$. Therefore
\[ \int_{-\pi}^{\pi} \frac{n^4}{x^4} \, dx \leq \frac{\pi^4}{n^3} \int_{-\pi}^{\pi} \frac{dx}{x^4} = \frac{\pi^4}{3n^3} \left( \frac{1}{\delta} - \frac{1}{n^3} \right) \leq \frac{C_4}{n^3\delta^3}, \]

and the lemma is established.

**Theorem 1.** Let \( r(x) = |x|, -\pi \leq x \leq \pi, \) and be defined periodically by \( r(x + 2k\pi) = r(x) \) for \( k = \pm 1, \pm 2, \ldots. \) Then there exists \( t \in T_n \) such that

\begin{enumerate}[1)
  \item \( \max_{-\pi \leq x \leq \pi} |t(x) - r(x)| < C_5/n. \)
  \item \( t(x) \) decreases on \([-\pi, 0]\) and increases on \([0, \pi]\).
  \item \( t'(x) \) increases on \([-\pi/2, \pi/2]\).
  \item If \( 0 < \delta < \pi/2, \) then for all \( x \in [\delta, \pi - \delta] \)
    \[ 1 - C_6/n^3\delta^3 \leq t'(x) \leq 1, \]
    and for all \( x \in (-\pi + \delta, -\delta), -1 \leq t'(x) \leq -1 + C_6/n^3\delta^3. \)
  \item \( \int_{-\pi}^{\pi} |t(x) - r(x)| \leq 4C_5/n. \)
\end{enumerate}

**Proof.** For \( n \) even let

\[ t(x) = \int_{-\pi}^{\pi} J_n(u)r(x - u) \, du. \]

By Jackson's Theorem, \( t(x) \) satisfies (i). By periodicity,

\[ t(x) = \int_{x-\pi}^{x+\pi} J_n(u)r(x - u) \, du = \int_{x-\pi}^{x} J_n(u)(x - u) \, du - \int_{x}^{x+\pi} J_n(u)(x - u) \, du. \]

Hence

\[ t'(x) = \int_{x-\pi}^{x} J_n(u) \, du - \pi J_n(x - \pi) - \int_{x}^{x+\pi} J_n(u) \, du + \pi J_n(x + \pi) \]

(7) \[ = \int_{x-\pi}^{x} J_n(u) \, du - \int_{x}^{x+\pi} J_n(u) \, du. \]

Let \( x \in [0, \pi]. \) By the periodicity and evenness of \( J_n(u) \) we then get

\[ t'(x) = 2 \int_{0}^{\pi} J_n(u) \, du - 2 \int_{\pi-x}^{\pi} J_n(u) \, du. \]

Applying (1), \( t'(x) \geq 0. \) By a similar argument \( t'(x) \leq 0 \) for \(-\pi \leq x < 0,\) proving (ii) for \( n \) even. For \( n \) odd choose

\[ t(x) = \int_{-\pi}^{\pi} J_{n-1}(u)r(x - u) \, du, \]

and the proof follows as above. (i) and (ii) are established.

**Proof of (iii).** Differentiating in (7),

\[ t''(x) = J_n(x) - J_n(x - \pi) - J_n(x + \pi) + J_n(x) = 2[J_n(x) - J_n(x + \pi)]. \]

This quantity is nonnegative for \( 0 \leq x \leq \pi/2,\) as shown in the proof of part (i).
of the Lemma. Then \( t''(x) \geq 0 \) for \(-\pi/2 \leq x \leq \pi/2\) by evenness and periodicity of \( f_n(x) \).

**Proof of (iv).** Let \( x \in [\delta, \pi - \delta] \). We use (7) to obtain the desired estimates. Since \( \int_{-\pi}^\pi f_n(u) \, du = 1 \) and \( f_n(u) \geq 0 \), \( t'(x) \leq 1 \). Now

\[
t'(x) \geq \int_{-\delta}^{\delta} f_n(u) \, du \quad \text{which, by (2), gives}
\]

\[
t'(x) \geq 1 - 2C_4/n^3\delta^3 - 2C_4/n^3\delta^3 = 1 - C_5/n^3\delta^3.
\]

The estimates for \( x \in (\pi - \delta, 0) \) are similarly obtained.

**Proof of (v).** If \( h(x) = t(x) - r(x) \), then \( h'(x) \leq 0 \) on \([0, \pi]\) by (5). Hence the total variation of \( h(x) \) on \([0, \pi]\) is bounded by

\[
2 \max_{0 \leq x \leq \pi} |t(x) - r(x)| \leq \frac{2C_5}{n}
\]

by (4). Similarly the total variation of \( h(x) \) on \([-\pi, 0]\) is bounded by \( 2C_5/n \).

Since \( \int_{-\pi}^\pi |dh(x)| \) is equal to the total variation of \( h(x) \) on \([-\pi, \pi] \), (v) is proved.

**Definition.** Let \( f \) be piecewise monotone on \([a, b]\) with peaks at \( a = x_1 < x_2 < \ldots < x_m = b \). A sequence of polynomials \( \{p_n\} \) is said to be **comonotone** with \( f \) on \([a, b]\) if, for \( n \) sufficiently large, \( p_n \) increases and decreases simultaneously with \( f \) on \([a, b]\). \( \{p_n\} \) is said to be **nearly comonotone** with \( f \) on \([a, b]\) if, for every \( \epsilon \) satisfying \( 0 < \epsilon < \frac{1}{2} \min \{x_{i+1} - x_i\} \), \( p_n \) is comonotone with \( f \) on \([x_i + \epsilon, x_{i+1} - \epsilon]\), \( i = 1, 2, \ldots, m - 1 \).

Using this terminology, the sequence of polynomials defined by (6) and (8) is comonotone with \( r(x) \) on \([-\pi, \pi]\).

**Definition.** Let \( a = y_0 < y_1 < \ldots < y_k = b \) and let \( L \in C[a, b] \) be linear on \([y_j, y_{j+1}]\), \( j = 0, 1, \ldots, k - 1 \). Then \( L \) will be called **piecewise linear** on \([a, b]\) and \( y_0, y_1, \ldots, y_k \) will be called the **nodes** of \( L \). Let \( S_j \) be the slope of \( L \) on \([y_j, y_{j+1}]\); we let \( M(L) = \max_j |S_j| \) and we let \( m(L) = \min_j |S_j| \). If \( m(L) > 0 \) then \( L \) will be called a **proper piecewise linear** function.

**Theorem 2.** Let \( L \) be a proper piecewise linear function on \([-\pi/2, \pi/2]\). Then there exists a nearly comonotone sequence \( \{t_n\} \), \( t_n \in T_n \), such that

\[
\|L - t_n\| \leq C_7 M(L)/n.
\]

**Proof.** Let \( -\pi/2 = x_1 < \ldots < x_m = \pi/2 \) be the peaks of \( L \). We will construct a sequence \( \{t_n\}, t_n \in T_n \), satisfying (9) and such that if \( 0 < \epsilon < \frac{1}{2} \min (x_{i+1} - x_i) \) then \( t_n \) will have the same monotonicity as \( L \) on \([x_i + \epsilon, x_{i+1} - \epsilon]\), \( i = 1, 2, \ldots, m - 1 \) for all \( n \geq \epsilon^{-1} [C_5 M(L)/m(L)]^{1/3} \).

Let \( -\pi/2 = y_0 < y_1 < \ldots < y_k = \pi/2 \) be the nodes of \( L \). Let
a_0 = \frac{1}{2} (S_0 + S_{k-1}), \quad a_j = \frac{1}{2} (S_j - S_{j-1}), \quad j = 1, 2, \ldots, k - 1.

Then

(10) \quad L(x) = A + \sum_{j=0}^{k-1} a_j |x - y_j|,

where $A$ is a constant. Let $t$ be defined as in Theorem 1 [(6) and (8)] and let

(11) \quad t_n(x) = A + \sum_{j=0}^{k-1} a_j t(x - y_j).

Then $t_n \in T_n$. It follows [3, p. 147] that

\begin{align*}
|L(x) - t_n(x)| &\leq \frac{C_7}{n} \max_i \left| \sum_{j=0}^{i} a_j \right| \leq \frac{C_7}{n} \max_i \left| S_i \right| \leq \frac{C_7 m(L)}{n},
\end{align*}

thus establishing (9).

Now let $x \in [x_i + \epsilon, x_{i+1} - \epsilon]$. We assume, without loss of generality, that $L$ is increasing on $(x_i, x_{i+1})$. Let $y_q, y_{q+1}, \ldots, y_p$ be those consecutive nodes (if any) of $L(x)$ which are within $\epsilon$ of $x$; i.e., such that

(12) \quad |x - y_q| < \epsilon, \quad |x - y_{q+1}| < \epsilon, \quad \ldots, \quad |x - y_p| < \epsilon.

Now, differentiating in (11), we get

(13) \quad t_n'(x) = \sum_{j=0}^{k-1} a_j r_{ij} = \sum_{j=0}^{q-1} a_j r_{ij} + \sum_{q}^{q'} a_j r_{ij} + \sum_{q'+1}^{k-1} a_j r_{ij},

where $r_{ij}$ denotes $t'(x - y_i)$. For $j = q, \ldots, q'$ we have $|x - y_j| < \epsilon$, hence, from Theorem 1 (iii), $r_{ij}$ is decreasing as $j$ goes from $q$ to $q'$. Also, from Theorem 1 (ii) and Theorem 1 (iv), $0 \leq r_{ij} \leq 1$ for $x - y_j \geq 0$ and $-1 \leq r_{ij} \leq 0$ for $x - y_j \leq 0$. Applying the estimates from Theorem 1 (iv) for $t'$ to the first and third sums in (13), we get

(14) \quad t_n'(x) = \sum_{0}^{q-1} a_j - \sum_{0}^{q-1} a_j E_j + \sum_{q}^{q'} a_j r_j - \sum_{q'+1}^{k-1} a_j + \sum_{q'+1}^{k-1} a_j E_j,

where, from (5), we have $0 \leq E_j \leq C_6/n^3 \epsilon^3$. Now, from the definition of $\{a_j\}$, rewriting the summands, we obtain

\begin{align*}
&\sum_{0}^{q-1} a_j - \sum_{q'}^{k-1} a_j + \sum_{q}^{q'} a_j r_j \\
&= \frac{1}{2} \left[ S_{q-1}(1 - r_q) + \sum_{q}^{q'-1} S_j (r_j - r_{j+1}) + S_q (1 + r_q) \right] \geq m(L)
\end{align*}
by positivity of each term. Since $0 \leq E_j \leq C_6/n^3 \epsilon^3$, by partial summation we obtain

$$\left| \sum_{q+1}^{q-1} a_j E_j - \sum_{q+1}^{k-1} a_j E_j \right| \leq \frac{C_8 M(L)}{n^3 \epsilon^3}.$$  

Combining (15) and (16) in (14), we get $t'_n(x) \geq m(L) - C_8 M(L)/n^3 \epsilon^3$. This quantity is $> 0$ for $n \geq \epsilon^{-1} [C_8 M(L)/m(L)]^{1/3}$. Q.E.D.

Remark. In the proof of Theorem 2 we showed that $t_n$ will have the same monotonicity as $L$ on $[x_i + \epsilon, x_{i+1} - \epsilon]$, $i = 1, 2, \ldots, m - 1$, for $n \geq \epsilon^{-1} [C_8 M(L)/m(L)]^{1/3}$.

It is not difficult to see that $t_n$ and $L$ will have the same monotonicity at a point $x \in [x_i + \epsilon, x_{i+1} - \epsilon]$ for $n \geq \epsilon^{-1} [C_8 M(L)/m(x, \epsilon, L)]^{1/3}$, where $m(x, \epsilon, L)$ denotes the minimum slope of $L$ in an $\epsilon$-neighborhood of $x$. Indeed, in view of the choice of $q, \ldots, q', S_q, \ldots, S_{q'}$, $S_q \geq m(x, \epsilon, L)$; hence, when making a local estimate in (15), $m(L)$ can be replaced by $m(x, \epsilon, L)$. In particular, if $\epsilon$ is less than the distance from $x$ to the nearest node, then $n$ depends only on the slope at $x$.

A piecewise monotone function $f$ will be called proper piecewise monotone if it satisfies the following: for any $\epsilon > 0$ and two successive peaks $x_i, x_{i+1}$ of $f$ there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \geq \delta$$

for all $x, y$ in $[x_i + \epsilon, x_{i+1} - \epsilon], x \neq y$.

Theorem 3. Let $f$ be a proper piecewise monotone function on $[-\pi/2, \pi/2]$ such that $f \in \text{Lip}_{M, 1}$ (i.e., such that $f$ satisfies $\omega(f, h) \leq M_h$). Then there is a nearly comonotone sequence $\{t_n\}, t_n \in T_n$, such that

$$\|f - t_n\| \leq C_9 M/n.$$  

Proof. Let $L_n$ be the proper piecewise linear function on $[-\pi/2, \pi/2]$ which has nodes at the peaks of $f$ and at the points $-\pi/2 + j\pi/n, j = 0, 1, \ldots, n$, such that $L_n(x) = f(x)$ at the nodes. Then $L_n$ and $f$ have the same peaks and the same monotonicity for all $x \in [-\pi/2, \pi/2]$. Also,

$$\|f - L_n\| \leq M\pi/n$$

and $M(L_n) \leq M/n$. Hence, by Theorem 2, there is a polynomial $t_n \in T_n$ such that

$$\|L_n - t_n\| \leq C_7 M/n.$$
Now \( \|f - t_n\| \leq \|f - L_n\| + \|L_n - t_n\| \leq C_9 M/n \) from (20) and (21), establishing (19).

Since \( f \) is proper piecewise monotone, for any \( \epsilon > 0 \) there exists \( \delta(\epsilon) > 0 \) such that the slope \( S_i \) of \( L_n \) in \([y_i, y_{i+1}]\) satisfies \( S_i \geq \delta \) whenever \([y_i, y_{i+1}]\) is not within \( \epsilon \) of a peak. Let \( n \geq \epsilon^{-1} \left[ C_8 M/\delta(\epsilon) \right]^{1/3} \). In view of the remark following Theorem 2, \( t_n \) has the same monotonicity as \( L_n \) (and, hence, as \( f \)) at all points not within \( \epsilon \) of a peak.

Remark. Theorems 2 and 3 were stated for the interval \([-\pi/2, \pi/2]\), but are easily extended to the interval \([a, a + \pi/2]\) for any real number \( a \) via the translation \( x = u + a + \pi/2 \).

Theorem 4. Let \( f \) be a proper piecewise monotone function on \([a, b]\) such that \( f \in \text{Lip}_M^1 \). Then there is a nearly comonotone sequence \( \{p_n\}, p_n \in P_n \), such that \( \|f - p_n\| \leq C_10 M/n \).

This theorem is proved by use of the standard transformation \( x = \cos \theta \), with some modifications.

Note that the class of functions for which Theorem 3 and Theorem 4 are proved includes all \( f \) which have a continuous derivative that does not vanish except at the peaks. If we view monotonicity more "locally" and less "globally", we can state our results more precisely, in a sense, than we have in Theorem 3 and Theorem 4. This is done in Theorem 3' and Theorem 4', which are actually corollary (indeed, equivalent) to the results already established.

Theorem 3'. Let \( f \in \text{Lip}_M^1 \) on \([-\pi/2, \pi/2]\). Then there is a sequence \( \{t_n\}, t_n \in T_n \), such that

\[
\|f - t_n\| < C_9 M/n.
\]

Moreover, \( f \) and \( t_n \) will have the same monotonicity at \( x \) for all

\[
n \geq \epsilon^{-1} \left[ C_8 M/\delta(\epsilon) \right]^{1/3},
\]

where \( \epsilon \) is the distance from \( x \) to the nearest peak of \( f \) and

\[
\delta(\epsilon) = \inf_{0<|b|<\epsilon} \frac{\left| f(x + b) - f(x) \right|}{b}.
\]

Theorem 4'. Let \( f \in \text{Lip}_M^1 \) on \([a, b]\). Then there is a sequence \( \{p_n\}, p_n \in P_n \), such that

\[
\|f - p_n\| \leq C_10 M/n.
\]

Moreover, \( f \) and \( p_n \) will have the same monotonicity at \( x \) for all \( n \geq \epsilon^{-1} \left[ C_8 M/\delta(\epsilon) \right]^{1/3} \), where \( \epsilon \) and \( \delta(\epsilon) \) are the same as in Theorem 3'.
REFERENCES


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